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A Note on the Paper "Some Determinantal Inequalities on Complex Positive Definite Matrices"

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Abstract By presenting a counterexample, the author of paper (ZHAO Li-feng. J. Math. Res. Exposition, 2007, 27(4): 949–954) declared that some assertions in papers of LÜ Yun-xia, ZHANG Shu-qing (J. Math. Res. Exposition, 1999, 19(3): 598–600), HE Gan-tong (J. Math. Res. Exposition, 2002, 22(1): 79–82) and YUAN Hui-ping (J. Math. Res. Exposition, 2001, 21(3): 464–468) are wrong. In this note, we point out that the counterexample is wrong. Further discussion on these assertions and some related results are also given.

Keywords positive semi-definite matrix; determinantal inequality; Hermitian part.

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Let $R^{n\times n}$ and $C^{n\times n}$ denote the sets of $n\times n$ real matrices and complex matrices, respectively. For a matrix $A\in C^{n\times n}$, let A^* denote the conjugate transpose of A, and define $H(A)=\frac{1}{2}(A+A^*)$, the Hermitian part of A, $S(A)=\frac{1}{2}(A-A^*)$, the skew-Hermitian part of A. For $n\times n$ Hermitian matrices A and B, $A\geq B$ ($A>B$) will mean that $A-B$ is positive semi-definite (positive definite).

First we quote several theorems which the author of [1] took to be wrong.

Theorem 1 ([2, Theorem 3]) *Let $A,B\in R^{n\times n}$, $n\geq 2$. If $H(A)\geq 0$, $B>0$, then*

$$|\det(A+B)|^k\geq|\det A|^k+(\det B)^k, \tag{1}$$

where k is a real number such that $k(n+t)\geq 2$, t is the number of real eigenvalues of AB^{-1}.

Theorem 2 ([3, Theorem 2]) *Let $A,B\in C^{n\times n}$, $n\geq 2$. If $H(A)\geq 0$, $B>0$, then*

$$|\det(A+B)|^{\frac{2}{2n-s}}\geq|\det A|^{\frac{2}{2n-s}}+(\det B)^{\frac{2}{2n-s}}, \tag{2}$$

where s is the number of nonreal eigenvalues of AB^{-1}.

Theorem 3 ([4, Theorem 1]) *Let $A,B\in C^{n\times n}$, $n\geq 2$. If $H(A)>0$, $B>0$, then*

$$|\det(A+B)|^{\frac{1}{n-m}}>|\det A|^{\frac{1}{n-m}}+(\det B)^{\frac{1}{n-m}}, \tag{3}$$

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where m is the number of conjugate pairs of AB^{-1}'s nonreal eigenvalues.

Here is the counterexample in [1]:

Example Let $A = \begin{pmatrix} 21/5 & 0 \\ -11/5 & 21/5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Then $H(A) \geq 0$, $B > 0$, and $AB^{-1} = \begin{pmatrix} 14/5 & 7/5 \\ -1/15 & 31/15 \end{pmatrix}$. The characteristic polynomial $\det(\lambda I - AB^{-1}) = \lambda^2 - (73/15)\lambda + (441/75)$ has two real roots. For these A, B, the three inequalities in Theorems 1, 2 and 3 are equal to

$$|\det(A+B)|^{\frac{1}{2}} \geq |\det A\,|^{\frac{1}{2}} + (\det B)^{\frac{1}{2}}.$$

(In the case of Theorem 3, we take $m = 0$). The calculating results presented in [1] are

$$|\det(A+B)|^{\frac{1}{2}} = \sqrt{35.4} < 4.2 + \sqrt{3} = |\det A|^{\frac{1}{2}} + (\det B)^{\frac{1}{2}}.$$

So the author of [1] concluded that the Theorems 1, 2, 3 and their corollaries are wrong.

It is obvious that the above inequality should turn backward in direction. As a matter of fact, $\det(A+B)|^{\frac{1}{2}}$ equals $\sqrt{35.24}$, less than $\sqrt{35.4}$, and even so, one should have

$$\begin{aligned} |\det(A+B)|^{\frac{1}{2}} &= \sqrt{35.24} = 5.93632883\cdots \\ &> 4.2 + \sqrt{3} = 5.93205080\cdots \\ &= |\det A|^{\frac{1}{2}} + (\det B)^{\frac{1}{2}}. \end{aligned}$$

Thus the corresponding theorems cannot be negated according to this counterexample.

The above three theorems discuss the same thing, to establish a determinantal inequality

$$|\det(A+B)|^s \geq |\det A|^s + (\det B)^s$$

for some positive real number s.

The authors of [2] did not say clearly that the matrices in their paper were real or complex. It seems that they deal with only real matrices since they made use of a condition in their proof that the nonreal eigenvalues of AB^{-1} occur in conjugate pairs. So Theorem 2 is the generalization of Theorem 1 that the matrices in Theorem 2 may be taken to be complex. Explicitly, if t and s are the numbers of real eigenvalues and nonreal eigenvalues of AB^{-1} respectively, then $s + t = n$ and $n + t = 2n - s$. Hence the inequalities (1) (when $k = 2/(n+t)$) and (2) are just the same. It is obvious that if $a^l \geq b^l + c^l$ for some positive real number a, b, c, l, one has $a^k \geq b^k + c^k$ for all $k \geq l$. So if the inequality (1) holds for $k = 2/(n+t)$, Theorem 1 will hold.

There is something to be questioned for Theorem 3 indeed. Under the conditions of Theorem 3, the nonreal eigenvalues of AB^{-1} need not occur in conjugate pairs. If so, what does the m mean?

For example, let

$$A = \begin{pmatrix} 1 & i & i \\ i & 1 & i \\ i & i & 1 \end{pmatrix}$$

and $B = I$, where $i = \sqrt{-1}$, then $H(A) = I > 0$ and $B > 0$. The eigenvalues of AB^{-1} are $1+2i$, $1-i$, $1-i$. The nonreal eigenvalues of AB^{-1} do not occur in conjugate pairs. If we take $m = 0$ in this case, the determinantal inequality is

$$\begin{aligned} |\det(A+B)|^{\frac{1}{3}} &= \sqrt[6]{200} = 2.418271175\cdots \\ &< \sqrt[6]{20} + 1 = 2.647548927\cdots \\ &= |\det A|^{\frac{1}{3}} + (\det B)^{\frac{1}{3}}. \end{aligned}$$

The inequality in Theorem 3 does not hold for these A and B.

Examining the proof of Theorem 3 carefully, we find the author made use of a condition that the nonreal eigenvalues of AB^{-1} occur in conjugate pairs. This is not always true when $H(A) \geq 0$ and $B > 0$ unless A, B are real.

We conclude that Theorems 1 and 2 are faultless while Theorem 3 is not true unless A and B are restricted to be real matrices.

In resent yeas, a lot of results on the Minkowski type determinantal inequalities have appeared in literatures. For example, one can see [6–10]. Here we will give a brief discussion about the results in [6]. The author of [6] dealt with only real matrices and established the following theorem.

Theorem 4 ([6, Theorem 1]) *Let $A, B \in R^{n\times n}$, $n \geq 2$, and $H(A) > 0$, $B > 0$.*

(a) If $k \geq \frac{1}{n}$, then

$$|\det(A+B)|^k \geq 2^{-km}(|\det A|^k + (\det B)^k); \tag{4}$$

(b) If $k \geq \frac{1}{n-m}$, then

$$|\det(A+B)|^k \geq |\det A|^k + (\det B)^k, \tag{5}$$

where $2m$ is the number of nonreal eigenvalues of AB^{-1}.

And two more theorems were also given in [6] to discuss the same inequalities as (4) and (5), of course, under different conditions that A and B satisfied [6, Theorem 2, Theorem 5]. Thus the analogous Minkowski type determinantal inequalities recently appearing in literatures were collected in the paper and were discussed by similar method.

Examining Theorems 1, 2, 4 carefully, we point out that Theorem 1 and the part (b) of Theorem 4 are the same, since here $t + 2m = n$ and then $1/(n-m) = 2/(n+t)$. And inequality (4) is a special case of inequality (6) in Theorem 5 below. In fact, when A, B are restricted to be real matrices and $H(A) > 0$, (6) is turned into (4) for $k = 1/n$ in (4), since here $s = 2m$.

Theorem 5 ([3, Theorem 3]) *Let $A, B \in C^{n\times n}$, $n \geq 2$. If $H(A) \geq 0$, $B > 0$, then*

$$|\det(A+B)|^{\frac{1}{n}} \geq 2^{-\frac{s}{2n}}(|\det A|^{\frac{1}{n}} + (\det B)^{\frac{1}{n}}), \tag{6}$$

where s is the number of nonreal eigenvalues of AB^{-1}.

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