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Semistrictly Convex Fuzzy Mappings

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Abstract In this paper, a new class of fuzzy mappings called semistrictly convex fuzzy mappings is introduced and we present some properties of this kind of fuzzy mappings. In particular, we prove that a local minimum of a semistrictly convex fuzzy mapping is also a global minimum. We also discuss the relations among convexity, strict convexity and semistrict convexity of fuzzy mapping, and give several sufficient conditions for convexity and semistrict convexity.

Keywords convexity; strict convexity; semistrict convexity; lower semicontinuity; fuzzy mapping.

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1. Introduction

In 1992, Nanda and Kar [1] firstly introduced and discussed the concepts of convex fuzzy mapping and strictly convex fuzzy mapping, obtained the criteria for convex fuzzy mapping, and considered several applications to convex fuzzy optimizations. Subsequently, some authors have investigated various aspects of theory and application of convex fuzzy mappings [2–5]. Especially, in 2003, Wang and Wu [4] studied the applications to convex fuzzy programming by establishing the fuzzy subdifferential of fuzzy mapping.

Motivated both by earlier works of [1–6] and by the importance of the concept of semistrict convexity in classical convex analysis, we propose and the consider semistrict convexity of fuzzy mappings in this paper. In particular, we get that a local minimum of a semistrictly convex fuzzy mapping is also a global minimum, also discuss the relations among convexity, strict convexity and semistrict convexity of fuzzy mapping, and infer several sufficient conditions for convexity and semitrict convexity.

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2. Preliminaries

Let R^n be the n-dimensional Euclidean space. The family of fuzzy numbers for R^1 will be denoted by $\mathcal{F}_0$. Since each $r \in R^1$ can be considered as a fuzzy number $\widetilde{r}$ defined by

$$\widetilde{r}(t) = \begin{cases} 1, & \text{if } t = r, \\ 0, & \text{if } t \neq r. \end{cases}$$

It follows that R^1 can be embedded in $\mathcal{F}_0$.

As is known in [7], the α-level set of a fuzzy number $u \in \mathcal{F}_0$ is a nonempty closed and bounded interval

$$[u_*(\alpha), u^*(\alpha)] = [u]_\alpha = \begin{cases} \{x \in R^1 | u(x) \geq \alpha\}, & \text{if } 0 < \alpha \leq 1, \\ \text{cl(suppu)}, & \text{if } \alpha = 0. \end{cases}$$

Again from Lemma 2.2 of [7], we see that a fuzzy number $u : R^1 \to [0, 1]$ is determined by the end-points of the interval $[u]_\alpha$. Thus we can identify a fuzzy number u with the parameterized triples

$$\{(u_*(\alpha), u^*(\alpha), \alpha)|\ 0 \leq \alpha \leq 1\},$$

where $u_*(\alpha)$ and $u^*(\alpha)$ denote the left-and right-hand endpoints of $[u]_\alpha$, respectively. We denote it as $u = \{(u_*(\alpha), u^*(\alpha), \alpha)|\ 0 \leq \alpha \leq 1\}$.

For any $u, v \in \mathcal{F}_0$ represented by $\{(u_*(\alpha), u^*(\alpha), \alpha)|\ 0 \leq \alpha \leq 1\}$ and $\{(v_*(\alpha), v^*(\alpha), \alpha)\ 0 \leq \alpha \leq 1\}$, respectively, and each nonnegative real number r, we define the addition $u + v$ and 'scalar' multiplication ru as follows:

$$\begin{aligned} u + v &= \{(u_*(\alpha) + v_*(\alpha), u^*(\alpha) + v^*(\alpha),\ \alpha)|0 \leq \alpha \leq 1\} \\ &= \{(u_*(\alpha), u^*(\alpha), \alpha)|0 \leq \alpha \leq 1\} + \{(v_*(\alpha), v^*(\alpha), \alpha)|\ 0 \leq \alpha \leq 1\}, \\ ru &= \{(ru_*(\alpha), ru^*(\alpha), \alpha)|\ 0 \leq \alpha \leq 1\}. \end{aligned}$$

Then $u + v \in \mathcal{F}_0,\ ru \in \mathcal{F}_0$ and

$$[u+v]_*(\alpha) = u_*(\alpha)+v_*(\alpha),\ [u+v]^*(\alpha) = u^*(\alpha)+v^*(\alpha);\ \ [ru]_*(\alpha) = ru_*(\alpha),\ \ [ru]^*(\alpha) = ru^*(\alpha).$$

Definition 2.1 *For u, $v \in \mathcal{F}_0$,*

(a) we say that $u \leq v$ if for every $\alpha \in [0, 1]$, $u^*(\alpha) \leq v^*(\alpha)$ and $u_*(\alpha) \leq v^*(\alpha)$;

(b) we say that $u < v$, if $u \leq v$ and there exists $\alpha_0 \in [0, 1]$ such that

$$u^*(\alpha_0) < v^*(\alpha_0) \quad \text{or} \quad u_*(\alpha_0) < v_*(\alpha_0);$$

(c) we say that $u = v$, if $u \leq v$ and $v \leq u$,

where $u = \{(u_*(\alpha), u^*(\alpha), \alpha)|0 \leq \alpha \leq 1\}$, $v = \{(v_*(\alpha), v^*(\alpha), \alpha)|0 \leq \alpha \leq 1\}$.

In the following, we recall the concept of convex fuzzy mappings. Throughout this paper, let C be a nonempty convex subset of R^n.

Definition 2.2 ([1]) *A fuzzy mapping $F : C \to \mathcal{F}_0$ is said to be convex if*

$$F(\lambda x + (1 - \lambda)y) \leq \lambda F(x) + (1 - \lambda)F(y),$$

for every $x, y \in C$ and $\lambda \in [0,1]$.

Definition 2.3 ([1]) *A fuzzy mapping $F : C \to \mathcal{F}_0$ is said to be strictly convex if*

$$F(\lambda x + (1-\lambda)y) < \lambda F(x) + (1-\lambda)F(y),$$

for every $x, y \in C,\ x \neq y$ and $\lambda \in [0,1]$.

Definition 2.4 *A fuzzy mapping $F : C \to \mathcal{F}_0$ is said to be lower semicontinuous at a point $x_0 \in C$, if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$F(x_0) \leq F(x) + \widetilde{\varepsilon},$$

for all $x \in C$ and $||x - x_0|| < \delta$. F is said to be lower semicontinuous on C if it is lower semicontinuous at each point of C.

3. Semistrictly convex fuzzy mappings

In this section, we introduce the concept of a new class of fuzzy mapping termed semistrictly convex fuzzy mapping, and discuss its properties. In particular, the following Theorem 3.1 shows that a local minimum of a semistrictly convex fuzzy mapping over a convex set is also a global minimum.

Definition 3.1 *A fuzzy mapping $F : C \to \mathcal{F}_0$ is said to be semistrictly convex if*

$$F(\lambda x + (1-\lambda)y) < \lambda F(x) + (1-\lambda)F(y),$$

for every $x, y \in C,\ F(x) \neq F(y)$ and $\lambda \in (0,1)$.

The following example illustrates that a semistrictly convex fuzzy mapping is not a convex fuzzy mapping.

Example 3.1 Let

$$F(x) = \begin{cases} \{(1,1,r) |\ 0 \leq r \leq 1\}, & x = 1, \\ \{(-\sqrt{1-r}, \sqrt{1-r}, r) |\ 0 \leq r \leq 1\}, & x \neq 1 \text{ and } x \in [0,3]. \end{cases}$$

Then $F : [0,3] \to \mathcal{F}_0$ is a semistrictly convex fuzzy mapping, but is not a convex fuzzy mapping. Therefore, F is not strictly convex fuzzy mapping either.

Proof Clearly, every $x, y \in [0,3]$, $F(x) \neq F(y)$ iff $x = 1,\ y \in [0,1) \cup (1,3]$ or $y = 1,\ x \in [0,1) \cup (1,3]$, and for $0 < \lambda < 1$ we have $\lambda x + (1-\lambda)y \in [0,1) \cup (1,3]$.

Without loss of generality, we assume that $x = 1$ and $y \in [0,1) \cup (1,3]$. Then

$$\begin{aligned} F(\lambda x + (1-\lambda)y) &= \{(-\sqrt{1-r}, \sqrt{1-r}, r) | 0 \leq r \leq 1\}, \\ \lambda F(x) + (1-\lambda)F(y) &= \lambda\{(1,1,r)|0 \leq r \leq 1\} + (1-\lambda)\{(-\sqrt{1-r}, \sqrt{1-r}, r)|0 \leq r \leq 1\} \\ &= \{(\lambda - (1-\lambda)\sqrt{1-r}, \lambda + (1-\lambda)\sqrt{1-r}, r)|0 \leq r \leq 1\}, \end{aligned}$$

which implies that $F(\lambda x + (1-\lambda)y) < \lambda F(x) + (1-\lambda)F(y)$. It follows that F is a semistrictly convex fuzzy mapping.

Let $x = 2$, $y = \frac{1}{2}$, $\lambda = \frac{1}{3}$. Then

$$F(\lambda x + (1-\lambda)y) = F(1) = \{(1, 1, r)|0 \leq r \leq 1\},$$

$$\begin{aligned}\lambda F(x) + (1-\lambda)F(y) &= \frac{1}{3}\{(-\sqrt{1-r}, \sqrt{1-r}, r)|0 \leq r \leq 1\} + \frac{2}{3}\{(-\sqrt{1-r}, \sqrt{1-r}, r)|0 \leq r \leq 1\} \\ &= \{(-\sqrt{1-r}, \sqrt{1-r}, r)|0 \leq r \leq 1\},\end{aligned}$$

which implies that $F(\lambda x + (1-\lambda)y) > \lambda F(x) + (1-\lambda)F(y)$. It follows that F is not a convex fuzzy mapping.

Theorem 3.1 *Let $F : C \to \mathcal{F}_0$ be a semistrictly convex fuzzy mapping. If $\overline{x} \in C$ is a local optimal solution to the problem of minimizing $F(x)$ subject to $x \in C$, then $\overline{x}$ is a global minimum.*

Proof Suppose that $\overline{x} \in C$ is a local minimum. Then there is a neighborhood $N(\overline{x})$ such that

$$F(\overline{x}) \leq F(x), \quad \forall x \in C \cap N(\overline{x}). \tag{1}$$

If $\overline{x}$ is not a global minimum of F, then there exists a $y \in C$ such that $F(y) \not\geq F(\overline{x})$. Hence there exists $r_0 \in [0, 1]$ such that

$$F(y)^*(r_0) < F(\overline{x})^*(r_0) \quad \text{or} \quad F(y)_*(r_0) < F(\overline{x})_*(r_0).$$

So that by the semistrict convexity of F, for every $\lambda \in (0, 1)$ we have

$$F(\lambda y + (1-\lambda)\overline{x}) < \lambda F(y) + (1-\lambda)F(\overline{x}),$$

which implies that

$$F(\lambda y + (1-\lambda)\overline{x})^*(r_0) \leq \lambda F(y)^*(r_0) + (1-\lambda)F(\overline{x})^*(r_0) < F(\overline{x})^*(r_0),$$

or

$$F(\lambda y + (1-\lambda)\overline{x})_*(r_0) \leq \lambda F(y)_*(r_0) + (1-\lambda)F(\overline{x})_*(r_0) < F(\overline{x})_*(r_0).$$

Therefore, we have $F(\lambda y + (1-\lambda)\overline{x}) \not\geq F(\overline{x})$, for all $0 < \lambda < 1$. For a sufficiently small $\lambda > 0$, it follows that $\lambda y + (1-\lambda)\overline{x} \in C \cap N(\overline{x})$, which is a contradiction to (1).

Remark 3.1 From Example 3.1 and Theorem 3.1, we can conclude that the class of semistrictly convex fuzzy mapping constitutes an important new class of convex fuzzy mapping in fuzzy programming.

Theorem 3.2 *Let $F : C \to \mathcal{F}_0$ be a semistrictly convex fuzzy mapping, and let $G : \mathcal{F}_0 \to \mathcal{F}_0$ be a convex and strictly increasing mapping. Then the composite mapping $G(F)$ is a semistrictly convex fuzzy mapping on C.*

Proof For any $x, y \in C$, $\lambda \in (0, 1)$, if $G(F(x)) \neq G(F(y))$, then $F(x) \neq F(y)$. Since F is a semistrictly convex fuzzy mapping, we have $F(\lambda x + (1-\lambda)y) < \lambda F(x) + (1-\lambda)F(y)$. From the convexity and strictly increasing property of G, we obtain

$$G(F(\lambda x + (1-\lambda)y) < G(\lambda F(x) + (1-\lambda)F(y)) \leq \lambda G(F(x)) + (1-\lambda)G(F(y)).$$

Hence, $G(F)$ is a semistrictly convex fuzzy mapping on C.

Naturally, we can get the following result.

Theorem 3.3 *Let $F : C \to \mathcal{F}_0$ be a semistrictly convex fuzzy mapping, and let $G : \mathcal{F}_0 \to \mathcal{F}_0$ be a strictly convex and increasing mapping. Then the composite mapping $G(F)$ is a semistrictly convex fuzzy mapping on C.*

4. The sufficient conditions for convex and strictly convex fuzzy mapping

We know that semistrict convexity cannot imply convexity and strict convexity. Nevertheless, we have the following interesting results.

Theorem 4.1 *Let $F : C \to \mathcal{F}_0$ be a semistrictly convex fuzzy mapping. If there exists $\alpha \in (0,1)$ such that*

$$F(\alpha x + (1-\alpha)y) \leq \alpha F(x) + (1-\alpha)F(y), \quad \forall x, y \in C.$$

Then F is a convex fuzzy mapping on C.

Proof Suppose that there exist $x, y \in C$ and $\lambda \in (0,1)$ such that

$$F(\lambda x + (1-\lambda)y) \not\leq \lambda F(x) + (1-\lambda)F(y).$$

Then there exists $r_0 \in (0,1)$ such that

$$F(\lambda x + (1-\lambda)y)^*(r_0) > \lambda F(x)^*(r_0) + (1-\lambda)F(y)^*(r_0) \tag{2}$$

or

$$F(\lambda x + (1-\lambda)y)_*(r_0) > \lambda F(x)_*(r_0) + (1-\lambda)F(y)_*(r_0).$$

Without loss of generality, we assume that (2) holds true.

(I) If $F(x) \neq F(y)$, then by semistrict convexity of F, for any $\lambda \in (0,1)$ we have

$$F(\lambda x + (1-\lambda)y) < \lambda F(x) + (1-\lambda)F(y),$$

which is a contradiction to (2).

(II) If $F(x) = F(y)$, let $z_\lambda = \lambda x + (1-\lambda)y$, then (2) implies that

$$\begin{aligned} F(z_\lambda)^*(r_0) &= F(\lambda x + (1-\lambda)y)^*(r_0) > \lambda F(x)^*(r_0) + (1-\lambda)F(y)^*(r_0) \\ &= F(x)^*(r_0) = F(y)^*(r_0). \end{aligned} \tag{3}$$

(i) If $0 < \alpha < \lambda \leq 1$, let $\mu = (\lambda - \alpha)/(1-\alpha)$, then $0 < \mu < \lambda \leq 1$ and

$$\begin{aligned} z_\lambda &= \lambda x + (1-\lambda)y = (\mu(1-\alpha) + \alpha)x + (1 - (\mu(1-\alpha)+\alpha))y \\ &= \alpha x + (1-\alpha)\mu x + (1-\alpha)y - \mu(1-\alpha)y \\ &= \alpha x + ((1-\alpha)\mu x + (1-\alpha)(1-\mu)y) = \alpha x + (1-\alpha)z_\mu. \end{aligned}$$

Hence by the hypothesis of the theorem and (3), we have

$$F(z_\lambda)^*(r_0) = F(\alpha x + (1-\alpha)z_\mu)^*(r_0)$$

$$\leq \alpha F(x)^*(r_0) + (1-\alpha)F(z_\mu)^*(r_0) < \alpha F(z_\lambda)^*(r_0) + (1-\alpha)F(z_\mu)^*(r_0),$$

which implies that $(1-\alpha)F(z_\lambda)^*(r_0) < (1-\alpha)F(z_\mu)^*(r_0)$. Therefore, we get

$$F(z_\lambda)^*(r_0) < F(z_\mu)^*(r_0). \tag{4}$$

On the other hand, let $\theta = (\lambda-\mu)/\lambda$. Then $0 < \theta < 1$ and by the same method we can get that $z_\mu = \theta y + (1-\theta)z_\lambda$. Hence by the hypothesis of the theorem and (3), we have

$$F(z_\mu)^*(r_0) = F(\theta y + (1-\theta)z_\lambda)^*(r_0) \leq \theta F(y)^*(r_0) + (1-\theta)F(z_\lambda)^*(r_0) < F(z_\lambda)^*(r_0),$$

which is a contradiction to (4).

(ii) If $0 < \lambda < \alpha < 1$, let $\mu = \lambda/\alpha$. Then $0 < \mu < 1$ and we can get that $z_\lambda = \alpha x + (1-\alpha)z_\mu$. Hence by the hypothesis of the theorem and (2), we have

$$F(z_\lambda)^*(r_0) \leq \alpha F(y)^*(r_0) + (1-\alpha)F(z_\mu)^*(r_0) < F(z_\mu)^*(r_0). \tag{5}$$

On the other hand, let $\theta = (\mu-\lambda)/(1-\lambda)$. Then $0 < \theta < 1$ and we can obtain that $z_\mu = \theta x + (1-\theta)z_\lambda$. Therefore, by the semistrict convexity of F and (3), we have

$$F(z_\mu)^*(r_0) = F(\theta x + (1-\theta)z_\lambda)^*(r_0) \leq \theta F(x)^*(r_0) + (1-\theta)F(z_\lambda)^*(r_0) < F(z_\lambda)^*(r_0),$$

which is a contradiction to (5). This completes the proof. □

Theorem 4.2 *Let $F: C \to \mathcal{F}_0$ be a semistrictly convex fuzzy mapping. If there exists $\alpha \in (0,1)$ such that for any $x, y \in C$ and $x \neq y$*

$$F(\alpha x + (1-\alpha)y) < \alpha F(x) + (1-\alpha)F(y),$$

then F is a strictly convex fuzzy mapping on C.

Proof Since F is semistrictly convex, we only need to show that if $F(x) = F(y)$ and $x \neq y$, then

$$F(\lambda x + (1-\lambda)y) < \lambda F(x) + (1-\lambda)F(y) = F(x) = F(y), \quad \forall \lambda \in (0,1).$$

By the hypothesis of the theorem and for any $x, y \in C$, $x \neq y$ and $F(x) = F(y)$, we have

$$F(\alpha x + (1-\alpha)y) < \alpha F(x) + (1-\alpha)F(y) = F(x) = F(y).$$

Let $z = \alpha x + (1-\alpha)y$. For each $\lambda \in (0,1)$, if $\lambda > \alpha$, then taking $\mu = (\lambda-\alpha)/(1-\alpha) \in (0,1)$, we have

$$\begin{aligned} \mu x + (1-\mu)z &= ((\lambda-\alpha)/(1-\alpha))x + (1-(\lambda-\alpha)/(1-\alpha))(\alpha x + (1-\alpha)y) \\ &= ((\lambda-\alpha)/(1-\alpha) + \alpha(1-\lambda)/(1-\alpha))x + ((1-\lambda)(1-\alpha)/(1-\alpha))y \\ &= \lambda x + (1-\lambda)y. \end{aligned}$$

Hence by the semistrict convexity of F and $F(z) < F(x)$, we have

$$F(\lambda x + (1-\lambda)y) = F(\mu x + (1-\mu)z) < \mu F(x) + (1-\mu)F(z) < F(x).$$

If $\lambda < \alpha$, then taking $\theta = \lambda/\alpha \in (0,1)$, we have

$$\theta z + (1-\theta)y = (\lambda/\alpha)(\alpha x + (1-\alpha)y) + (1-\lambda/\alpha)y = \lambda x + (1-\lambda)y.$$

Again by the semistrict convexity of F and $F(z) < F(y)$, we have

$$F(\lambda x + (1-\lambda)y) = F(\theta z + (1-\theta)y) < \theta F(z) + (1-\theta)F(y) < F(y).$$

This completes the proof. □

5. The sufficient conditions for semistrictly convex fuzzy mappings

We know that convexity can not imply semistrict convexity. Nevertheless, we have the following interesting results.

Theorem 5.1 *Let $F : C \to \mathcal{F}_0$ be a convex fuzzy mapping. If there exists $\alpha \in (0,1)$ such that for any $x, y \in C$, $F(x) \neq F(y)$*

$$F(\alpha x + (1-\alpha)y) < \alpha F(x) + (1-\alpha)F(y),$$

then F is a semistrictly convex fuzzy mapping on C.

Proof Suppose that there exist $x, y \in C$ and $\lambda \in (0,1)$ such that

$$F(x) \neq F(y) \quad \text{and} \quad F(\lambda x + (1-\lambda)y) \not< \lambda F(x) + (1-\lambda)F(y).$$

Then $F(\lambda x + (1-\lambda)y) \not\leq \lambda F(x) + (1-\lambda)F(y)$ or $F(\lambda x + (1-\lambda)y) \geq \lambda F(x) + (1-\lambda)F(y)$.

(I) If $F(\lambda x + (1-\lambda)y) \not\leq \lambda F(x) + (1-\lambda)F(y)$, then it contradicts the convexity of F.

(II) If $F(\lambda x + (1-\lambda)y) \geq \lambda F(x) + (1-\lambda)F(y)$, then by $F(x) \neq F(y)$ we have $F(x) \not\leq F(y)$ or $F(x) \not\geq F(y)$. Hence there exists $r_0 \in [0,1]$ such that

$F(x)^*(r_0) > F(y)^*(r_0)$ or $F(x)_*(r_0) > F(y)_*(r_0)$ or $F(x)^*(r_0) < F(y)^*(r_0)$ or $F(x)_*(r_0) < F(y)_*(r_0)$. Without loss of generality, suppose that $F(x)^*(r_0) < F(y)^*(r_0)$ and let $z = \lambda x + (1-\lambda)y$. Then

$$F(z)^*(r_0) \geq \lambda F(x)^*(r_0) + (1-\lambda)F(y)^*(r_0) > F(x)^*(r_0). \tag{6}$$

Hence by the convexity of F and (6) we have

$$F(\alpha x + (1-\alpha)z)^*(r_0) \leq \alpha F(x)^*(r_0) + (1-\alpha)F(z)^*(r_0) < F(z)^*(r_0),$$

$$F(\alpha^2 x + (1-\alpha^2)z)^*(r_0) = F(\alpha(\alpha x + (1-\alpha)z) + (1-\alpha)z)^*(r_0) < F(z)^*(r_0),$$

$$\cdots$$

$$F(\alpha^k x + (1-\alpha^k)z)^*(r_0) = F(\alpha(\alpha^{k-1}x + (1-\alpha^{k-1})z) + (1-\alpha)z)^*(r_0) < F(z)^*(r_0),\ \forall k \in N. \tag{7}$$

From $z = \lambda x + (1-\lambda)y$, we have

$$\alpha^k x + (1-\alpha^k)z = \alpha^k x + (1-\alpha^k)(\lambda x + (1-\lambda)y) = (\lambda - \alpha^k\lambda + \alpha^k)x + (1-\lambda-\alpha^k+\alpha^k\lambda)y.$$

Take $k_1 \in N$ such that $\alpha^{k_1}/(1-\alpha) < \lambda/(1-\lambda)$, and let

$\beta_1 = \lambda + \alpha^{k_1}(1-\lambda)$, $\beta_2 = \lambda - (\alpha^{k_1+1}/(1-\alpha))(1-\lambda)$; $\overline{x} = \beta_1 x + (1-\beta_1)y$, $\overline{y} = \beta_2 x + (1-\beta_2)y$.

Then

$$\beta_1, \beta_2 \in (0,1) \quad \text{and} \quad \alpha^{k_1}x + (1-\alpha^{k_1})z = \beta_1 x + (1-\beta_1)y = \overline{x}. \tag{8}$$

Hence by (7) and (8), we have

$$F(\overline{x})^*(r_0) = F(\beta_1 x + (1-\beta_1)y)^*(r_0) = F(\alpha^{k_1}x + (1-\alpha^{k_1})z)^*(r_0) < F(z)^*(r_0). \tag{9}$$

(i) If $F(\overline{x})^*(r_0) \geq F(\overline{y})^*(r_0)$, then from $z = \lambda x + (1-\lambda)y = \alpha\overline{x} + (1-\alpha)\overline{y}$ and the convexity of F, we obtain that

$$F(z)^*(r_0) \leq \alpha F(\overline{x})^*(r_0) + (1-\alpha)F(\overline{y})^*(r_0) \leq F(\overline{x})^*(r_0),$$

which contradicts inequality (9).

(ii) If $F(\overline{x})^*(r_0) < F(\overline{y})^*(r_0)$, then from $z = \alpha\overline{x} + (1-\alpha)\overline{y}$ and the hypothesis of the theorem, we get that

$$\begin{aligned} F(z) &< \alpha F(\overline{x}) + (1-\alpha)F(\overline{y}) \leq \alpha(\beta_1 F(x) + (1-\beta_1)F(y)) + (1-\alpha)(\beta_2 F(x) + (1-\beta_2)F(y)) \\ &= (\alpha\beta_1 + (1-\alpha)\beta_2)F(x) + (\alpha(1-\beta_1) + (1-\alpha)(1-\beta_2))F(y) = \lambda F(x) + (1-\lambda)F(y), \end{aligned}$$

which contradicts inequality (6). This completes the proof. □

Theorem 5.2 *Let C be a closed set, and let $F : C \to \mathcal{F}_0$ be a lower semicontinuous fuzzy mapping. If there exists $\alpha \in (0,1)$ such that for any $x, y \in C$, $F(x) \neq F(y)$ we have*

$$F(\alpha x + (1-\alpha)y) < \alpha F(x) + (1-\alpha)F(y),$$

then F is a semistrictly convex fuzzy mapping on C.

Proof (I) At first, we show that for any $x, y \in C$ there exists $\lambda \in (0,1)$ such that

$$F(\lambda x + (1-\lambda)y) \leq \lambda F(x) + (1-\lambda)F(y).$$

Suppose that there exist $x, y \in C$ such that $\forall \lambda \in (0,1)$

$$F(\lambda x + (1-\lambda)y) \not\leq \lambda F(x) + (1-\lambda)F(y). \tag{10}$$

If $F(x) \neq F(y)$, then by the hypothesis of the theorem, there exists $\alpha \in (0,1)$ such that

$$F(\alpha x + (1-\alpha)y) < \alpha F(x) + (1-\alpha)F(y),$$

which is a contradiction to (10).

If $F(x) = F(y)$, then for given $\lambda \in (0,1)$, by (10) we know that $F(\lambda x + (1-\lambda)y) \neq F(x) = F(y)$ and there exists $r_\lambda \in [0,1]$ such that

$$F(\lambda x + (1-\lambda)y)^*(r_\lambda) > \lambda F(x)^*(r_\lambda) + (1-\lambda)F(y)^*(r_\lambda) = F(x)^*(r_\lambda) = F(y)^*(r_\lambda) \tag{11}$$

or

$$F(\lambda x + (1-\lambda)y)_*(r_\lambda) > \lambda F(x)_*(r_\lambda) + (1-\lambda)F(y)_*(r_\lambda) = F(x)_*(r_\lambda) = F(y)_*(r_\lambda).$$

Without loss of generality, we assume that (11) holds true. Then by the hypothesis of the theorem and (11), we have

$$\begin{aligned} F(x)^*(r_{1-\alpha(1-\lambda)}) = F(y)^*(r_{1-\alpha(1-\lambda)}) &< F((1-\alpha(1-\lambda))x + \alpha(1-\lambda)y)^*(r_{1-\alpha(1-\lambda)}) \\ &= F(\alpha(\lambda x + (1-\lambda)y) + (1-\alpha)x)^*(r_{1-\alpha(1-\lambda)}) \end{aligned}$$

$$\leq \alpha F(\lambda x + (1-\lambda)y)^*(r_{1-\alpha(1-\lambda)}) + (1-\alpha)F(x)^*(r_{1-\alpha(1-\lambda)}),$$

which implies that $\alpha F(x)^*(r_{1-\alpha(1-\lambda)}) < \alpha F(\lambda x + (1-\lambda)y)^*(r_{1-\alpha(1-\lambda)})$. Hence we have

$$F(y)^*(r_{1-\alpha(1-\lambda)}) = F(x)^*(r_{1-\alpha(1-\lambda)}) < F(\lambda x + (1-\lambda)y)^*(r_{1-\alpha(1-\lambda)}).$$

Therefore, we obtain

$$\begin{aligned} &F(x)^*(r_{1-\alpha(1-\lambda)}) = F(y)^*(r_{1-\alpha(1-\lambda)}) \\ &\quad < F((1-\alpha(1-\lambda))x + \alpha(1-\lambda)y)^*(r_{1-\alpha(1-\lambda)}) < F(\lambda x + (1-\lambda)y)^*(r_{1-\alpha(1-\lambda)}). \end{aligned}$$

Again by the hypothesis of the theorem and the above inequality, we have

$$\begin{aligned} &F(\alpha((1-\alpha(1-\lambda))x + \alpha(1-\lambda)y) + (1-\alpha)y)^*(r_{1-\alpha(1-\lambda)}) \\ &\quad \leq \alpha F((1-\alpha(1-\lambda))x + \alpha(1-\lambda)y)^*(r_{1-\alpha(1-\lambda)}) + (1-\alpha)F(y)^*(r_{1-\alpha(1-\lambda)}) \\ &\quad < F((1-\alpha(1-\lambda))x + \alpha(1-\lambda)y)^*(r_{1-\alpha(1-\lambda)}) \\ &\quad < F(\lambda x + (1-\lambda)y)^*(r_{1-\alpha(1-\lambda)}). \end{aligned}$$

Let $\lambda = \alpha/(1+\alpha) \in (0,1)$. Then the above inequality implies that

$$F\Big(\frac{\alpha}{1+\alpha}x + \frac{1}{1+\alpha}y\Big)^*\Big(\frac{1}{1+\alpha}\Big) < F\Big(\frac{\alpha}{1+\alpha}x + \frac{1}{1+\alpha}y\Big)^*\Big(\frac{1}{1+\alpha}\Big),$$

which is a contradiction.

(II) Secondly, we show that F is a convex fuzzy mapping on C. By the lower semicontinuity of F on C, we know that for any $r \in [0,1]$, both $F(x)^*(r)$ and $F(x)_*(r)$ are lower semicontinuous real valued functions. It follows from the closeness of C, we can easily check that the epigraphs of $F(x)^*(r)$ and $F(x)_*(r)$:

$$\mathrm{epi}(F(\cdot)^*(r)) = \{(x,a) | x \in C \ \text{ and } \ F(x)^*(r) \leq a\},$$

$$\mathrm{epi}(F(\cdot)_*(r)) = \{(x,a) |\ x \in C \ \text{ and } \ F(x)_*(r) \leq a\}$$

both are closed in R^{n+1}. Since by (I) there exists $\lambda \in (0,1)$ such that

$$F(\lambda x + (1-\lambda)y)(r) \leq \lambda F(x) + (1-\lambda)F(y),$$

for any $r \in [0,1]$ we get

$$F(\lambda x + (1-\lambda)y)^*(r) \leq \lambda F(x)^*(r) + (1-\lambda)F(y)^*(r),$$

$$F(\lambda x + (1-\lambda)y)_*(r) \leq \lambda F(x)_*(r) + (1-\lambda)F(y)_*(r).$$

It follows that for any $u, v \in \mathrm{epi}(F(\cdot)^*(r))(u, v \in \mathrm{epi}(F(\cdot)_*(r)))$, there exists $\lambda \in (0,1)$ such that

$$\lambda u + (1-\lambda)v \in \mathrm{epi}(F(\cdot)^*(r))(\lambda u + (1-\lambda)v \in \mathrm{epi}(F(\cdot)_*(r))). \tag{12}$$

Otherwise, suppose that F is not a convex mapping on C. Then there exist distinct $x, y \in C$ and $\alpha \in (0,1)$ such that

$$F(\alpha x + (1-\alpha)y) \not\preceq \alpha F(x) + (1-\alpha)F(y).$$

Hence there exists $r_0 \in [0,1]$ such that

$$F(\alpha x+(1-\alpha)y)^*(r_0) > \alpha F(x)^*(r_0)+(1-\alpha)F(y)^*(r_0)$$

or

$$F(\alpha x+(1-\alpha)y)_*(r_0) > \alpha F(x)_*(r_0)+(1-\alpha)F(y)_*(r_0).$$

Without loss of generality, we assume that

$$F(\alpha x+(1-\alpha)y)^*(r_0) > \alpha F(x)^*(r_0)+(1-\alpha)F(y)^*(r_0).$$

Then for $u=(x,F(x)^*(r_0)) \in \mathrm{epi}(F\cdot)^*(r_0))$, $v=(y,F(y)^*(r_0)) \in \mathrm{epi}(F(\cdot)^*(r_0))$, we have

$$\omega = \alpha u+(1-\alpha)v = (\alpha x+(1-\alpha)y,\ \alpha F(x)^*(r_0)+(1-\alpha)F(y)^*(r_0))\overline{\in}\mathrm{epi}(F(\cdot)^*(r_0)).$$

Let $u_t = tu+(1-t)\omega,\ r=\inf\{t\in[0,1]|\ u_t \in \mathrm{epi}(F(\cdot)^*(r_0))\}$. Then there exists a sequence of points $\{t_n\}\subset[0,1]$ such that

$$u_{t_n} \in \mathrm{epi}(F(\cdot)^*(r_0)) \ \text{ and } \ t_n \to r\ (n\to\infty).$$

From the continuity of u_t at r and the closednees of $\mathrm{epi}(F(\cdot)^*(r_0))$, we have $u_r \in \mathrm{epi}(F(\cdot)^*(r_0))$. Since $u_0=\omega\overline{\in}\mathrm{epi}(F(\cdot)^*(r_0))$, $r>0$, by the definition of r, we have

$$u_t\overline{\in}\mathrm{epi}(F(\cdot)^*(r_0)) \ \text{ for any } \ t\in[0,r).$$

Similarly, let $v_t = tv+(1-t)\omega,\ s=\inf\{t\in(0,1]|\ v_t\in\mathrm{epi}(F(\cdot)^*(r_0))\}$. Then $s>0$, $v_s \in \mathrm{epi}(F(\cdot)^*(r_0))$ and $v_t\overline{\in}\mathrm{epi}(F(\cdot)^*(r_0))$ for all $t\in[0,s)$. Hence for any $\lambda\in(0,1)$ we have

$$\lambda u_r+(1-\lambda)v_s\overline{\in}\mathrm{epi}(F(\cdot)^*(r_0)),$$

which is a contradiction to (12).

(III) Finally, by the hypothesis of the theorem, the conclusion of (II), and Theorem 5.1, we obtain that F is a semistrictly convex fuzzy mapping on C. □

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