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A Kind of Weingarten Surfaces in E^3 with Prescribed Principal Curvatures

Zhong Hua HOU[1,*], Cai Ling KONG[2]

1. *School of Mathematical Sciences, Dalian University of Technology, Liaoning* 116024*, P. R. China;*
2. *Department of Mathematics and Physics, North China Electric Power University, Hebei* 071000*, P. R. China*

Abstract In this paper, we construct a kind of Weingarten surfaces in E^3 and study its geometric properties. We first derive an explicit differential relationship between the principal curvatures of them. Then we prove an existence theorem of this kind of surfaces with prescribed principal curvatures. At last, we present two examples involving the rotation surfaces as the special case, and present several figures to the second example.

Keywords the principal curvatures; the Weingarten surfaces; the rotation surfaces.

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1. Introduction

A surface S in the 3 dimensional Euclidean space E^3 is called the Weingarten surface (briefly W-surface) if there exists a relationship $\phi(k_1, k_2) = 0$ between the principal curvatures k_1 and k_2 of S. There are many consequences on the study of W-surfaces in E^3 or in the 3-dimensional Minkowski space [2–7].

A rotational surface is an important W-surface. Let $f(s)$ and $g(s)$ be the principal curvatures of a rotation surface S where s is the arc-length parameter of the Meridian of S. Huang [1] proved the following results:

(1) $f(s)$ and $g(s)$ satisfy

$$(2f' - g')f' = (f - g)\{f'' - fg(f - g)\}, \tag{1.1}$$

where f', g' and g'' denote the first and second derivatives of f and g;

(2) Let $f(s)$ and $g(s)$ be two smooth functions satisfying (1.1). Then there exist a family of rotational surfaces around the Oz axis taking f and g as the principal curvatures.

The purpose of this paper is to construct a new kind of W-surfaces whose principal curvatures satisfy the differential relationship similar to (1.1). We first discuss the theory of curves

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* Corresponding author
E-mail address: zhhou@dlut.edu.cn (Z. H. HOU)

in SO(3). Then we construct W-surfaces and derive the differential relationship between the principal curvatures of them. Furthermore, we study the existence of this kind of W-surfaces with prescribed principal curvatures satisfying the given differential relationship. At last, we give two examples. We find that the rotational surfaces are just special cases of the first example. The second example is a new kind of W-surfaces satisfying the given differential relationship but different from the rotational surfaces. So our result generalizes that in paper [1].

2. The theory of curves in SO(3)

Let $\mathrm{SO}(3)=\{A\in \mathrm{GL}(3,R)|AA^{\mathrm{T}}=I,\det A=1\}$, where $\mathrm{GL}(3,R)$ is the general linear group and $\mathrm{so}(3)=\{A|A+A^{\mathrm{T}}=0, A\in \mathrm{gl}(3,R)\}$, be the Lie algebra of SO(3). For any $A,B\in \mathrm{so}(3)$, the inner product and the exterior product of A and B are defined respectively by

$$\langle A,B\rangle=\frac{1}{2}\mathrm{tr}(AB^{\mathrm{T}}),\quad A\wedge B=AB-BA.$$

Let

$$A=\begin{pmatrix}0 & x_1 & y_1\\ -x_1 & 0 & z_1\\ -y_1 & -z_1 & 0\end{pmatrix},\quad B=\begin{pmatrix}0 & x_2 & y_2\\ -x_2 & 0 & z_2\\ -y_2 & -z_2 & 0\end{pmatrix}.$$

Then it follows that

$$\langle A,B\rangle=x_1x_2+y_1y_2+z_1z_2,$$

and

$$A\wedge B=\begin{pmatrix}0 & y_2z_1-y_1z_2 & x_1z_2-x_2z_1\\ -(y_2z_1-y_1z_2) & 0 & x_2y_1-x_1y_2\\ -(x_1z_2-x_2z_1) & -(x_2y_1-x_1y_2) & 0\end{pmatrix}.$$

Let $C: A=A(s)$ $(s\in[0,L])$ be a regular curve in SO(3) parameterized by arc-length s and $B=\frac{\mathrm{d}A}{\mathrm{d}s}A^{\mathrm{T}}$. Then B is an anti-symmetric matrix and

$$\frac{\mathrm{d}A}{\mathrm{d}s}=BA,\quad B=\begin{pmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{pmatrix}, \tag{2.1}$$

where a, b and c are smooth functions with $a^2+b^2+c^2=1$.

Let $e_1(s)=B(s)A(s)$. Then $e_1(s)$ is the unit tangent vector of C at $A(s)$, which is also the covariant derivative $DA/\mathrm{d}s$ of C in $T_{A(s)}\mathrm{SO}(3)$.

In order to obtain the covariant derivative of $e_1(s)$, we take the usual derivative

$$\frac{\mathrm{d}}{\mathrm{d}s}(\frac{\mathrm{D}A}{\mathrm{d}s})=\frac{\mathrm{d}}{\mathrm{d}s}(BA)=B'A+B^2A.$$

Since $\langle B'A,B^2A\rangle=0$ and $B'\in \mathrm{so}(3), B^2\in \mathrm{so}(3)^{\perp}$, we obtain that

$$\frac{\mathrm{D}^2A}{\mathrm{d}s^2}=B'A. \tag{2.2}$$

It follows from (2.2) that $(B'\wedge B)A\in T_{A(s)}\mathrm{SO}(3)$, $|(B'\wedge B)A|=|B'A|$ and

$$\langle (B'\wedge B)A,BA\rangle=\langle (B'\wedge B)A,B'A\rangle=0,$$

which means that $(B'\wedge B)A$ is normal to $\frac{\mathrm{D}A}{\mathrm{d}s}$ and $\frac{\mathrm{D}^2A}{\mathrm{d}s^2}$ in $T_{A(s)}\mathrm{SO}(3)$.

Definition 2.1 *Let $C : A = A(s)$ $(s \in [0, L])$ be a regular curve in* SO(3) *parameterized by arc-length s. $k(s) = |\mathrm{D}^2 A(s)/\mathrm{d}s^2|$ is called the interior curvature of C in* SO(3). *$e_2(s) = B'A/k(s)$ and $e_3(s) = [B', B]A/k(s)$ are respectively called the normal and bi-normal vectors of C in* SO(3). *The function $\tau(s) = -\langle \mathrm{D}e_3/\mathrm{d}s, e_2\rangle$ is called the interior torsion of C in* SO(3).

Remark 2.1 The regular curve C in SO(3) with $k(s) \equiv 0$ $(s \in [0, L])$ is a one parameter subgroup of SO(3). In the rest of this paper, we consider only the regular curves in SO(3) with $k(s) \neq 0$ for all $s \in [0, L]$.

Remark 2.2 It is easy to see that $\langle e_i, e_j\rangle = \delta_{ij}$ for any $1 \leq i,\ j \leq 3$ and

$$k(s) = \sqrt{(a')^2 + (b')^2 + (c')^2}. \tag{2.3}$$

And the moving frame $\{A(s); e_1(s), e_2(s), e_3(s)\}$ along C satisfies

$$\frac{\mathrm{D}e_1}{\mathrm{d}s} = ke_2, \quad \frac{\mathrm{D}e_2}{\mathrm{d}s} = -ke_1 + \tau e_3, \quad \frac{\mathrm{D}e_3}{\mathrm{d}s} = -\tau e_2. \tag{2.4}$$

3. Geometry of a kind of W-surfaces

3.1 Construction of a kind of W-surfaces

Let C_1 be a smooth regular plane curve with arc-length parameter s defined by

$$C_1 : r = r(s) = (x(s), 0, z(s)),\ x(s) > 0,\ (\dot{x})^2 + (\dot{z})^2 = 1,\ s \in [0, L]. \tag{3.1}$$

Let $C_2 : A = A(t)$ $(t \in [0, T])$ be a smooth regular curve in SO(3) defined by

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = B(t)A(t),\ A(0) = I,\ B(t) = \begin{pmatrix} 0 & a(t) & b(t) \\ -a(t) & 0 & c(t) \\ -b(t) & -c(t) & 0 \end{pmatrix}, \tag{3.2}$$

where $a^2 + b^2 + c^2 > 0$ for all $t \in [0, T]$. Consider surface S defined by

$$S : X(t, s) = r(s)A(t), \quad s \in [0, L],\ t \in [0, T]. \tag{3.3}$$

Take the unit normal vector of S at $X(s, t)$ to be $n = X_t \times X_s/|X_t \times X_s|$. After direct computation, we obtain that

$$E = b^2(x^2 + z^2) + (ax - cz)^2,\ F = b(x\dot{z} - \dot{x}z),\ G = 1,$$

where $E = X_t \cdot X_t$, $F = X_t \cdot X_s$ and $G = X_s \cdot X_s$. Let $\lambda = \sqrt{EG - F^2}$. Then

$$L = \frac{1}{\lambda}X_t \times X_s \cdot X_{tt},\ M = \frac{1}{\lambda}X_t \times X_s \cdot X_{ts},\ N = \frac{1}{\lambda}X_t \times X_s \cdot X_{ss}.$$

If $x\dot{z} - \dot{x}z = 0$ for all $s \in [0, L]$, we can see that $\dot{r}$ is parallel to r. It follows that C_1 is a part of straight line. In this case, S becomes a ruled surface.

From now on, we assume that $b = 0$ for all $t \in [0, T]$, which implies $F = 0$. In this case, the principal curvatures of S are

$$f = \frac{\varepsilon(c\dot{x} + a\dot{z})}{cz - ax}; \quad g = \varepsilon(\ddot{x}\dot{z} - \dot{x}\ddot{z}). \tag{3.4}$$

Remark 3.1 Without loss of generality, we assume that $\varepsilon = 1$. Then $g(s)$ is nothing but the relative curvature of C_1.

Remark 3.2 Let $a = 1$ and $b = c = 0$. Then the surface S defined by (3.3) is the rotational surface around Oz axis.

We consider the relationship between the principal curvatures f and g. We will prove the following:

Theorem 3.1 *Let S be a surface defined by (3.1)–(3.3). Then the principal curvatures $f(t, s)$ and $g(s)$ of S satisfy the following differential relationship*

$$\{f_{ss} - fg(f - g)\}(f - g) - f_s(2f_s - \dot{g}) = 0, \tag{3.5}$$

$$\alpha f_s(t, 0) + \{f(t, 0) - g(0)\}\{1 - \beta f(t, 0)\} = 0, \tag{3.6}$$

where $\alpha = x(0)\dot{x}(0) + z(0)\dot{z}(0)$, $\beta = \dot{x}(0)z(0) - x(0)\dot{z}(0)$.

Proof It follows from (3.1) that there exists a smooth function $\theta(s)$ on $[0, L]$ such that

$$\dot{x}(s) = \cos\theta(s), \quad \dot{z}(s) = \sin\theta(s). \tag{3.7}$$

Thus we have

$$\ddot{x}(s) = -\dot{\theta}(s)\sin\theta(s), \quad \ddot{z}(s) = \dot{\theta}(s)\cos\theta(s). \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.4), we obtain

$$\dot{\theta}(s) = -g(s). \tag{3.9}$$

From (3.7), (3.8) and (3.9) we have

$$\ddot{x} = \dot{z}g, \ \ddot{z} = -\dot{x}g; \ \dddot{x} = -\dot{x}g^2 + \dot{z}\dot{g}, \ \dddot{z} = -\dot{z}g^2 - \dot{x}\dot{g}. \tag{3.10}$$

Denote $\gamma(t) = a(t)/c(t)$. Then it follows from (3.4) with $\varepsilon = 1$ that

$$\gamma(t) = \frac{fz - \dot{x}}{fx + \dot{z}}. \tag{3.11}$$

Taking partial derivatives two times about s on both sides of (3.11) and applying (3.10), we obtain

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} f_s & f - g \\ f_{ss} + fg^2 - f^2 g & 2f_s - \dot{g} \end{pmatrix} \begin{pmatrix} x\dot{x} + z\dot{z} \\ 1 + (x\dot{z} - z\dot{x})f \end{pmatrix}, \tag{3.12}$$

where $f_s = \partial f/\partial s$, $f_{ss} = \partial^2 f/\partial s^2$, $\dot{g} = \mathrm{d}g/\mathrm{d}s$.

If $x\dot{x} + z\dot{z} \neq 0$ for all $s \in [0, L]$, then (3.12) has nonzero solutions, which implies

$$f_s(2f_s - \dot{g}) - \{f_{ss} - fg(f - g)\}(f - g) = \begin{vmatrix} f_s & f - g \\ f_{ss} + fg^2 - f^2 g & 2f_s - \dot{g} \end{vmatrix} = 0. \tag{3.13}$$

If $x\dot{x} + z\dot{z} = 0$ for all $s \in [0, L]$, then $x^2 + z^2 = \rho^2 = \text{const}$. It follows that

$$\begin{cases} x = \rho\cos\varphi(s), & \dot{x} = -\rho\dot{\varphi}\sin\varphi(s), \\ z = \rho\sin\varphi(s), & \dot{z} = +\rho\dot{\varphi}\cos\varphi(s), \end{cases} \quad (\rho\dot{\varphi})^2 = 1. \tag{3.14}$$

Substituting (3.14) into (3.4), we obtain $f = g = \pm\rho^{-1}$. In this case, (3.13) also holds. (3.5) follows from (3.13). (3.6) follows from (3.12) with $s = 0$. □

3.2 The existence of the kind of W-surfaces

In this section, we proceed to prove the following:

Theorem 3.2 *Let $f(t,s)$ and $g(s)$ be two smooth functions such that $f(t,s) \neq g(s)$ for all $(t,s) \in [0,T] \times [0,L]$. Suppose that $f(t,s)$ and $g(s)$ satisfy*

$$\{f_{ss} - fg(f-g)\}(f-g) - f_s(2f_s - \dot{g}) = 0, \tag{3.15}$$

$$\alpha f_s(t,0) + \{f(t,0) - g(0)\}\{1 - \beta f(t,0)\} = 0, \tag{3.16}$$

where α and β are constants. Then there exists a W-surface in R^3 defined by (3.1)–(3.3) with principal curvatures $f(t,s)$ and $g(s)$ such that s is the arc-length parameter of the generating curve.

Proof Let

$$\theta(s) = \theta_0 - \int_0^s g(u)\mathrm{d}u,\ x(s) = \int_0^s \cos\theta(u)\mathrm{d}u + x_0,\ z(s) = \int_0^s \sin\theta(u)\mathrm{d}u + z_0,$$

where θ_0, x_0 and z_0 are integral constants to be determined. Let

$$F(t,s) = \sin\theta + h(t)\cos\theta + f\{x - h(t)z\}, \tag{3.17}$$

where $h(t)$ $(t \in [0,T])$ is a function to be determined.

Differentiating (3.17) about s two times and applying (3.15), we obtain

$$F_s = f_s(x - hz) + (f-g)(\cos\theta - h\sin\theta), \tag{3.18}$$

$$0 = (f-g)F_{ss} - (2f_s - \dot{g})F_s - Fg(f-g)^2. \tag{3.19}$$

(3.19) is a second-order linear ODE of $F(t,s)$ about s.

Denote $f^0 = f(t,0)$, $f_s^0 = f_s(t,0)$ and $g_0 = g(0)$ and put

$$F(t,0) = 0,\ F_s(t,0) = 0,\ t \in [0,T]. \tag{3.20}$$

Then it is easy to check that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\theta_0 + x_0 f^0 & \cos\theta_0 - z_0 f^0 \\ -x_0 f_s^0 - (f^0 - g_0)\cos\theta_0 & z_0 f_s^0 + (f^0 - g_0)\sin\theta_0 \end{pmatrix} \begin{pmatrix} 1 \\ h(t) \end{pmatrix}. \tag{3.21}$$

Equations (3.21) have non-zero solutions if and only if

$$(x_0\cos\theta_0 + z_0\sin\theta_0)f_s^0 + (f^0 - g_0)\{1 - f^0(z_0\cos\theta_0 - x_0\sin\theta_0)\} = 0. \tag{3.22}$$

We choose θ_0, x_0 and z_0 so that

$$\begin{pmatrix} \cos\theta_0 & \sin\theta_0 \\ -\sin\theta_0 & \cos\theta_0 \end{pmatrix} \begin{pmatrix} x_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{3.23}$$

From (3.16) and (3.23) we can see that (3.22) holds. It follows from (3.21) that

$$h(t) = \frac{x_0 f^0 + \sin\theta_0}{z_0 f^0 - \cos\theta_0} = \frac{\alpha f^0\cos\theta_0 + (1 - \beta f^0)\sin\theta_0}{\alpha f^0\sin\theta_0 - (1 - \beta f^0)\cos\theta_0}.$$

From the existence and uniqueness theorem on ordinary differential equations we have that the Cauchy problem (3.19) and (3.20) has only trivial solution $F(t,s)=0$ for all $t\in[0,T]$ and $s\in[0,L]$, which implies that

$$f(t,s)=\frac{\dot z+h(t)\dot x}{h(t)z-x}. \tag{3.24}$$

Let $C_1: r(s)=(x(s),0,z(s))$ $(s\in[0,L])$ be a smooth plane curve. Let $C_2: A=A(t)$, $(t\in[0,T])$ be a smooth curve in SO(3). Here $A(t)$ is the unique solution of (3.2) with

$$B(t)=a(t)\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & h(t)\\ 0 & -h(t) & 0\end{pmatrix}, \tag{3.25}$$

where $a(t)$ is a nonzero smooth function in t for all $t\in[0,T]$. Consider the W-surface $S: X(t,s)=r(s)A(t)$ defined by (3.3).

It follows from (3.4) and (3.25) that the principal curvatures of S are

$$k_1(t,s)=\frac{\dot z(s)+h(t)\dot x(s)}{h(t)z(s)-x(s)}=f(t,s),\ k_2(t,s)=\ddot x(s)\dot z(s)-\dot x(s)\ddot z(s)=g(s),$$

which implies that S is the desired surface. The theorem is proved. $\square$

Remark 3.3 The generating curve C_1 is uniquely determined by $\{\alpha,\beta,\theta_0\}$ and C_2 in SO(3) is uniquely determined by $\{\alpha,\beta,\theta_0,f_0\}$. Therefore the W-surfaces given in Theorem 3.2 are one parameter family of W-surfaces S_{θ_0}.

3.3 Examples of the kind of W-surfaces

We proceed to give some explicit examples of W-surfaces defined by (3.1)–(3.3). Figures 1 and 2 represent some graphics of W-surfaces defined in Example 2.

Example 1 Let $a(t)$ be a smooth function over $[0,T]$ and λ be a constant. Let

$$B_\lambda(t)=a(t)\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & \lambda\\ 0 & -\lambda & 0\end{pmatrix}, \tag{3.26}$$

and $a(t)$ be a nonzero smooth function over $[0,T]$. Solving the Cauchy problem:

$$\frac{\mathrm{d}A_\lambda(t)}{\mathrm{d}t}=B_\lambda(t)A_\lambda(t),\quad A_\lambda(0)=I, \tag{3.27}$$

we obtain that

$$A_\lambda(t)=\frac{1}{\rho^2}\begin{pmatrix}\lambda^2+\cos\gamma(t) & \rho\sin\gamma(t) & \lambda-\lambda\cos\gamma(t)\\ -\rho\sin\gamma(t) & \rho^2\cos\gamma(t) & \rho\lambda\sin\gamma(t)\\ \lambda-\lambda\cos\gamma(t) & -\rho\lambda\sin\gamma(t) & 1+\lambda^2\cos\gamma(t)\end{pmatrix} \tag{3.28}$$

where

$$\rho=\sqrt{1+\lambda^2},\quad \gamma(t)=\rho\int_0^t a(u)\mathrm{d}u.$$

It is easy to check that the curve

$$C_2^\lambda: A_\lambda=A_\lambda(t),\quad t\in[0,T],$$

lies in SO(3), with interior curvature $k(t) \equiv 0$ (cf. (2.4)).

Take $w = \int_0^t a(t)\mathrm{d}t$. Then (3.27) becomes

$$\frac{\mathrm{d}A_\lambda(w)}{\mathrm{d}w} = B_\lambda(w)A_\lambda(w),\ A_\lambda(0) = I,\ B_\lambda(w) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \lambda \\ 0 & -\lambda & 0 \end{pmatrix}. \tag{3.29}$$

And the solution to (3.29) becomes

$$A_\lambda(w) = \frac{1}{\rho^2}\begin{pmatrix} \lambda^2 + \cos\rho w & \rho\sin\rho w & \lambda - \lambda\cos\rho w \\ -\rho\sin\rho w & \rho^2\cos\rho w & \rho\lambda\sin\rho w \\ \lambda - \lambda\cos\rho w & -\rho\lambda\sin\rho w & 1 + \lambda^2\cos\rho w \end{pmatrix}.$$

The W-surface with generating curve C_1 defined by (3.1) is defined to be

$$S_\lambda : X(w,s) = r(s)A_\lambda(w), \quad w \in [0,W],\ s \in [0,L]. \tag{3.30}$$

Let

$$U_\lambda = \begin{pmatrix} \rho^{-1} & 0 & \lambda\rho^{-1} \\ 0 & 1 & 0 \\ -\lambda\rho^{-1} & 0 & \rho^{-1} \end{pmatrix}, \quad M = \begin{pmatrix} \cos\rho w & \sin\rho w & 0 \\ -\sin\rho w & \cos\rho w & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then it is easy to check that

$$A_\lambda(w) = U_\lambda M U_\lambda^{\mathrm{T}},$$

where U_λ^{T} is the transpose of U_λ. It follows that S_λ constructed in (3.30) is a rotational surface in the coordinates determined by the frame

$$e_1 = (\rho^{-1}, 0, -\lambda\rho^{-1}),\ e_2 = (0,1,0),\ e_3 = (\lambda\rho^{-1}, 0, \rho^{-1}).$$

Remark 3.4 Let $\lambda = 0$, $a(t) \equiv 1$. Then the surface S_0 is nothing but the rotational surface mentioned in paper [1].

Example 2 Let C_2 be a regular curve in SO(3) defined by (3.2) with $b(t) \equiv 0$, where t is the arc-length parameter of C_2. Suppose that the interior curvature $k(t)$ of C_2 is equal to 1. Then it follows from (2.2) and (2.4) that

$$a(t)^2 + c(t)^2 = 1, \quad [a'(t)]^2 + [c'(t)]^2 = 1. \tag{3.31}$$

By (3.31), we may assume that

$$a(t) = \cos t, \quad c(t) = \sin t.$$

Solving the Cauchy problem (3.2), we obtain

$$A_2(t) = \frac{1}{2}\begin{pmatrix} \alpha\sin t + (\beta-\gamma)\cos t & \gamma\sin t + \alpha\cos t & \beta\sin t - \alpha\cos t \\ -\alpha & \beta & \gamma \\ \alpha\cos t - (\beta-\gamma)\sin t & \gamma\cos t - \alpha\sin t & \beta\cos t + \alpha\sin t \end{pmatrix}, \tag{3.32}$$

where $\alpha(t) = \sqrt{2}\sin(\sqrt{2}t)$, $\beta(t) = 1 + \cos(\sqrt{2}t)$, $\gamma(t) = 1 - \cos(\sqrt{2}t)$.

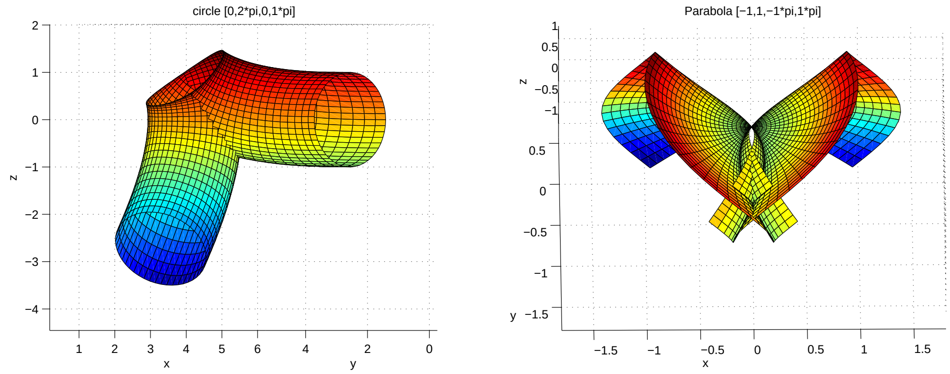

Figure 1 W-surfaces generated by $(4 + \cos s, 0, \sin s)$ (left) and $(s, 0, s^2)$ (right).

It is easy to check that $A_2(t) = \Phi V$, where

$$\Phi = \begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix} \quad V = \frac{1}{2}\begin{pmatrix} \alpha' & \gamma' & \beta' \\ -\alpha & \beta & \gamma \\ \alpha & \gamma & \beta \end{pmatrix}.$$

It follows that the W-surface S_2 constructed in (3.3) with $A_2(t)$ defined by (3.32) is derived via a prescribed rotation around e_2-axis in the coordinates determined by the moving frame $[e_1, e_2, e_3]$, where

$$e_1 = (\alpha', \gamma', \beta'),\ \ e_2 = (-\alpha, \beta, \gamma),\ \ e_3 = (\alpha, \gamma, \beta).$$

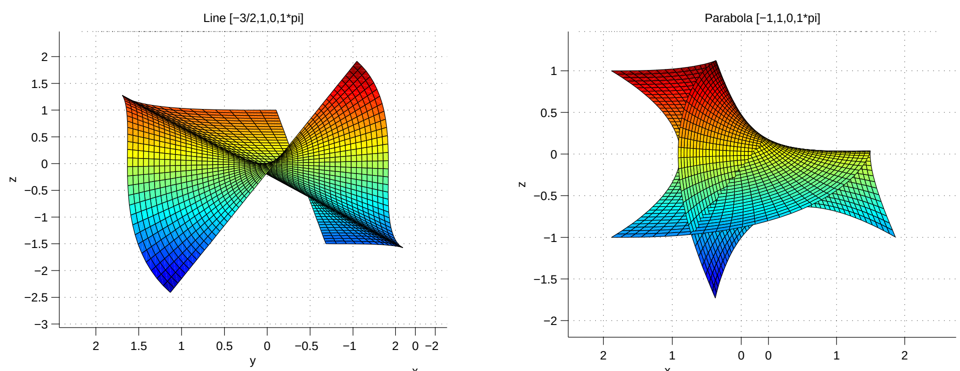

Figure 2 W-surfaces generated by $(1 + 2s, 0, s)$ (left) and $(1 + s^2, 0, s)$ (right).

References

[1] HUANG Xuanguo. *A theorem on the existence of surfaces with a prescribed principal curvature function* [J]. Chinese Ann. Math. Ser. A, 1997, **18**(6): 743–750. (in Chinese)

[2] DO CARMO M, DAJCZER M. *Rotation hypersurfaces in spaces of constant curvature* [J]. Trans. Amer. Math. Soc., 1983, **277**(2): 685–709.

[3] DILLEN F, KÜHNEL W. *Ruled Weingarten surfaces in Minkowski 3-space* [J]. Manuscripta Math., 1999, **98**(3): 307–320.

[4] JI Fenghui, HOU Zhonghua. *On Lorentzian surfaces with $H^2 = K$ in Minkowski 3-space* [J]. J. Math. Anal. Appl., 2007, **334**(1): 54–58.

[5] HOU Zhonghua, JI Fenghui. *Helicoidal surfaces with $H^2 = K$ in Minkowski 3-space* [J]. J. Math. Anal. Appl., 2007, **325**(1): 101–113.

[6] JI Fenghui, HOU Zhonghua. *Helicoidal surfaces under the cubic screw motion in Minkowski 3-space* [J]. J. Math. Anal. Appl., 2006, **318**(2): 634–647.

[7] JI Fenghui, HOU Zhonghua. *A kind of helicoidal surfaces in 3-dimensional Minkowski space* [J]. J. Math. Anal. Appl., 2005, **304**(2): 632–643.