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Weak Convergence of a Projection Algorithm for Variational Inequalities and Relatively Nonexpansive Mappings in a Banach Space

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Abstract In this paper, we introduce an iterative sequence for finding a common element of the set of fixed points of a relatively nonexpansive mapping and the set of solutions of the variational inequality for an inverse-strongly-monotone mapping in a Banach space. Then we show that the sequence converges weakly to a common element of two sets.

Keywords relatively nonexpansive mapping; generalized projection; inverse-strongly-monotone mapping; weakly sequential continuity; p-uniformly convexity constant.

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1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$, let E^* denote the dual of E and let $\langle x, f\rangle$ denote the value of $f \in E^*$ at $x \in E$. Suppose that C is a nonempty, closed convex subset of E and A is a monotone operator of C into E^*. Then we study the problem of finding a point $u \in C$ such that

$$\langle v-u, Au\rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

This problem is called the variational inequality problem [1]. The set of solutions of the variational inequality problem is denoted by $\mathrm{VI}(C, A)$. Such a problem is connected with the convex minimization problem, the complementarity problem, the problem of finding a point $u \in E$ satisfying $0 = Au$ and so on. An operator A of C into E^* is said to be inverse-strongly-monotone [2–4] if there exists a positive real number α such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2$$

for all $x, y \in C$. In such a case, A is said to be α-inverse-strongly-monotone. If A is an α-inverse-strongly-monotone mapping of C into E^*, then it is obvious that A is $\frac{1}{\alpha}$-Lipschitz continuous.

In 2005, Iiduka and Takahashi [5] proved strong convergence theorems for finding a common element of the set of solution of the variational inequality problem for an inverse-strongly-monotone mapping and the set of fixed points of a nonexpansive mapping in a Hilbert space. In

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the same year, Matsushita, and Takahashi [6] proved a strong convergence theorem for relatively nonexpansive mappings in a Banach space by using generalized projection algorithm. Recently, Iiduka and Takahashi [7] introduced the following iteration process:

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n). \tag{1.2}$$

They proved the sequence $\{x_n\}$ converges weakly to a solution of the variational inequality problem (1.1) for an operator A that satisfies the following conditions in a 2-uniformly convex and uniformly smooth Banach space E :

(1) A is α-inverse-strongly-monotone; (2) $\mathrm{VI}(C, A) \neq \emptyset$;

(3) $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in \mathrm{VI}(C, A)$.

Inspired and motivated by these facts, our purpose in this paper is to obtain a weak convergence theorem for finding a common element of the set of solutions of a variational inequality problem and the set of fixed points of a relatively nonexpansive mapping in a Banach space.

2. Preliminaries

Throughout this paper, we denote by N and R the sets of positive integers and real numbers, respectively. When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$.

A multi-valued operator $T : E \to 2^{E^*}$ with domain $D(T) = \{z \in E : Tz \neq \emptyset\}$ and range $R(T) = \bigcup\{Tz \in E^* : z \in D(T)\}$ is said to be monotone if $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$ for each $x_i \in D(T)$ and $y_i \in Tx_i$, $i = 1, 2$. A monotone operator T is said to be maximal if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operator.

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be strictly convex if for any $x, y \in U$, $x \neq y$ implies $\|\frac{x+y}{2}\| < 1$. It is also said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$, $\|x - y\| \geq \epsilon$ implies $\|\frac{x+y}{2}\| \leq 1 - \delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex. And we define a function $\delta : [0, 2] \to [0, 1]$ called the modulus of convexity of E as follows:

$$\delta(\epsilon) = \inf\{1 - \|\frac{x + y}{2}\| : x, y \in U, \|x - y\| \geq \epsilon\}.$$

Then E is a uniformly convex if and only if $\delta(\epsilon) > 0$ for all $\epsilon \in (0, 2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in [0, 2]$. For example, see [8] and [9] for more details. We know the following fundamental characterization [7, 8] of p-uniformly convex Banach spaces:

Lemma 2.1 ([8]) *Let p be a real number with $p \geq 2$ and E a Banach space. Then E is p-uniformly convex if and only if there exists a constant $0 < c \leq 1$ such that*

$$\frac{1}{2}(\|x + y\|^p + \|x - y\|^p) \geq \|x\|^p + c^p\|y\|^p \quad \text{for all } x, y \in E. \tag{2.1}$$

The best constant $1/c$ in Lemma 2.1 is called the p-uniformly convexity constant of E ([8]).

A Banach space E is said to be smooth if the limit

$$\lim_{n\to\infty} \frac{\|x+ty\|-\|x\|}{t} \tag{2.2}$$

exists for all $x, y \in U$. It is also said to be uniformly smooth if the limit (2.2) is attained uniformly for $x, y \in U$. One should note that no Banach space is p-uniformly convex for $1 < p < 2$; see [9] for more details. It is well known that Hilbert and the Lebesgue L^q $(1 < q \le 2)$ spaces are 2-uniformly convex, uniformly smooth.

On the other hand, with each $p > 1$, the (generalized) duality mapping J_p from E into 2^{E^*} is defined by

$$J_p(x) := \{v \in E^* : \langle x, v\rangle = \|x\|^p, \|v\| = \|x\|^{p-1}\}, \quad \forall x \in E.$$

In particular, $J = J_2$ is called the normalized duality mapping. If E is a Hilbert space, then $J = I$, where I is the identity mapping. The duality mapping J has the following properties:

(i) If E is smooth, then J is single-valued;

(ii) If E is strictly convex, then J is one-to-one;

(iii) If E is reflexive, then J is surjective.

(iv) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

The duality mapping J from a smooth Banach space E into E^* is said to be weakly sequentially continuous [7] if $x_n \rightharpoonup x$ implies $Jx_n \rightharpoonup^* Jx$, where $\rightharpoonup^*$ implies the weak* convergence.

Lemma 2.2 ([7]) *Let p be a given real number with $p \ge 2$ and E a p-uniformly convex Banach space. Then, for all $x, y \in E, j_x \in J_p x$ and $j_y \in J_p y$,*

$$\langle x-y, j_x - j_y\rangle \ge \frac{c^p}{2^{p-2}p}\|x-y\|^p,$$

where J_p is the generalized duality mapping of E and $1/c$ is the p-uniformly convexity constant of E.

Let E be a smooth Banach space. The function $\phi : E \times E \to R$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2$$

for all $x, y \in E$. It is obvious from the definition of the function ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y, x) \le (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{2.3}$$

Remark 2.1 From the Remark 2.1 of [6], we can know that if E is a strictly convex and smooth Banach space, then for $x, y \in E, \phi(y, x) = 0$ if and only if $x = y$.

Lemma 2.3 ([6]) *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}, \{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$, and either $\{y_n\}$, or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Let C be a nonempty closed convex subset of E. Suppose that E is reflexive, strictly convex and smooth. Then, for any $x \in E$, there exists a unique element $x_0 \in C$ such that

$$\phi(x_0, x) = \min_{y\in C} \phi(y, x).$$

The mapping $\Pi_C : E \to C$ defined by $\Pi_C x = x_0$ is called the generalized projection [6, 7, 10]. In a Hilbert space, $\Pi_C = P_C$ (metric projection). The following are well-known results.

Lemma 2.4 ([6, 7, 10]) *Let C be a nonempty closed convex subset of a smooth Banach space E, $x \in E$ and $x_0 \in C$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \textit{for all} \quad y \in C.$$

Lemma 2.5 ([6, 7, 10]) *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 2.6 ([7]) *Let S be a nonempty, closed convex subset of a uniformly convex, smooth Banach space E. Let $\{x_n\}$ be a sequence in E. Suppose that, for all $u \in S$,*

$$\phi(u, x_{n+1}) \leq \phi(u, x_n)$$

for every $n \in N$. Then $\{\Pi_S(x_n)\}$ is a Cauchy sequence.

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point $p \in C$ is said to be an asymptotic fixed point of T if there exists $\{x_n\}$ in C which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$. A mapping T of C into itself is said to be relatively nonexpansive [6, 11] if the following conditions are satisfied:

(i) $F(T)$ is nonempty;

(ii) $\phi(u, Tx) \leq \phi(u, x), \forall u \in F(T), x \in C$; (iii) $\hat{F}(T) = F(T)$.

Lemma 2.7 ([6]) *Let E be a strictly convex and smooth Banach space, let C be a closed convex subset of E, and let T be a relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

Let E be a reflexive, strictly convex, smooth Banach space and J the duality mapping from E into E^*. Then J^{-1} is also single-valued, one-to-one, surjective, and it is the duality mapping from E^* into E. We make use of the following mapping V studied in Alber [12]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2 \tag{2.4}$$

for all $x \in E$ and $x^* \in E^*$. In other words, $V(x, x^*) = \phi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$. For each $x \in E$, the mapping g defined by $g(x^*) = V(x, x^*)$ for all $x^* \in E^*$ is a continuous, convex function from E^* into R. We know the following lemma [12]:

Lemma 2.8 ([12]) *Let E be a reflexive, strictly convex, smooth Banach space and let V be as in (2.4). Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq V(x, x^* + y^*), \quad \textit{for all } x \in E \textit{ and } x^*, y^* \in E^*.$$

An operator A of C into E^* is said to be hemicontinuous if for all $x, y \in C$, the mapping f of $[0,1]$ into E^* defined by $f(t) = A(tx+(1-t)y)$ is continuous with respect to the weak* topology of E^*. We denote by $N_C(v)$ the normal cone for C at a point $v \in C$, that is,

$$N_C(v) = \{x^* \in E^* : \langle v-y, x^*\rangle \geq 0 \ \text{ for all } \ y \in C\}.$$

We know the following theorem ([13]):

Theorem 2.9 ([13]) *Let C be a nonempty, closed convex subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^*. Let $T \subset E \times E^*$ be an operator defined as follows:*

$$Tv = \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & v \bar{\in} C. \end{cases}$$

Then T is maximal monotone and $T^{-1}0 = \mathrm{VI}(C, A)$.

Lemma 2.10 ([7]) *Let C be a nonempty, closed convex subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^*. Then*

$$\mathrm{VI}(C, A) = \{u \in C : \langle v-u, Av\rangle \geq 0 \ \textit{ for all } \ v \in C\}.$$

It is obvious from Lemma 2.10 that the set $\mathrm{VI}(C, A)$ is a closed convex subset of C.

Lemma 2.11 ([14]) *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x-y\|),$$

for all $x, y \in B_r$ and $t \in [0,1]$, where $B_r = \{z \in E : \|z\| \leq r\}$.

3. Main results

Theorem 3.1 *Let E be a 2-uniformly convex, uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous. Let C be a nonempty, closed convex subset of E. Assume that A is an operator of C into E^* that satisfies the conditions (1)–(3). Assume that S is a relatively nonexpansive mapping from C into itself such that $F = F(S)\bigcap \mathrm{VI}(C, A) \neq \emptyset$. Suppose that $x_1 = x \in C$ and $\{z_n\}$ is defined by*

$$\begin{cases} z_n = \Pi_C(J^{-1}(Jx_n - \lambda_n Ax_n)), \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n Jz_n + (1-\alpha_n)JSz_n), \end{cases} \tag{3.1}$$

$n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence of positive numbers, $\{\alpha_n\} \subset (0,1)$, satisfies $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, then the sequence $\{z_n\}$ converges weakly to some element $z \in F$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. Further $z = \lim_{n\to\infty} \Pi_F(z_n)$.

Proof Put $y_n = J^{-1}(Jx_n - \lambda_n Ax_n)$ for every $n \in N$. Let $p \in F$. It follows from Lemmas 2.5

and 2.8 that

$$\begin{aligned}\phi(p, z_{n+1}) =&\phi(p, \Pi_C y_{n+1}) \leq \phi(p, y_{n+1}) = V(p, Jx_{n+1} - \lambda_{n+1}Ax_{n+1})\\ \leq& V(p, (Jx_{n+1} - \lambda_{n+1}Ax_{n+1}) + \lambda_{n+1}Ax_{n+1}) -\\ &2\langle J^{-1}(Jx_{n+1} - \lambda_{n+1}Ax_{n+1}) - p, \lambda_{n+1}Ax_{n+1}\rangle\\ =&V(p, Jx_{n+1}) - 2\lambda_{n+1}\langle y_{n+1} - p, Ax_{n+1}\rangle\\ =&\phi(p, x_{n+1}) - 2\lambda_{n+1}\langle x_{n+1} - p, Ax_{n+1}\rangle + 2\langle y_{n+1} - x_{n+1}, -\lambda_{n+1}Ax_{n+1}\rangle \quad (3.2)\end{aligned}$$

for every $n \in N$. From the condition (1) and $p \in \mathrm{VI}(C, A)$, we have

$$\begin{aligned}-2\lambda_{n+1}\langle x_{n+1} - p, Ax_{n+1}\rangle =& - 2\lambda_{n+1}\langle x_{n+1} - p, Ax_{n+1} - Ap\rangle - 2\lambda_{n+1}\langle x_{n+1} - p, Ap\rangle\\ \leq& - 2\lambda_{n+1}\alpha\|Ax_{n+1} - Ap\|^2 \qquad (3.3)\end{aligned}$$

for every $n \in N$. By Lemma 2.2 and the condition (3), we also have

$$\begin{aligned}&2\langle y_{n+1} - x_{n+1}, -\lambda_{n+1}Ax_{n+1}\rangle \leq 2\|y_{n+1} - x_{n+1}\|\|\lambda_{n+1}Ax_{n+1}\|\\ &\leq \frac{4}{c^2}\|Jy_{n+1} - Jx_{n+1}\|\|\lambda_{n+1}Ax_{n+1}\| = \frac{4}{c^2}\| - \lambda_{n+1}Ax_{n+1}\|\|\lambda_{n+1}Ax_{n+1}\|\\ &= \frac{4}{c^2}\lambda_{n+1}^2\|Ax_{n+1}\|^2 \leq \frac{4}{c^2}\lambda_{n+1}^2\|Ax_{n+1} - Ap\|^2 \qquad (3.4)\end{aligned}$$

for each $n \in N$. Therefore, from (3.3), (3.4), (3.2), and the convexity of $\| \cdot \|^2$, we get

$$\begin{aligned}\phi(p, z_{n+1}) \leq&\phi(p, x_{n+1}) + 2a(\frac{2}{c^2}b - \alpha)\|Ax_{n+1} - Ap\|^2\\ \leq&\phi(p, J^{-1}(\alpha_n Jz_n + (1 - \alpha_n)JSz_n)) + 2a(\frac{2}{c^2}b - \alpha)\|Ax_{n+1} - Ap\|^2\\ \leq&\|p\|^2 - 2\langle p, \alpha_n Jz_n + (1 - \alpha_n)JSz_n\rangle +\\ &\alpha_n\|z_n\|^2 + (1 - \alpha_n)\|Sz_n\|^2 + 2a(\frac{2}{c^2}b - \alpha)\|Ax_{n+1} - Ap\|^2\\ =&\alpha_n\phi(p, z_n) + (1 - \alpha_n)\phi(p, Sz_n) + 2a(\frac{2}{c^2}b - \alpha)\|Ax_{n+1} - Ap\|^2\\ \leq&\phi(p, z_n) + 2a(\frac{2}{c^2}b - \alpha)\|Ax_{n+1} - Ap\|^2 \leq \phi(p, z_n) \qquad (3.5)\end{aligned}$$

for each $n \in N$. Therefore, $\{\phi(p, z_n)\}$ is nonincreasing and hence there exists $\lim_{n\to\infty} \phi(p, z_n)$. So, $\{z_n\}$ is bounded. It follows from (3.5) that

$$-2a(\frac{2}{c^2}b - \alpha)\|Ax_{n+1} - Ap\|^2 \leq \phi(p, z_n) - \phi(p, z_{n+1}),$$

which implies $\lim_{n\to\infty} \|Ax_{n+1} - Ap\| = 0$. From Lemmas 2.5, 2.8 and (3.4), we have

$$\begin{aligned}\phi(x_n, z_n) &= \phi(x_n, \Pi_C y_n) \leq \phi(x_n, y_n) = V(x_n, Jx_n - \lambda_n Ax_n)\\ &\leq V(x_n, Jx_n - \lambda_n Ax_n + \lambda_n Ax_n) - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, \lambda_n Ax_n\rangle\\ &= \phi(x_n, x_n) - 2\langle y_n - x_n, \lambda_n Ax_n\rangle\\ &= 2\langle y_n - x_n, -\lambda_n Ax_n\rangle \leq \frac{4}{c^2}\lambda_n^2\|Ax_n - Ap\|^2\end{aligned}$$

for each $n \in N$. By $\lim_{n\to\infty} \|Ax_{n+1} - Ap\| = 0$, we get

$$\lim_{n\to\infty} \phi(x_n, z_n) = 0. \tag{3.6}$$

Applying Lemma 2.3, we obtain from (3.6) that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.7}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Jz_n\| = 0. \tag{3.8}$$

Since $\{z_n\}$ is bounded, $\{Sz_n\}$ is also bounded. Let $r = \sup_{n\in N}\{\|z_n\|, \|Sz_n\|\}$. Since E is a uniformly smooth Banach space, we know that E^* is a uniformly convex Banach space. Therefore, from Lemma 2.11 there exists a continuous, strictly increasing, convex function g with $g(0) = 0$ such that

$$\|\alpha x^* + (1-\alpha)y^*\|^2 \le \alpha\|x^*\|^2 + (1-\alpha)\|y^*\|^2 - \alpha(1-\alpha)g(\|x^* - y^*\|)$$

for $x^*, y^* \in B_r^*$ and $\alpha \in [0,1]$. So, for $p \in F$, from (3.1) and (3.5), we have

$$\begin{aligned}
\phi(p, z_{n+1}) \le & \phi(p, x_{n+1}) \le \phi(p, J^{-1}(\alpha_n Jz_n + (1-\alpha_n)JSz_n)) \\
\le & \|p\|^2 - 2\langle p, \alpha_n Jz_n + (1-\alpha_n)JSz_n\rangle + \\
& \alpha_n\|z_n\|^2 + (1-\alpha_n)\|Sz_n\|^2 - \alpha_n(1-\alpha_n)g(\|Jz_n - JSz_n\|) \\
\le & \phi(p, z_n) - \alpha_n(1-\alpha_n)g(\|Jz_n - JSz_n\|).
\end{aligned}$$

Therefore, we have

$$\alpha_n(1-\alpha_n)g(\|Jz_n - JSz_n\|) \le \phi(p, z_n) - \phi(p, z_{n+1}), \quad \forall n \in N.$$

Since there exists $\lim_{n\to\infty} \phi(p, z_n)$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, we have $\lim_{n\to\infty} g(\|Jz_n - JSz_n\|) = 0$. Therefore, from the property of g we have $\lim_{n\to\infty} \|Jz_n - JSz_n\| = 0$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|z_n - Sz_n\| = 0. \tag{3.9}$$

Since $\{z_n\}$ is bounded, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \rightharpoonup z$. Since S is relatively nonexpansive, we have $z \in \hat{F}(S) = F(S)$. Next, we show that $z \in \mathrm{VI}(C, A)$. Let $T \subset E \times E^*$ be an operator as follows:

$$Tv = \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & v \bar{\in} C. \end{cases}$$

By Theorem 2.9, T is maximal monotone and $T^{-1}0 = \mathrm{VI}(C, A)$. Let $(v, w) \in G(T)$. Since $w \in Tv = Av + N_C(v)$, we have $w - Av \in N_C(v)$. From $z_n \in C$, we get

$$\langle v - z_n, w - Av\rangle \ge 0. \tag{3.10}$$

On the other hand, from $z_n = \Pi_C y_n$ and Lemma 2.4, we have $\langle v - z_n, Jz_n - (Jx_n - \lambda_n Ax_n)\rangle \ge 0$ and hence

$$\langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n} - Ax_n\rangle \le 0. \tag{3.11}$$

Then it follows from (3.10) and (3.11) that

$$\begin{aligned}\langle v - z_n, w\rangle \geq &\langle v - z_n, Av\rangle \\ \geq &\langle v - z_n, Av\rangle + \langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n} - Ax_n\rangle \\ = &\langle v - z_n, Av - Ax_n\rangle + \langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n}\rangle \\ = &\langle v - z_n, Av - Az_n\rangle + \langle v - z_n, Az_n - Ax_n\rangle + \\ &\langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n}\rangle \\ \geq &- \|v - z_n\| \frac{\|z_n - x_n\|}{\alpha} - \|v - z_n\| \frac{\|Jz_n - Jx_n\|}{a} \\ \geq &- M(\frac{\|z_n - x_n\|}{\alpha} + \frac{\|Jz_n - Jx_n\|}{a})\end{aligned}$$

for every $n \in N$, where $M = \sup\{\|v - z_n\| : n \in N\}$. Taking $n = n_k$, from (3.7) and (3.8), we have $\langle v - z, w\rangle \geq 0$ as $k \to \infty$. By the maximality of T, we obtain $z \in T^{-1}0$ and hence $z \in \mathrm{VI}(C, A)$. Therefore, $z \in F$.

Put $u_n = \Pi_F(z_n)$. It follows from (3.5) and Lemma 2.6 that $\{u_n\}$ is a Cauchy sequence. Since F is closed, $\{u_n\}$ converges strongly to $w \in F$. By the uniform smoothness of E, we also have $\lim_{n\to\infty} \|Ju_n - Jw\| = 0$. Finally, we prove $z = w$. It follows from Lemma 2.4, $u_n = \Pi_F(z_n)$ and $z \in F$ that $\langle z - u_{n_k}, Ju_{n_k} - Jz_{n_k}\rangle \geq 0$. Since J is weakly sequentially continuous, we have $\langle z - w, Jw - Jz\rangle \geq 0$, as $k \to \infty$. On the other hand, since J is monotone, we have $\langle z - w, Jw - Jz\rangle \leq 0$. Hence we have $\langle z - w, Jw - Jz\rangle = 0$. From the strict convexity of E, we have $z = w$. Therefore, the sequence $\{z_n\}$ converges weakly to $z = \lim_{n\to\infty} \Pi_F(z_n)$. This completes the proof. □

Corollary 3.1 ([7, Theorem 3.1]) *Let E be a 2-uniformly convex, uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous and C a nonempty, closed convex subset of E. Assume that A is an operator of C into E^* that satisfies the conditions (1)–(3). Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \Pi_C(J^{-1}(Jx_n - \lambda_n Ax_n))$$

for every $n \in N$, where $\{\lambda_n\}$ is a sequence of positive numbers. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, then the sequence $\{x_n\}$ converges weakly to some element $z \in \mathrm{VI}(C, A)$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. Further $z = \lim_{n\to\infty} \Pi_{\mathrm{VI}(C,A)}(x_n)$.

Proof Taking $S = I$ in Theorem 3.1, we have $z_n = x_{n+1}$. Thus, it is obvious that the conclusion is true.

Remark 3.1 From Corollary 3.1, we can see Theorem 3.1 in this paper generalizes the Theorem 3.1 in [7].

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