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# Quantization of Dimodule Algebras and Quantum Yang-Baxter Module Algebras

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**Abstract** In this paper, we mainly construct quantization of dimodule algebras and quantum Yang-Baxter $H$-module algebras, and give some results of smash products and braided products.

**Keywords** Long bialgebras; dimodule algebras; quantum Yang-Baxter module algebras; smash products; braided products.

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## 1. Introduction and preliminaries

The concept of Long bialgebras was introduced in [1]. In [2] the author gave the relations between dimodule algebras and Long bialgebras, and in [3], the authors gave a necessary and sufficient condition for a smash product constructed by dimodule algebras to be a bialgebra (Hopf algebra), and gave two interesting examples to show that the conditions in [3, Theorem 2.4] "(H1) and (H2)" weaken the commutativity and cocommutativity of $H$ in [4].

In this paper, we mainly construct quantization of dimodule algebras and quantum Yang-Baxter $H$-module algebras, and give some results of smash products and braided products.

We always work over a fixed field $k$ and follow [5] for terminologies on algebras, coalgebras, comodules, bialgebras and Hopf algebras.

We recall some definitions used in this paper.

For a coalgebra $C$, a right $C$-comodule is a $k$-space $M$ with a linear map $\rho: M \to M \otimes C$, such that

$$(\rho \otimes \mathrm{id})\rho = (\mathrm{id} \otimes \Delta)\rho,\ \ (\mathrm{id} \otimes \varepsilon)\rho = \mathrm{id}.$$

In what follows, the comodule structure $\rho$ of $M$ is written as $\rho(m) = \Sigma m_{(0)} \otimes m_{(1)}$.

A right $H$-comodule algebra $A$ is both an algebra and a right $H$-comodule with the comodule structure $\rho$, such that for any $a, b \in A$, $\rho(ab) = \rho(a)\rho(b)$, $\rho(1_A) = 1_A \otimes 1_H$.

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A left $H$-module algebra $B$ is both an algebra and a left $H$-module with the module structure "$\cdot$", satisfying

$$h\cdot(ab)=\Sigma(h_1\cdot a)(h_2\cdot b), h\cdot 1_A=\varepsilon(h)1_A,$$

for any $a,b\in B, h\in H$.

A bialgebra $H$ is called a Long bialgebra in [1, 2] if there exists a linear map $\sigma: H\otimes H\to k$, such that for any $x,y,z\in H$:

(L1) $\Sigma\sigma(x_1,y)x_2=\Sigma\sigma(x_2,y)x_1$,

(L2) $\sigma(x,1)=\varepsilon(x)$,

(L3) $\sigma(x,yz)=\Sigma\sigma(x_1,y)\sigma(x_2,z)$,

(L4) $\sigma(1,x)=\varepsilon(x)$,

(L5) $\sigma(xy,z)=\Sigma\sigma(x,z_2)\sigma(y,z_1)$.

A linear map $\sigma: H\otimes H\to k$ is called a two-cocycle on a bialgebra $H$, if for any $x,y,z\in H$,

$$\Sigma\sigma(x_2,y_2)\sigma(z,x_1y_1)=\Sigma\sigma(z_2,x_2)\sigma(z_1x_1,y).$$

Assume that $(H,\sigma)$ is a Long bialgebra. Then, by [2], $\sigma$ is the two-cocycle. If for any $x,y\in H$,

(L1$'$) $\Sigma\sigma(x,y_1)y_2=\Sigma\sigma(x,y_2)y_1$,

then we call $(H,\sigma)$ a strongly Long bialgebra.

Assume that $(H,\sigma)$ is a strongly Long bialgebra. Then, by $(L1')$ and (L5), for any $h,g\in H$, we have

$$\sigma(hg,-)=\sigma(gh,-). \tag{A}$$

So, by Proposition 2.1 in [2], $(H,\sigma)$ is a Yang-Baxter coalgebra given in [6] if $\sigma$ is invertible.

In what follows, we give some examples of strongly Long bialgebras.

**Example 1.1** Let $H=H_4$ be the Sweedler's 4-dimensional Hopf algebra for a given field $k$ of chark$\neq 2$. Then, by [7], it is described as follows:

$$H_4=k\langle 1,x,y,xy|x^2=1,y^2=0,yx=-xy\rangle$$

with the coalgebra structure

$$\Delta(x)=x\otimes x,\Delta(y)=y\otimes 1+x\otimes y,\varepsilon(x)=1,\varepsilon(y)=0$$

whose antipode is given by $S(x)=x^{-1}$, $S(y)=-xy$.

Let $\sigma: H\otimes H\to k$ be a $k$-linear map as follows:

$$\begin{aligned}\sigma(x,1)&=1=\sigma(1,x)=\sigma(x,x),\\ \sigma(x,y)&=0=\sigma(y,x)=\sigma(y,1)=\sigma(1,y)=\sigma(y,y),\\ \sigma(xy,-)&=0=\sigma(-,xy).\end{aligned}$$

Then, $(H,\sigma)$ is a strongly Long bialgebra.

In fact, by [1], $(H,\sigma)$ is a Long bialgebra. So, in order to show that $(H,\sigma)$ is a strongly Long bialgebra, we have to prove that the conditions "$\Sigma\sigma(x,h_1)h_2=\Sigma\sigma(x,h_2)h_1, \Sigma\sigma(y,h_1)h_2=\Sigma\sigma(y,h_2)h_1, \Sigma\sigma(xy,h_1)h_2=\Sigma\sigma(xy,h_2)h_1$" hold, for any $h\in H$.

If taking $h = y$, we have $\Sigma\sigma(x, h_1)h_2 = \sigma(x, y)1 + \sigma(x, x)y = y = \sigma(x, 1)y + \sigma(x, y)x = \Sigma\sigma(x, h_2)h_1$; if taking $h = x$, it is easy to see that $\Sigma\sigma(x, h_1)h_2 = \Sigma\sigma(x, x)x = x = \Sigma\sigma(x, h_2)h_1$.

In a similar way, we can show that the conditions

$$\Sigma\sigma(y, h_1)h_2 = \Sigma\sigma(y, h_2)h_1 \text{ and } \Sigma\sigma(xy, h_1)h_2 = \Sigma\sigma(xy, h_2)h_1$$

are satisfied, for any $h \in H$.

**Example 1.2** Let $H = k\langle x_i | i = 1, 2, \ldots, 6\rangle$ be a free algebra generated by six generators. Its comultiplication $\Delta$ and counity $\varepsilon$ are given by

$$\begin{aligned}
&\Delta(x_1) = x_1 \otimes x_1, \Delta(x_2) = x_2 \otimes x_2,\\
&\Delta(x_3) = x_3 \otimes x_2 + x_4 \otimes x_3 + (x_2 - x_3 - x_4) \otimes x_5,\\
&\Delta(x_4) = x_4 \otimes x_4 + (x_2 - x_3 - x_4) \otimes (x_2 - x_5 - x_6),\\
&\Delta(x_5) = x_5 \otimes x_2 + (x_2 - x_5 - x_6) \otimes x_3 + x_6 \otimes x_5,\\
&\Delta(x_6) = (x_2 - x_5 - x_6) \otimes (x_2 - x_3 - x_4) + x_6 \otimes x_6,\\
&\varepsilon(x_1) = \varepsilon(x_2) = \varepsilon(x_4) = \varepsilon(x_6) = 1, \varepsilon(x_3) = \varepsilon(x_5) = 0.
\end{aligned}$$

Now we denote $c_{11} = x_1$, $c_{22} = x_2$, $c_{32} = x_3$, $c_{33} = x_4$, $c_{42} = x_5$, $c_{44} = x_6$ and define the map $\phi : \{x_1, x_2, x_3, x_4\} \to k$ by

$$\phi(x_1) = 1, \phi(x_2) = \phi(x_3) = \phi(x_4) = 2,$$

and $\sigma(c_{iv}, c_{ju}) = \delta_{uv}\delta_{\phi(x_i)v}\delta_{\phi(x_j)v}$. Then, by [1], $(H, \sigma)$ is a Long bialgebra. Moreover, it is not difficult to show that $(H, \sigma)$ satisfies the condition "$\Sigma\sigma(x, y_1)y_2 = \Sigma\sigma(x, y_2)y_1$", for any $x, y \in H$, so, it is a strongly Long bialgebra.

## 2. Twisted dimodule algebras and their smash products

In this section, we always think that $(H, \sigma)$ is a strongly Long bialgebra.

A $k$-module $M$ which is both a left $H$-module and a right $H$-comodule is called a left, right $H$-dimodule if for any $m \in M$, $h \in H$,

(DM) $\Sigma(h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)} = \Sigma h \cdot m_{(0)} \otimes m_{(1)}$.

Let $M$ be a left, right $H$-dimodule. If $M$ is both a left $H$-module algebra and a right $H$-comodule algebra, then $M$ is called an $H$-dimodule algebra.

Let $(H, \sigma)$ be a strongly Long bialgebra, and $A$ a right $H$-comodule algebra. Define a new multiplication "$\cdot_\sigma$" on $A$:

$$a \cdot_\sigma b = \Sigma\sigma(a_{(1)}, b_{(1)})a_{(0)}b_{(0)}.$$

Then, by [5], $(A, \cdot_\sigma)$ is an algebra. In what follows, we denote the algebra by $A_\sigma$, and hence we have

**Lemma 2.1** *$(A_\sigma, \rho)$ is a right $H$-comodule algebra.*

**Proof** For any $a, b \in A$,

$$\begin{aligned}
\rho(a \cdot_\sigma b) &= \Sigma\sigma(a_{(1)}, b_{(1)})\rho(a_{(0)}b_{(0)}) = \Sigma\sigma(a_{(1)}, b_{(1)})\rho(a_{(0)})\rho(b_{(0)}) \\
&= \Sigma\sigma(a_{(1)}, b_{(1)})a_{(0)(0)}b_{(0)(0)} \otimes a_{(0)(1)}b_{(0)(1)} \\
&= \Sigma\sigma(a_{(1)2}, b_{(1)2})a_{(0)}b_{(0)} \otimes a_{(1)1}b_{(1)1} \\
&\overset{(L1,L1')}{=} \Sigma\sigma(a_{(0)(1)}, b_{(0)(1)})a_{(0)(0)}b_{(0)(0)} \otimes a_{(1)}b_{(1)} \\
&= \Sigma a_{(0)} \cdot_\sigma b_{(0)} \otimes a_{(1)}b_{(1)}.
\end{aligned}$$

Assume that $A$ is an $H$-dimodule algebra. Then, via the left module structure "$\rightharpoonup_\sigma$" on $A$:

$$h \rightharpoonup_\sigma a = \Sigma\sigma(a_{(1)}, h_1)h_2 \cdot a_{(0)},$$

we have

**Lemma 2.2** *$(A_\sigma, \rightharpoonup_\sigma)$ is a left $H$-module algebra.*

**Proof** For $h$, $\ell \in H$, $a, b \in A_\sigma$,

$$\begin{aligned}
\ell \cdot_\sigma (h \cdot_\sigma a) &= \Sigma\sigma(a_{(1)}, h_1)\ell \cdot_\sigma (h_2 \cdot a_{(0)}) \\
&= \Sigma\sigma(a_{(1)}, h_1)\sigma((h_2 \cdot a_{(0)})_{(1)}, \ell_1)\ell_2 \cdot (h_2 \cdot a_{(0)})_{(0)} \\
&= \Sigma\sigma(a_{(1)}, h_1)\sigma(a_{(0)(1)}, \ell_1)(\ell_2 h_2) \cdot a_{(0)(0)} \\
&\overset{(L3)}{=} \Sigma\sigma(a_{(1)}, (\ell h)_1)(\ell h)_2 \cdot a_{(0)} = (\ell h) \cdot_\sigma a, \\
h \rightharpoonup_\sigma (a \cdot_\sigma b) &= \Sigma h \rightharpoonup_\sigma (\sigma(a_{(1)}, b_{(1)})a_{(0)}b_{(0)}) \\
&= \Sigma\sigma(a_{(1)}, b_{(1)})\sigma((a_{(0)}b_{(0)})_{(1)}, h_1)h_2 \cdot (a_{(0)}b_{(0)})_{(0)} \\
&= \Sigma\sigma(a_{(1)}, b_{(1)})\sigma(a_{(0)(1)}b_{(0)(1)}, h_1)h_2 \cdot (a_{(0)(0)}b_{(0)(0)}) \\
&= \Sigma\sigma(a_{(0)(1)}, b_{(0)(1)})\sigma(a_{(1)}b_{(1)}, h_1)h_2 \cdot (a_{(0)(0)}b_{(0)(0)}) \\
&= \Sigma\sigma(a_{(0)(1)}, b_{(0)(1)})\sigma(a_{(1)}b_{(1)}, h_1)(h_2 \cdot (a_{(0)(0)})(h_3 \cdot b_{(0)(0)}) \\
&= \Sigma\sigma(a_{(1)}b_{(1)}, h_1)\sigma((h_2 \cdot a_{(0)})_{(1)}, (h_3 \cdot b_{(0)})_{(1)})(h_2 \cdot a_{(0)})_{(0)}(h_3 \cdot b_{(0)})_{(0)} \\
&\overset{(L5,L1')}{=} \Sigma\sigma(a_{(1)}, h_1)\sigma(b_{(1)}, h_2)(h_3 \cdot a_{(0)}) \cdot_\sigma (h_4 \cdot b_{(0)}) \\
&= \Sigma(h_1 \cdot_\sigma a) \cdot_\sigma (h_2 \cdot_\sigma b).
\end{aligned}$$

It is obvious that $1_H \rightharpoonup_\sigma a = a$ and $h \rightharpoonup_\sigma 1_A = \varepsilon(h)1_A$, so, $(A_\sigma, \rightharpoonup_\sigma)$ is a left $H$-module algebra.

**Proposition 2.3** *(1) $(A_\sigma, \rightharpoonup_\sigma, \rho)$ is an $H$-dimodule algebra.*

*(2) If $A$ is also a bialgebra, then, the tensor product coalgebra structure on $A$ is compatible with the twisted product structure making $A_\sigma$ into a bialgebra if*

$$f_A : A \otimes A \to A, a \otimes x \mapsto \Sigma\sigma(a_{(1)}, x_{(1)})a_{(0)}x_{(0)}$$

*is a coalgebra map.*

**Proof** (1) It is straightforward by Lemmas 2.1 and 2.2. (2) It is easy to show that $\Delta_{A_\sigma}$ is an algebra map if and only if $f_A$ is a comultiplication map, and $\varepsilon_{A_\sigma}$ is an algebra map if and only if $f_A$ is a counit map.

**Example 2.4** (1) Let $(H,\sigma)$ be a strongly Long bialgebra, and let $h \bullet_\sigma x = \Sigma\sigma(h,x_1)x_2$, for any $h$, $x \in H$. Then, via the left $H$-module action "$\bullet_\sigma$", it is easy to show that $(H,\bullet_\sigma,\Delta)$ is an $H$-dimodule algebra. Again by Proposition 2.3, $(H_\sigma,\rightharpoonup_\sigma,\Delta)$ is also an $H$-dimodule algebra whose module action is given by $h \rightharpoonup_\sigma x = \Sigma\sigma(x_2,h_1)h_2 \bullet_\sigma x_1$, where the multiplication of $H_\sigma$ is given by $x \cdot_\sigma y = \Sigma\sigma(x_1,y_1)x_2y_2$.

(2) Let $H = H_4$ be the Sweedler's 4-dimensional Hopf algebra. Then, by Example 1.1, $(H,\sigma)$ is a strongly Long bialgebra. So, by (1), $(H_\sigma,\rightharpoonup_\sigma,\Delta)$ is an $H$-dimodule algebra.

(3) Let $(H,\sigma)$ be a coquasitriangular Hopf algebra in [5]. Define a measuring action of $H$ on $H$: $h \bullet_\sigma x = \Sigma\sigma(h,x_1)x_2$. Then, by Example 2.3 in [3], $(H,\bullet_\sigma,\Delta)$ is an $H$-dimodule algebra.

It is obvious that any cocommutative coquasitriangular Hopf algebra is a strongly Long bialgebra. So, by (1), we know that $(H_\sigma,\rightharpoonup_\sigma,\Delta)$ is an $H$-dimodule algebra whose module action is given by $h \rightharpoonup_\sigma x = \Sigma\sigma(x_2,h_1)h_2 \bullet_\sigma x_1$.

Let $A$ and $B$ be two $H$-dimodule algebras. A smash product $A\#B$ in [4] is defined as follows: $A\#B = A \otimes B$ as $k$-modules and its multiplication is given by

$$(a\#b)(c\#d) = \Sigma a(b_{(1)} \cdot c)\#b_{(0)}d,$$

for any $a, c \in A$; $b, d \in B$.

By Theorem 2.4 in [3] (weaken the commutativity and cocommutativity of $H$ in [4]), we have

**Lemma 2.5** *Let $H$ be a Hopf algebra and let $A$ and $B$ be two $H$-dimodule algebras such that the following conditions hold,*

*(H1)* $\Sigma h_1 \cdot a \otimes h_2 = \Sigma h_2 \cdot a \otimes h_1$, *for all* $a \in A, h \in H$,

*(H2)* $\Sigma x_{(0)} \otimes x_{(1)}h = \Sigma x_{(0)} \otimes hx_{(1)}$, *for all* $h \in H, x \in B$,

*then we have the following conclusions.*

*(1)* $A\#B$ *is an* $H$*-dimodule algebra,*

*where the left* $H$*-module and the right* $H$*-comodule of* $A\#B$ *are respectively defined by*

*(M1)* $h \cdot (a\#x) = \Sigma h_1 \cdot a\#h_2 \cdot x$,

*(M2)* $\rho_{A\otimes B}(a\#x) = \Sigma a_{(0)}\#x_{(0)} \otimes a_{(1)}x_{(1)}$.

*(2) If* $A$ *and* $B$ *are two bialgebras, then the tensor product coalgebra structure on* $A\#B$ *is compatible with the smash product structure making* $A\#B$ *into a bialgebra if and only if the map*

$$f : A\#B \to A\#B, a\#x \mapsto \Sigma x_{(1)} \cdot a\#x_{(0)}$$

*is a coalgebra map.*

By the above lemma, we get

**Proposition 2.6** *Let $A$ and $B$ be two $H$-dimodule algebras, such that the following conditions hold,*

*(H1)* $h_1 \cdot a \otimes h_2 = h_2 \cdot a \otimes h_1$, *for all* $h \in H, a \in A$,

*(H2)* $\Sigma b_{(0)} \otimes b_{(1)} h = \Sigma b_{(0)} \otimes h b_{(1)}$, *for all* $h \in H, b \in B$.

*Then we have the following conclusions.*

*(1)* $A_\sigma \# B_\sigma$ *is an* $H$*-dimodule algebra whose multiplication is given by*

$$(a\#b)(c\#d) = \Sigma a \cdot_\sigma (b_{(1)} \rightharpoonup_\sigma c)\# b_{(0)} \cdot_\sigma d,$$

*where the left* $H$*-module and the right* $H$*-comodule of* $A_\sigma \# B_\sigma$ *are respectively defined by*

*(M1′)* $h \cdot (a\#x) = \Sigma h_1 \rightharpoonup_\sigma a \# h_2 \rightharpoonup_\sigma x$,

*(M2)* $\rho(a\#x) = \Sigma a_{(0)} \# x_{(0)} \otimes a_{(1)} x_{(1)}$.

*In what follows, we call the smash product* $A_\sigma \# B_\sigma$ *a twisted smash product.*

*(2) If* $A$ *and* $B$ *are two bialgebras such that* $f_A$ *and* $f_B$ *are coalgebra maps, then the tensor product coalgebra structure on* $A_\sigma \# B_\sigma$ *is compatible with the smash product structure making* $A_\sigma \# B_\sigma$ *into a bialgebra if and only if the map*

$$f : A_\sigma \# B_\sigma \to A_\sigma \# B_\sigma, a\#x \mapsto \Sigma x_{(1)} \rightharpoonup_\sigma a \# x_{(0)}$$

*is a coalgebra map.*

According to the above proposition and Example 2.4, we get

**Corollary 2.7** *(1) Let* $(H, \sigma)$ *be a strongly Long bialgebra. If* $H$ *is commutative, then* $H_\sigma \# H_\sigma$ *is an* $H$*-dimodule algebra whose multiplication is given by*

$$(h\#x)(g\#y) = \Sigma h \cdot_\sigma (x_2 \rightharpoonup_\sigma g) \# x_1 \cdot_\sigma y.$$

*(2) Let* $(H, \sigma)$ *be a cocommutative coquasitriangular Hopf algebra. Then,* $H_\sigma \# H_\sigma$ *is an* $H$*-dimodule algebra whose multiplication is given by*

$$(h\#x)(g\#y) = \Sigma h \bullet_\sigma (x_2 \rightharpoonup_\sigma g) \# x_1 \bullet_\sigma y.$$

**Proof** (1) is straightforward by Proposition 2.6. (2) By Corollary 2.9 in [3], the cocommutativity of coquasitriangular Hopf algebra $(H, \sigma)$ implies that $H$ is commutative, so, the condition (H2) holds. So, by Example 2.4 and Proposition 2.6, we know that $H_\sigma \# H_\sigma$ is an $H$-dimodule algebra.

**Proposition 2.8** *Let* $(H, \sigma)$ *be a strongly Long bialgebra where* $\sigma$ *is invertible with inverse* $\sigma^{-1}$*. Let* $A$ *and* $B$ *be two* $H$*-dimodule algebras, such that the conditions (H1) and (H2) hold. If* $\sigma^2 = \sigma \circ \tau$*, then, there exists an isomorphism of dimodule algebras as follows:*

$$f : A_\sigma \# B_\sigma \to (A\#B)_\sigma, a\#b \mapsto \Sigma \sigma(a_{(1)}, b_{(1)}) a_{(0)} \# b_{(0)}$$

*with the inverse* $g : (A\#B)_\sigma \to A_\sigma \# B_\sigma, c\#d \mapsto \Sigma \sigma^{-1}(c_{(1)}, d_{(1)}) c_{(0)} \# d_{(0)}$*, where* $A_\sigma \# B_\sigma$ *and* $(A\#B)_\sigma$ *are* $H$*-dimodule algebras given in Proposition 2.6.*

**Proof** For any $a\#b, c\#d \in A_\sigma \# B_\sigma$, we have

$$\begin{aligned} f(a\#b) \cdot_\sigma f(c\#d) =& \Sigma \sigma(a_{(1)}, b_{(1)}) \sigma(c_{(1)}, d_{(1)}) (a_{(0)} \# b_{(0)}) \cdot_\sigma (c_{(0)} \# d_{(0)}) \\ =& \Sigma \sigma(a_{(1)}, b_{(1)}) \sigma(c_{(1)}, d_{(1)}) \sigma(a_{(0)(1)} b_{(0)(1)}, c_{(0)(1)} d_{(0)(1)}) \times \\ & a_{(0)(0)} (b_{(0)(0)(1)} \cdot c_{(0)(0)}) \# b_{(0)(0)(0)} d_{(0)(0)} \\ =& \Sigma \sigma(a_{(1)2}, b_{(1)3}) \sigma(c_{(1)2}, d_{(1)2}) \sigma(a_{(1)1} b_{(1)2}, c_{(1)1} d_{(1)1}) \times \end{aligned}$$

$$a_{(0)}(b_{(1)1}\cdot c_{(0)})\#b_{(0)}d_{(0)}.$$

And, by $\sigma^2=\sigma\circ\tau$, we get

$$\begin{aligned}
f((a\#b)(c\#d)) &= \Sigma f(a\cdot_\sigma (b_{(1)}\rightharpoonup_\sigma c)\#b_{(0)}\cdot_\sigma d)\\
&= \Sigma\sigma(c_{(1)},b_{(1)1})\sigma(a_{(1)},c_{(0)(1)})\sigma(b_{(0)(1)},d_{(1)})f(a_{(0)}(b_{(1)2}\cdot c_{(0)(0)})\#b_{(0)(0)}d_{(0)})\\
&= \Sigma\sigma(c_{(1)},b_{(1)1})\sigma(a_{(1)},c_{(0)(1)})\sigma(b_{(0)(1)},d_{(1)})\sigma(a_{(0)(1)}c_{(0)(0)(1)},b_{(0)(0)(1)}d_{(0)(1)})\times\\
&\quad a_{(0)(0)}(b_{(1)2}\cdot c_{(0)(0)(0)})\#b_{(0)(0)(0)}d_{(0)(0)}\\
&= \Sigma\sigma(c_{(1)3},b_{(1)3})\sigma(a_{(1)2},c_{(1)2})\sigma(b_{(1)2},d_{(1)2})\sigma(a_{(1)1}c_{(1)1},b_{(1)1}d_{(1)1})\times\\
&\quad a_{(0)}(b_{(1)4}\cdot c_{(0)})\#b_{(0)}d_{(0)}\\
&\overset{(L3)}{=} \Sigma\sigma(c_{(1)4},b_{(1)3})\sigma(a_{(1)3},c_{(1)3})\sigma(b_{(1)2},d_{(1)2})\sigma(a_{(1)1}c_{(1)1},b_{(1)1})\times\\
&\quad \sigma(a_{(1)2}c_{(1)2},d_{(1)1})a_{(0)}(b_{(1)4}\cdot c_{(0)})\#b_{(0)}d_{(0)}\\
&\overset{(L5)}{=} \Sigma\sigma(c_{(1)4},b_{(1)4})\sigma(a_{(1)3},c_{(1)3})\sigma(b_{(1)3},d_{(1)3})\sigma(a_{(1)1},b_{(1)1})\sigma(c_{(1)1},b_{(1)2})\times\\
&\quad \sigma(a_{(1)2},d_{(1)1})\sigma(c_{(1)2},d_{(1)2})a_{(0)}(b_{(1)5}\cdot c_{(0)})\#b_{(0)}d_{(0)}\\
&\overset{(L1,1')}{=} \Sigma\underbrace{\sigma(c_{(1)3},b_{(1)3})\sigma(c_{(1)2},b_{(1)2})}\sigma(a_{(1)3},c_{(1)4})\sigma(b_{(1)4},d_{(1)3})\sigma(a_{(1)1},b_{(1)1})\times\\
&\quad \sigma(a_{(1)2},d_{(1)1})\sigma(c_{(1)1},d_{(1)2})a_{(0)}(b_{(1)5}\cdot c_{(0)})\#b_{(0)}d_{(0)}\\
&\overset{(L1')}{=} \Sigma\sigma(b_{(1)2},c_{(1)2})\sigma(a_{(1)2},c_{(1)3})\sigma(b_{(1)3},d_{(1)2})\sigma(a_{(1)1},b_{(1)1})\times\\
&\quad \sigma(a_{(1)3},d_{(1)3})\sigma(c_{(1)1},d_{(1)1})a_{(0)}(b_{(1)4}\cdot c_{(0)})\#b_{(0)}d_{(0)}\\
&\overset{(L3)}{=} \Sigma\sigma(b_{(1)2},c_{(1)2}d_{(1)2})\sigma(a_{(1)2},c_{(1)3}d_{(1)3})\sigma(a_{(1)1},b_{(1)1})\sigma(c_{(1)1},d_{(1)1})\times\\
&\quad a_{(0)}(b_{(1)3}\cdot c_{(0)})\#b_{(0)}d_{(0)}\\
&\overset{(L5)}{=} \Sigma\sigma(a_{(1)2}b_{(1)2},c_{(1)2}d_{(1)2})\sigma(a_{(1)1},b_{(1)1})\sigma(c_{(1)1},d_{(1)1})a_{(0)}(b_{(1)3}\cdot c_{(0)})\#b_{(0)}d_{(0)}\\
&= \Sigma\sigma(a_{(1)1}b_{(1)2},c_{(1)1}d_{(1)1})\sigma(a_{(1)2},b_{(1)3})\sigma(c_{(1)2},d_{(1)2})a_{(0)}(b_{(1)1}\cdot c_{(0)})\#b_{(0)}d_{(0)}.
\end{aligned}$$

So, $f((a\#b)(c\#d))=f(a\#b)\cdot_\sigma f(c\#d)$ and hence $f$ is an algebra map.

The left proof is easy.

## 3. Twisted quantum Yang-Baxter module algebras and their braided products

In this section, we always assume that $H$ is a Hopf algebra with bijective antipode $S$, and $(H,\sigma)$ a strongly Long bialgebra. So, by [2], we know that $\sigma$ is invertible with inverse $\sigma\circ(S\otimes I)$ or $\sigma\circ(I\otimes S)$. Hence $\sigma=\sigma\circ(S\otimes S)$.

A left, right quantum Yang-Baxter $H$-module $M$ is a $k$-module which is a left $H$-module and a right $H$-comodule satisfying the following equivalent compatibility conditions:

(YB1) $\Sigma h_1\cdot m_{(0)}\otimes h_2m_{(1)}=\Sigma(h_2\cdot m)_{(0)}\otimes(h_2\cdot m)_{(1)}h_1$;

(YB2) $\Sigma(h\cdot m)_{(0)}\otimes(h\cdot m)_{(1)}=\Sigma h_2\cdot m_{(0)}\otimes h_3m_{(1)}S^{-1}(h_1)$.

A $k$-algebra $A$ which is a quantum Yang-Baxter $H$-module is said to be a quantum Yang-Baxter $H$-module algebra in [7] if it is both a left $H$-module algebra and a right $H^{op}$-comodule algebra, where $H^{op}$ denotes the Hopf algebra with the opposite underlying algebra of $H$.

For every left, right quantum Yang-Baxter $H$-module $A$, since $(H,\sigma)$ is a strongly Long bialgebra, we have

$$\Sigma\sigma(S((h\cdot a)_{(1)}),\ell)(h\cdot a)_{(0)} = \Sigma\sigma(S(a_{(1)}),\ell)h\cdot a_{(0)}, \tag{B}$$

for any $h,\ell\in H$, $a\in A$.

According to the above equality (B), we easily get

$$\Sigma\sigma((h\cdot a)_{(1)},\ell)(h\cdot a)_{(0)} = \Sigma\sigma(a_{(1)},\ell)h\cdot a_{(0)}. \tag{C}$$

**Lemma 3.1** *(1) Let $A$ be a quantum Yang-Baxter $H$-module algebra. Then $A_\sigma$ is a quantum Yang-Baxter $H$-module algebra with the left $H$-module structure: $h \twoheadrightarrow_\sigma a = \Sigma\sigma(S(a_{(1)}),h_1)h_2\cdot a_{(0)}$ and the multiplication of $A_\sigma$ is given by $a\cdot_\sigma b = \Sigma\sigma(a_{(1)},b_{(1)})a_{(0)}b_{(0)}$.*

*(2) If $A$ is also a bialgebra, then, the tensor product coalgebra structure on $A$ is compatible with the twisted product structure making $A_\sigma$ into a bialgebra if there exists a map*

$$f_A : A\otimes A\to A, a\otimes x\mapsto \Sigma\sigma(a_{(1)},x_{(1)})a_{(0)}x_{(0)}$$

*such that it is a coalgebra map.*

**Proof** In a similar way of Proposition 2.3(2), we can show the second statement.

It is easy to show that $(A_\sigma,\twoheadrightarrow_\sigma)$ is a left $H$-module algebra by equalities $(A)$ and $(C)$ and $(A_\sigma,\rho)$ a right $H^{op}$-comodule algebra. So, $A_\sigma$ is a left, right quantum Yang-Baxer $H$-module algebra.

Let $A$ and $B$ be two quantum Yang-Baxter $H$-module algebras. Define a new multiplication on $A\otimes B$ as follows:

$$(a\otimes x)(b\otimes y) = \Sigma ab_{(0)}\otimes (b_{(1)}\cdot x)y.$$

$A\otimes B$ with the above multiplication is denoted by $A\propto B$ and called the braided product of $A$ and $B$ in [4].

By Theorem 4.3 in [3], we get

**Lemma 3.2** *Let $A$ and $B$ be two quantum Yang-Baxter $H$-module algebras. Then*

*(1) $A\propto B$ is a quantum Yang-Baxter $H$-module algebra;*

*(2) if $A$ and $B$ are bialgebras, then $A\propto B$ is a bialgebra if and only if*

$$f : A\propto B\to A\propto B, a\propto x\mapsto \Sigma a_{(0)}\propto a_{(1)}\cdot x$$

*is a coalgebra map, where the braided product $A\propto B$ is a tensor product coalgebra as a coalgebra.*

According to Lemmas 3.1 and 3.2, we obtain

**Proposition 3.3** *Let $A$ and $B$ be two quantum Yang-Baxter $H$-module algebras. Then*

*(1) $A_\sigma\propto B_\sigma$ is a quantum Yang-Baxter $H$-module algebra whose multiplication is given by*

$$(a\otimes x)(b\otimes y) = \Sigma\sigma(a_{(1)},b_{(1)1})a_{(0)}b_{(0)}\otimes\sigma(S(x_{(1)2}),b_{(1)2})\sigma(x_{(1)1},y_{(1)})(b_{(1)3}\cdot x_{(0)})y_{(0)}.$$

*(2) If $A$ and $B$ are bialgebras such that $f_A$ and $f_B$ are coalgebra maps, then $A_\sigma \propto B_\sigma$ is a bialgebra if and only if*

$$f : A_\sigma \propto B_\sigma \to A_\sigma \propto B_\sigma, a \propto x \mapsto \Sigma a_{(0)} \propto a_{(1)} \twoheadrightarrow_\sigma x$$

*is a coalgebra map.*

**Example 3.4** Let $H$ be a finite dimensional Hopf algebra. Then, by [4], $(H, \cdot, \rho)$ is a quantum Yang-Baxter $H$-module algebra with the $H$-structures as follows:

$$h \cdot x = \Sigma h_1 x S(h_2); \quad \rho(h) = \Sigma h_2 \otimes S^{-1}(h_1).$$

By Lemma 3.1, we know that $H_\sigma$ is a quantum Yang-Baxter $H$-module algebra whose module structure is given by

$$\begin{aligned} h \twoheadrightarrow_\sigma x &= \Sigma\sigma(S(x_{(1)}), h_1) h_2 \cdot x_{(0)} = \Sigma\sigma(S(S^{-1}(x_1)), h_1) h_2 \cdot x_2 \\ &= \Sigma\sigma(x_1, h_1) h_2 x_2 S(h_3), \end{aligned}$$

and whose multiplication is given by

$$\begin{aligned} h \cdot_\sigma g &= \Sigma\sigma(h_{(1)}, g_{(1)}) h_{(0)} \cdot g_{(0)} = \Sigma\sigma(S^{-1}(h_1), S^{-1}(g_1)) h_2 \cdot g_2 \\ &= \Sigma\sigma(S^{-1}(h_1), S^{-1}(g_1)) h_2 g_2 S(h_3) \\ &= \Sigma\sigma(h_1, g_1) h_2 g_2 S(h_3). \end{aligned}$$

Note here that $\sigma \circ (S^{-1} \otimes S^{-1}) = \sigma$ by $\sigma \circ (S \otimes S) = \sigma$. Hence by Proposition 3.3, the braided product $H_\sigma \propto H_\sigma$ is a quantum Yang-Baxter $H$-module algebra.

In a similar way of Proposition 2.8, we can show the following proposition.

**Proposition 3.5** *Let $A$ and $B$ be quantum Yang-Baxter $H$-module algebras. If $\sigma = (\sigma \circ \tau) * \sigma^{-1}$, then, there exists an isomorphism of quantum Yang-Baxter $H$-module algebras as follows:*

$$f : A_\sigma \propto B_\sigma \to (A \propto B)_\sigma, a \propto b \mapsto \Sigma\sigma(a_{(1)}, b_{(1)}) a_{(0)} \propto b_{(0)}$$

*with the inverse $g : (A \propto B)_\sigma \to A_\sigma \propto B_\sigma$ given by $g(c \propto d) = \Sigma\sigma^{-1}(c_{(1)}, d_{(1)}) c_{(0)} \propto d_{(0)}$, where $A_\sigma \propto B_\sigma$ and $(A \propto B)_\sigma$ are quantum Yang-Baxter $H$-module algebras given in Proposition 3.3.*

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