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A Note on Eigenvalues of Hermite Matrix

Zhi Zhen WANG*, **Xiao Lin WU**, **Li XIA**
Department of Applied Mathematics, Shanghai Normal University, Shanghai 200234*, P. R. China*

Abstract This note investigates the relationship of eigenvalues of Hermitian matrices P and UPU^* with $UU^* = I_k$ and $k \le n$. We present several equivalent conditions for $\lambda_i(UPU^*) = \lambda_i(P)$ $(i \le k \le n)$.

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1. Introduction

In matrix theory, the term UPU^* appears in many results [1–11]. Such matrix also appears in the algebraic Riccati equation which arises in the fields of optimal control and filtering theory [1]. Estimate of bounds on the solution of this equation is important for analysis and synthesis of linear systems. Much work is done particularly on the bounds of traces and the extreme eigenvalues of matrix product. In [3–11], some lower and upper bounds for the trace of the solutions to the equation were presented. In [4], some eigenvalue inequalities for matrix product were obtained. In those results, an important term is the eigenvalue of UPU^* with $UU^* = I_k$ and $k \le n$. But, the class of UPU^* with $UU^* = I_k$ and $k < n$ fails to have many important properties of the class of UPU^* with $UU^* = I_k$ and $k = n$. It is well known that $\lambda_i(UPU^*) = \lambda_i(P)$ if $UU^* = I_n$. However, this is not true if $UU^* = I_k$ and $k < n$. For this reason, we shall return to study the further relationship of eigenvalues of UPU^* and P.

2. Preliminaries

Denote by $\mathbb{R}$ and $\mathbb{C}$ the sets of the real numbers and the complex numbers, respectively, and $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the n-tuples with components from $\mathbb{R}$ and $\mathbb{C}$, respectively.

The set of all m-by-n matrices over $\mathbb{C}$ is denoted by $M_{m,n}$, and $M_{m,n}$ is abbreviated to M_n if $m = n$. The rank of matrix $A \in M_{m,n}$ is denoted by $\mathrm{R}(A)$. The symbol $I_n \in M_n$ denotes the

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* Corresponding author
E-mail: wzhzh@shnu.edu.cn (Z. Z. WANG)

identity matrix.

Definition 1 *Given $A \in M_n$. If a scalar λ and a nonzero vector $x \in \mathbb{C}^n$ happen to satisfy the equation $Ax = \lambda x$, then λ is called an eigenvalue of A and x is called an eigenvector of A associated with λ.*

Definition 2 *A matrix $A \in M_n$ is called Hermitian if $\overline{A}^{\mathrm{T}} = A$ and the Hermitian adjoint A^* of $A \in M_{m,n}$ is defined by $A^* = \overline{A}^{\mathrm{T}}$.*

The following result is known as the Poincaré Separation Theorem [2].

Proposition 1 *Let $P \in M_n$ be Hermitian, and $U \in M_{k,n}$ have k orthonormal rows. If the eigenvalues of P and UPU^* are arranged in decreasing order, we have*

$$\lambda_{n-k+i}(P) \le \lambda_i(UPU^*) \le \lambda_i(P).$$

3. Main results

In this section, we shall study the relationship of eigenvalues of Hermitian matrices P and UPU^* while $UU^* = I_k$ and $k \le n$.

Lemma 1 *If λ is an eigenvalue of UPU^*, then λ is an eigenvalue of U^*UP. And if $\lambda \ne 0$ is an eigenvalue of U^*UP, then λ is an eigenvalue of UPU^*.*

Proof The result is well known since matrices AB and BA have the same nonzero eigenvalues. $\square$

Example 1 Take

$$U = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & 0 \\ \frac{-1}{2\sqrt{6}} & \frac{-1}{2\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} \\ \frac{3}{2\sqrt{26}} & \frac{3}{2\sqrt{26}} & \frac{-3}{\sqrt{26}} & \frac{7}{2\sqrt{26}} & \frac{-1}{2\sqrt{26}} \end{pmatrix}, \quad P = \begin{pmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 5 & 2 & 3 & 1 \\ 3 & 2 & 5 & 4 & 1 \\ 2 & 3 & 4 & 5 & 1 \\ 1 & 1 & 1 & 1 & 5 \end{pmatrix}$$

we have

$$UPU^* = \begin{pmatrix} 11 & \frac{11}{2\sqrt{2}} & \frac{23\sqrt{3}}{2\sqrt{26}} \\ \frac{11}{2\sqrt{2}} & 6 & \frac{2\sqrt{3}}{\sqrt{13}} \\ \frac{23\sqrt{3}}{2\sqrt{26}} & \frac{2\sqrt{3}}{\sqrt{13}} & \frac{34}{13} \end{pmatrix}, \quad U^*UP = \begin{pmatrix} \frac{57}{13} & \frac{57}{13} & \frac{37}{13} & \frac{37}{13} & \frac{4}{13} \\ \frac{57}{13} & \frac{57}{13} & \frac{37}{13} & \frac{37}{13} & \frac{4}{13} \\ \frac{42}{13} & \frac{29}{13} & \frac{56}{13} & \frac{43}{13} & \frac{31}{13} \\ \frac{29}{13} & \frac{42}{13} & \frac{43}{13} & \frac{56}{13} & \frac{31}{13} \\ \frac{7}{13} & \frac{7}{13} & \frac{31}{13} & \frac{31}{13} & \frac{29}{13} \end{pmatrix}.$$

All eigenvalues of P are $\frac{19+\sqrt{97}}{2}$, $\frac{19-\sqrt{97}}{2}$, 4, 2, 0; eigenvalues of UPU^* are $\frac{121+21\sqrt{10}}{13}$, $\frac{121-21\sqrt{10}}{13}$, 1 and eigenvalues of U^*UP are $\frac{121+21\sqrt{10}}{13}$, $\frac{121-21\sqrt{10}}{13}$, 1, 0, 0.

Lemma 2 *The rank of U^*UP is the same as the rank of UPU^*, i.e., $R(U^*UP) = R(UPU^*)$.*

Lemma 3 *Given $A \in M_n$, then λ is an eigenvalue of A if and only if $R(A - \lambda I_n) < n$.*

Lemma 4 *Given $\lambda \in \mathbb{C}$, then*

(1) $\mathrm{R}\begin{pmatrix} I_n - U^*U \\ P - \lambda I_n \end{pmatrix} = \mathrm{R}\begin{pmatrix} (U^*UP - \lambda I_n)^* \\ I_n - U^*U \end{pmatrix}$;

(2) $\mathrm{R}\begin{pmatrix} I_n - U^*U \\ P - \lambda I_n \end{pmatrix} < n \Leftrightarrow$ *The system* $\begin{cases} Py = \lambda y \\ y = U^*Uy \end{cases}$ *has a nonzero solution* $y \neq 0$.

Proof (1) This result follows from the relationship

$$\begin{pmatrix} I_n - U^*U \\ P - \lambda I_n \end{pmatrix} = \begin{pmatrix} O & I_n \\ I_n & P \end{pmatrix} \begin{pmatrix} (U^*UP - \lambda I_n)^* \\ I_n - U^*U \end{pmatrix}.$$

(2) Since $\mathrm{R}\begin{pmatrix} I_n - U^*U \\ P - \lambda I_n \end{pmatrix} < n \Leftrightarrow$ The system $\begin{pmatrix} I_n - U^*U \\ P - \lambda I_n \end{pmatrix} y = 0$ has a nonzero solution $y \neq 0$, $y \in \mathbb{R}^n$, and the latter is equivalent to that the system $\begin{cases} Py = \lambda y \\ y = U^*Uy \end{cases}$ has a nonzero solution $y \neq 0$. □

Theorem 1 *Let λ be an eigenvalue of Hermitian matrix P, and $U \in M_{k,n}$ such that $UU^* = I_k$.*

(1) λ is an eigenvalue of UPU^* if and only if $\mathrm{R}\begin{pmatrix} I_n - U^*U \\ P - \lambda I_n \end{pmatrix} < n$;

(2) λ is an eigenvalue of UPU^* if and only if there exists a $y \neq 0$, $Py = \lambda y$ satisfying $(I_n - U^*U)y = 0$;

(3) λ is an eigenvalue of UPU^* if $\mathrm{R}(P - \lambda I_n) < k$.

Proof (1) Sufficiency. If there exists a $y \neq 0$, $Py = \lambda y$ satisfying $(I_n - U^*U)y = 0$, we have

$$\lambda y = Py = P(U^*Uy) = PU^*(Uy)$$

and $UPU^*(Uy) = \lambda(Uy)$. Since $y \neq 0$, $y = U^*Uy$ and $UU^* = I_k$, we know $Uy \neq 0$. So λ is an eigenvalue of UPU^*.

Necessity. The equation

$$\begin{pmatrix} I_n - U^*U \\ P - \lambda I_n \end{pmatrix} = \begin{pmatrix} O & I_n \\ I_n & P \end{pmatrix} \begin{pmatrix} (U^*UP - \lambda I_n)^* \\ I_n - U^*U \end{pmatrix}$$

shows $\mathrm{R}\begin{pmatrix} I_n - U^*U \\ P - \lambda I_n \end{pmatrix} = \mathrm{R}\begin{pmatrix} (U^*UP - \lambda I_n)^* \\ I_n - U^*U \end{pmatrix}$.

Using Lemma 2, we have

$$\mathrm{R}((U^*UP - \lambda I_n)^*) = \mathrm{R}(U^*UP - \lambda I_n) = R(UPU^* - \lambda I_k)$$

and $\mathrm{R}(UPU^* - \lambda I_k) < k$. This gives $\mathrm{R}((U^*UP - \lambda I_n)^*) < k$ and

$$\mathrm{R}\begin{pmatrix} (U^*UP - \lambda I_n)^* \\ I_n - U^*U \end{pmatrix} \leq \mathrm{R}(U^*UP - \lambda I_n)^* + \mathrm{R}(I_n - U^*U) < k + (n-k) = n.$$

So, we have $\mathrm{R}\begin{pmatrix} (U^*UP - \lambda I_n)^* \\ I_n - U^*U \end{pmatrix} < n$.

(2) By Lemma 4, it is clear that (2) is equivalent to (1).

(3) Since $UPU^* - \lambda I_k = U(P - \lambda I_n)U^*$, $\mathrm{R}(P - \lambda I_n) < k$ and $\mathrm{R}(U) = \mathrm{R}(U^*) = k$, we have $\mathrm{R}(UPU^* - \lambda I_k) < k$. By Lemma 3, λ is an eigenvalue of UPU^*. $\square$

Example 2 Take the same matrix P as that in Example 1, and

$$U = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{2\sqrt{11}} & \frac{5}{2\sqrt{11}} & \frac{3}{2\sqrt{11}} & \frac{1}{2\sqrt{11}} & 0 \\ \frac{2}{\sqrt{209}} & -\frac{4}{\sqrt{209}} & \frac{2}{\sqrt{209}} & \frac{8}{\sqrt{209}} & \frac{11}{\sqrt{209}} \end{pmatrix}.$$

A direct calculation shows that

$$\mathrm{R}\begin{pmatrix} I_5 - U^*U \\ P - \lambda I_5 \end{pmatrix} = \begin{cases} 5, & \lambda = \frac{19+\sqrt{97}}{2}; \\ 5, & \lambda = \frac{19-\sqrt{97}}{2}; \\ 5, & \lambda = 4; \\ 4, & \lambda = 2; \\ 5, & \lambda = 0. \end{cases}$$

So $\frac{19+\sqrt{97}}{2}$, $\frac{19-\sqrt{97}}{2}$, 4, 0 are not eigenvalues of UPU^*, and 2 must be an eigenvalue of UPU^*. In fact, we have

$$UPU^* = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{130}{\sqrt{11}} & \frac{210}{11\sqrt{19}} \\ 0 & \frac{210}{11\sqrt{19}} & \frac{1149}{209} \end{pmatrix},$$

and the eigenvalues of UPU^* are $\frac{329+\sqrt{42121}}{38}$, $\frac{329-\sqrt{42121}}{38}$, 2.

Remark The condition (3) in Theorem 1 is sufficient, but not necessary.

Take

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$UPU^* = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

And we have $\mathrm{R}(P - \lambda I_3) = 2 = k$ with $\lambda = 1$.

Theorem 2 *Assume that P is a Hermitian matrix.*

(1) If λ_1, λ_2 are eigenvalues of P and $\lambda_1 > \lambda_2$, then there exists an $x \neq 0$, $x \in \mathbb{R}^n$ such that $x^*Px = \lambda_1$, but x is not an eigenvector of P associated with λ_1;

(2) If λ is an eigenvalue of P and $x \neq 0$, $x^*x = 1$ such that $x^*Px = \lambda$, then x is an eigenvector of P associated with λ.

Proof (1) Let $\{\xi_1, \xi_2, \ldots, \xi_n\}$ be an orthonormal set of eigenvectors of P with $P\xi_i = \lambda_i \xi_i$. Take

$$\lambda = \begin{cases} 1, & \lambda_2 = 0 \\ \frac{2}{\sqrt{5}}, & \lambda_1\lambda_2 > 0 \\ \frac{2}{\sqrt{3}}, & \lambda_1\lambda_2 < 0 \end{cases},$$

we have

$$\lambda(\xi_1 + \lambda_1\xi_2)^* P \lambda(\xi_1 + \lambda_1\xi_2) = \lambda^2 \left(\lambda_1 + \lambda_1^2\lambda_2\right) = \lambda_1, \quad \lambda_2 = 0$$
$$\lambda(\xi_1 + \frac{1}{2}\sqrt{|\frac{\lambda_1}{\lambda_2}|}\xi_2)^* P \lambda(\xi_1 + \frac{1}{2}\sqrt{|\frac{\lambda_1}{\lambda_2}|}\xi_2) = \lambda^2(\lambda_1 + \frac{1}{4}|\frac{\lambda_1}{\lambda_2}|\lambda_2) = \lambda_1, \quad \lambda_2 \neq 0.$$

But $\lambda(\xi_1 + \lambda_1\xi_2), \lambda(\xi_1 + \frac{1}{2}\sqrt{|\frac{\lambda_1}{\lambda_2}|}\xi_2)$ are not eigenvectors of P associated with λ_1.

(2) Since λ is an eigenvalue of UPU^* with $U = x^*$, using Theorem 1 gives a $y \neq 0$, $Py = \lambda y$ satisfying $(I_n - xx^*)y = 0$. This shows $x^*y \neq 0$ and $(Px - \lambda x)(x^*y) = 0$, which means $Px = \lambda x$, i.e., x is an eigenvector of P associated with λ. □

4. Conclusion

This note investigates the relationship of eigenvalues of Hermitian matrices P and UPU^* with $UU^* = I_k$ and $k \leq n$. A theorem guarantees $\lambda_i(UPU^*) = \lambda_i(P)$ $(i \leq k)$ if $k = n$. In this case, P is unitarily equivalent to UPU^*. In general, it is not true. The Poincaré Separation Theorem gives an eigenvalue inequality for UPU^*. We present several equivalent conditions for $\lambda_i(UPU^*) = \lambda_i(P)$ $(i \leq k)$ if $k \leq n$. The results in this note deserve a pure theoretical interest as well as computational purpose.

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