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A Controlled Convergence Theorem for the C-Pettis Integral

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Abstract In this paper, we give the Riemann-type extensions of Dunford integral and Pettis integral, C-Dunford integral and C-Pettis integral. We discuss the relationship between the C-Pettis integral and Pettis integral, and prove a controlled convergence theorem for the C-Pettis integral.

Keywords C-integral; C-Pettis integral; controlled convergence.

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1. Introduction

In 1996, Bongiorno provided a new solution to the problem of recovering a function from its derivative by integration by introducing a constructive minimal integration process of Riemann type, called C-integral, which includes the Lebesgue integral and also integrates the derivatives of differentiable function. Bongiorno and Piazza [1–3] discussed some properties of the C-integral of real-valued functions. In [8–10], we studied the Banach-valued C-integral.

The Dunford integral and the Pettis integral are generalizations of Lebegue integral to the Banach-valued functions. In this paper, we give the Riemann-type extensions of Dunford integral and Pettis integral, C-Dunford integral and C-Pettis integral. We discuss the relationship between the C-Pettis integral and Pettis integral. If a function f is C-integrable on $[a,b]$, then f is C-Pettis integrable on $[a,b]$, but an example shows that the converse is not true. Finally, we prove a controlled convergence theorem for the C-Pettis integral.

2. Definitions and basic properties

Throughout this paper, $[a,b]$ is a compact interval in R. X will denote a real Banach space with norm $\|\cdot\|$ and its dual X^*. A partition D is a finite collection of interval-point pairs $\{([u_i,v_i],\xi_i)\}_{i=1}^n$, where $\{[u_i,v_i]\}_{i=1}^n$ are non-overlapping subintervals of $[a,b]$. $\delta(\xi)$ is a positive function on $[a,b]$, i.e., $\delta(\xi):[a,b]\to \mathrm{R}^+$. We say that $D=\{([u_i,v_i],\xi_i)\}_{i=1}^n$ is

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1) a partition of $[a,b]$ if $\bigcup_{i=1}^n [u_i, v_i] = [a,b]$,

2) δ-fine McShane partition of $[a,b]$ if $[u_i, v_i] \subset B(\xi_i, \delta(\xi_i)) = (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ and $\xi_i \in [a,b]$ for all $i = 1, 2, \dots, n$,

3) δ-fine C-partition of $[a,b]$ if it is a δ-fine McShane partition of $[a,b]$, satisfying the condition $\sum_{i=1}^n \mathrm{dist}(\xi_i, [u_i, v_i]) < \frac{1}{\varepsilon}$, here $\mathrm{dist}(\xi_i, [u_i, v_i]) = \inf\{|t_i - \xi_i| : t_i \in [u_i, v_i]\}$.

Given a δ-fine C-partition $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$, we write $S(f, D) = \sum_{i=1}^n f(\xi_i)(v_i - u_i)$ for integral sums over D, whenever $f : [a,b] \to X$.

Definition 1 *A function $f : [a,b] \to X$ is C-integrable if there exists a vector $A \in X$ such that for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : [a,b] \to R^+$ such that*

$$\|S(f, D) - A\| < \epsilon$$

for each δ-fine C-partition $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[a,b]$. A is called the C-integral of f on $[a,b]$, and we write $A = \int_a^b f$ or $A = (C)\int_a^b f$.

The basic properties of the C-integral, say, linearity and additivity with respect to intervals, can be found in [9]. We do not present them here. We refer to [9] for the details.

Definition 2 *A function $f : [a,b] \to X$ is C-Dunford integrable on $[a,b]$ if x^*f is C-integrable on $[a,b]$ for each $x^* \in X^*$ and if for every subinterval $[c,d] \subset [a,b]$ there exists an element $x^{**}_{[c,d]} \in X^{**}$ such that $\int_c^d x^* f = x^{**}_{[c,d]}(x^*)$ for each $x^* \in X^*$. We write $(CD)\int_c^d f = x^{**}_{[c,d]} \in x^{**}$.*

Definition 3 *A function $f : [a,b] \to X$ is C-Pettis integrable on $[a,b]$ if f is C-Dunford integrable on $[a,b]$ and $(CD)\int_c^d f \in X$ for every interval $[c,d] \subset [a,b]$. We write $(CP)\int_c^d f = (CD)\int_c^d f \in X$.*

The function f is C-Pettis integrable on the set $E \subset [a,b]$ if the function $f\chi_E$ is C-Pettis integrable on $[a,b]$. We write $(CP)\int_E f = (CP)\int_a^b f\chi_E$.

Lemma 1 *If a function $f : [a,b] \to X$ is C-integrable on $[a,b]$, then f is C-Pettis integrable on $[a,b]$.*

Proof f is C-integrable on $[a,b]$, then x^*f is C-integrable on $[a,b]$ for each $x^* \in X^*$ and $(C)\int_a^b x^* f = x^*((C)\int_a^b f)$ from [8,Theorem 2.7]. For each subinterval $[c,d] \subset [a,b]$, we have $(C)\int_c^d f \in X$. Then f is C-Pettis integrable on $[a,b]$ and $(CP)\int_a^b f = (C)\int_a^b f$.

Remark The following example shows that the converse of Lemma 1 is not true. In other words, there exists a function which is C-Pettis integrable but is not C-integrable.

Example (a) Define a function $f : [0,1] \longrightarrow l_\infty(\omega_1)$ by

$$f(t)(\alpha) = \begin{cases} 1, & \text{if } t \in N_\alpha \backslash C_\alpha, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where ω_1 is the first uncountable ordinal, and $\{N_\alpha\}_{\alpha \in \omega_1}$ and $\{C_\alpha\}_{\alpha \in \omega_1}$ are two collection of subsets of $[0,1]$ satisfying the following properties:

1) For each $\alpha \in \omega_1$, N_α is a set of zero Lebesgue measure;

2) $N_\alpha \subset N_\beta$, if $\alpha < \beta$;

3) Every subset of $[0,1]$ of zero Lebesgue measure is contained in some set N_α;

4) For each $\alpha \in \omega_1$, C_α is a countable set;

5) $C_\alpha \subset C_\beta$, if $\alpha < \beta$;

6) Every countable subset of $[0,1]$ is contained in some set C_α.

In [5, Example(CH)], Di Piazza and Preiss proved that f is Pettis integrable but is not McShane integrable on $[0,1]$. It is easy to know that f is C-Pettis integrable on $[0,1]$ from Lemma 1. In [8,Theorem 3.4], we proved that f is McShane integrable if and only if f is C-integrable and Pettis integrable. Suppose that f is C-integrable on $[0,1]$, then f is McShane integrable on $[0,1]$. This is a contradiction, so f is not C-integrable on $[0,1]$.

3. Main results

Definition 4 *Let $F_n, F : [a,b] \to R$ and let E be a subset of $[a,b]$.*

(a) F is said to be AC_c on E if for each $\varepsilon > 0$ there is a constant $\eta > 0$ and a positive function $\delta(\xi) : E \to R^+$ such that $\sum_i |F([u_i,v_i])| < \epsilon$ for each δ-fine partial C-partition $D = \{([u_i,v_i],\xi_i)\}$ of $[a,b]$ satisfying $\xi_i \in E$ for each i and $\sum_i (v_i - u_i) < \eta$.

(b) F_n is said to be UAC_c on E if for each $\varepsilon > 0$ there is a constant $\eta > 0$ and a positive function $\delta(\xi) : E \to R^+$ such that $\sum_i |F_n([u_i,v_i])| < \epsilon$ for all n and for each δ-fine partial C-partition $D = \{([u_i,v_i],\xi_i)\}$ of $[a,b]$ satisfying $\xi_i \in E$ for each i and $\sum_i (v_i - u_i) < \eta$.

(c) F is said to be ACG_c on E if F is continuous on E and E can be expressed as a countable union of sets on each of which F is AC_c.

(d) F is said to be $UACG_c$ on E if F is continuous on E and E can be expressed as a countable union of sets on each of which F is UAC_c.

Lemma 2 *Let $f : [a,b] \to X$ and assume that $\{f_n\}$ is a sequence of C-integrable functions. Assume that the following conditions are satisfied:*

1) $f_n \to f$ almost everywhere on $[a,b]$;

2) F_n are $UACG_c$ on $[a,b]$.

Then f is C-integrable on $[a,b]$ and $\lim_{n\to\infty} (C)\int_a^b f_n = (C)\int_a^b f$.

Proof The proof is standard and similar to [7, Theorem 5.5.2].

Theorem 1 (Controlled Convergence Theorem) *Let $f : [a,b] \to X$ and assume that $\{f_n\}$ is a sequence of C-Pettis integrable functions. Assume that the following conditions are satisfied:*

1) For each $x^* \in X^*$, $x^* f_n \to x^* f$ almost everywhere on $[a,b]$;

2) The family $\{x^* F_n : x^* \in X^*, n \in \mathbb{N}\}$ is $UACG_c$ on $[a,b]$.

Then f is C-Pettis integrable on $[a,b]$ and

$$\lim_{n\to\infty} (CP)\int_a^b f_n = (CP)\int_a^b f \quad (\textit{weakly}).$$

Proof We will prove the Theorem in two steps.

Step 1. The sequence $\{f_n\}$ is C-Pettis integrable on $[a,b]$, then for each $x^* \in X^*$, $x^* f_n$ is C-integrable on $[a,b]$. From Lemma 2 we have that $x^* f$ is C-integrable on $[a,b]$ and

$$\lim_{n\to\infty}(C)\int_a^b x^* f_n = (C)\int_a^b x^* f.$$

Step 2. Assume $[c,d]$ is an arbitrary subinterval of $[a,b]$. Let $\mathcal{C}$ denote the weak closure of $\{(CP)\int_c^d f_n : n \in \mathbb{N}\}$. It is easy to see that $\mathcal{C}$ is bounded and that $\mathcal{C}\backslash\{(CP)\int_c^d f_n : n \in \mathbb{N}\}$ contains at most one point. We claim that $\mathcal{C}$ is weakly compact.

Suppose $\mathcal{C}$ is not weakly compact, then there exists a bounded sequence $(x_k^*) \subset X^*$, a sequence $(x_n) \subset \mathcal{C}$ and $\epsilon > 0$ such that

$$\begin{cases} x_k^*(x_n) = 0, & \text{if } \ k > n, \\ x_k^*(x_n) > \epsilon, & \text{if } \ k \le n. \end{cases} \tag{2}$$

We can take subsequence $(g_n) \subset (f_n)$ and a sequence $(y_k^*) \subset x_k^*$ such that

$$\begin{cases} (C)\displaystyle\int_c^d y_k^* g_n = 0, & \text{if } \ k > n, \\ (C)\displaystyle\int_c^d y_k^* g_n > \epsilon, & \text{if } \ k \le n, \\ \displaystyle\lim_{n\to\infty}(C)\int_c^d x^* g_n = (C)\int_c^d x^* f, & \text{for each } \ x^* \in X^*. \end{cases} \tag{3}$$

From [4, Lemma 1], we can find a subsequence $(y_{k_j}^*) \subset (y_k^*)$ such that $\lim_{j\to\infty} y_{k_j}^* f$ exists almost everywhere. Assume y_0^* is a weak* cluster point of $(y_{k_j}^*) \subset (y_k^*)$, then we have $\lim_{j\to\infty} y_{k_j}^* f = y_0^* f$ almost everywhere on $[a,b]$. It is not difficult to get that $\lim_{j\to\infty}(C)\int_c^d y_{k_j}^* f = (C)\int_c^d y_0^* f$. To force a contradiction, note that for each j, we have that

$$\lim_{n\to\infty}(C)\int_c^d y_{k_j}^* g_n = (C)\int_c^d y_{k_j}^* f.$$

When $n \ge k_j$, from (3) we have that $(C)\int_c^d y_{k_j}^* g_n > \epsilon$ and $(C)\int_c^d y_{k_j}^* f \ge \epsilon$. Therefore

$$\lim_{j\to\infty}(C)\int_c^d y_{k_j}^* f = (C)\int_c^d y_0^* f \ge \epsilon.$$

On the other hand, g_n is C-Pettis integrable for each n, the functional $x^* \longrightarrow (C)\int_c^d x^* g_n$ is weak*-continuous. Then if (y_α^*) is a subset of $(y_{k_j}^*)$ weak* converging to y_0^*, by (3) and passing to the limit with $n \to \infty$ we have that

$$\begin{aligned} \lim_{n\to\infty}\lim_{\alpha}(C)\int_c^d y_\alpha^* g_n &= \lim_{n\to\infty}\lim_{\alpha} y_\alpha^*(CP)\int_c^d g_n = \lim_{n\to\infty} y_0^*(CP)\int_c^d g_n \\ &= \lim_{n\to\infty}(C)\int_c^d y_0^* g_n = (C)\int_c^d y_0^* f = 0 \end{aligned}$$

which contradicts the inequality $(C)\int_c^d y_0^* f \ge \epsilon$. Therefore, the set $\mathcal{C}$ is weakly compact.

Since $\lim_{n\to\infty}(C)\int_c^d x^* f_n = (C)\int_c^d x^* f$, the sequence $\{(CP)\int_c^d f_n\}$ is weak Cauchy. It follows from the weak compactness of $\mathcal{C}$ that $\lim_{n\to\infty}(CP)\int_c^d f_n$ exists weakly in X. Moreover

$[c,d]$ is an arbitrary subinterval of $[a,b]$, then f is C-Pettis integrable on $[a,b]$ and

$$\lim_{n\to\infty}(CP)\int_a^b f_n = (CP)\int_a^b f \quad \text{(weakly)}.$$

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References

[1] BONGIORNO B. *On the minimal solution of the problem of primitives* [J]. J. Math. Anal. Appl., 2000, **251**(2): 479–487.

[2] BONGIORNO B, DI PIAZZA L, PREISS D. *A constructive minimal integral which includes Lebesgue integrable functions and derivatives* [J]. J. London Math. Soc. (2), 2000, **62**(1): 117–126.

[3] DI PIAZZA L. *A Riemann-type minimal integral for the classical problem of primitives* [J]. Rend. Istit. Mat. Univ. Trieste, 2002, **34**(1-2): 143–153.

[4] GEITZ R F. *Pettis integration* [J]. Proc. Amer. Math. Soc., 1981, **82**(1): 81–86.

[5] DI PIAZZA L, PREISS D. *When do McShane and Pettis integrals coincide?* [J]. Illinois J. Math., 2003, **47**(4): 1177–1187.

[6] YE Guoju, AN Tianqing. *On Henstock-Dunford and Henstock-Pettis integrals* [J]. Int. J. Math. Math. Sci., 2001, **25**(7): 467–478.

[7] LEE P Y, VÝBORNÝ R. *Integral: An Easy Approach After Kurzweil and Henstock* [M]. Cambridge University Press, Cambridge, 2000.

[8] ZHAO Dafang, YE Guoju. *C-integral and Denjoy-C integral* [J]. Commun. Korean Math. Soc., 2007, **22**(1): 27–39.

[9] ZHAO Dafang, YE Guoju. *On C-integral of Banach-valued functions* [J]. Kangweon-Kyungki J. Math., 2006, **14**(2): 169–183.

[10] ZHAO Dafang, YE Guoju. *On strong C-integral of Banach-valued functions* [J]. J. Chungcheong Math. Soc., 2007, **20**(1): 1–10.