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Maximal Armendariz Subrings Relative to a Monoid of Matrix Rings

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Abstract Let M be a monoid. Maximal M-Armendariz subrings of upper triangular matrix rings are identified when R is M-Armendariz and reduced. Consequently, new families of M-Armendariz rings are presented.

Keywords M-Armendariz ring; matrix ring; reduced ring.

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1. Introduction

Throughout this paper R denotes an associative ring with identity. According to Rege and Chhawchharia [1], a ring R is called Armendariz if, whenever $(\sum_{i=0}^{m} a_ix^i)(\sum_{j=0}^{n} b_jx^j) = 0$ in $R[x]$, $a_ib_j = 0$ for all i and j. A ring is called reduced if it has no non-zero nilpotent elements. Every reduced ring is Armendariz by Armendariz [2], but the more comprehensive study of the notion of Armendariz rings was carried out just recently (see Anderson and Camillo [3], Kim and Lee [4], Hong, Kim and Kwak [5], Huh, Lee and Smoktunowicz [6], Lee and Wong [7], Lee and Zhou [8], Liu [9], Hong, Kim and Twak [10]).

According to Liu [11], a ring R is called an M-Armendariz ring (an Armendariz ring relative to a monoid) if, whenever $(\sum_{i=1}^{m} a_i\alpha_i)(\sum_{j=1}^{n} b_j\beta_j) = 0$ in $R[M]$, $a_ib_j = 0$ for all i and j.

In this paper, we continue the study of M-Armendariz rings, and focus on the M-Armendariz property of certain subrings of upper triangular matrix rings.

We denote by $T_n(R)$ and $M_n(R)$ the $n\times n$ upper triangular matrix ring and matrix ring over R, respectively.

Let R be a ring. Define a subring F_n of upper triangular matrix ring $T_n(R)$ over R as follows:

$$F_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} |a, a_{ij} \in R \right\}.$$

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Let M be a monoid. It was proved in Liu [11], Proposition 1.7 and Remark 1.8 that if R is M-Armendariz and reduced, then the ring F_3 is M-Armendariz but F_n is not M-Armendariz for $n \geqslant 4$.

So, for an M-Armendariz and reduced ring R, it is interesting to find some maximal M-Armendariz subrings of $T_n(R)$. For this purpose, in this paper, we define $S_{n,m}(R) =$

$$\left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_{m-1} & a_m & a_{1,m+1} & \cdots & a_{1,n-1} & a_{1,n} \\ 0 & a_1 & \ddots & & a_{m-1} & a_{2,m+1} & \cdots & a_{2,n-1} & a_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & a_1 & a_2 & a_{m-1,m+1} & \cdots & a_{m-1,n-1} & a_{m-1,n} \\ 0 & \cdots & \cdots & 0 & a_1 & \overline{a}_2 & \cdots & \overline{a}_{n-m} & \overline{a}_{n-m+1} \\ \vdots & & & \vdots & 0 & a_1 & \ddots & & \overline{a}_{n-m} \\ \vdots & & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \vdots & \vdots & & \ddots & a_1 & \overline{a}_2 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & a_1 \end{pmatrix} \right\},$$

where $a_h, \overline{a}_g, a_{i,j} \in R$, and show that for any M-Armendariz and reduced ring R and all $2 \leqslant m \leqslant n-1$, the ring $S_{n,m}(R)$ is a maximal M-Armendariz subring of $T_n(R)$. This is a generalization of Liu [11], Proposition 1.7 and Remark 1.8.

By the term "ring" we mean an associative ring with identity, and by a general ring we mean an associative ring with or without identity. For clarity, R will always denote a ring while a general ring will be denoted by I.

Let I be a general ring. Define a subring $D_n(I)$ of matrix ring $M_n(I)$ over I as follows:

$$D_n(I) = \left\{ \begin{pmatrix} a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & \cdots & a_n \end{pmatrix} | a_i \in I \right\} \text{ for } n \geqslant 2.$$

We show that for any general M-Armendariz and reduced ring I, and $|M| \geqslant 2$, $|I| \geqslant 2$, the general ring $D_n(I)$ is a maximal general M-Armendariz subring of $M_n(I)$ for $n \geqslant 2$.

2. Maximal M-Armendariz subrings of $T_n(R)$

Lemma 2.1 *Let M be a monoid with $|M| \geqslant 2$. If R is M-Armendariz and reduced, then $R[M]$ is reduced.*

Proof The proof has been shown in the proof of Liu [11, Proposition 2.1]. □

According to [8], for $A = (a_{i,j})$, $B = (b_{i,j}) \in M_n(R)$, we write $[A \cdot B]_{i,j} = 0$ to mean that $a_{i,l}b_{l,j} = 0$ for $l = 1, \ldots, n$. We identify $M_n(R)[M]$ with $M_n(R[M])$ canonically.

For $n \geqslant 2$, let $V = \sum_{i=1}^{n-1} E_{i,i+1}$ where $E_{i,j} : 1 \leqslant i, j \leqslant n$ are the matrix units.

Lemma 2.2 *Let M be a monoid. For $u = \sum_{i=1}^{m} A_i\alpha_i$, $v = \sum_{j=1}^{k} B_j\beta_j \in M_n(R)[M]$, let*

$f_{i,j}=\sum_{s=1}^{m}a_{i,j}^{(s)}\alpha_s$ and $g_{i,j}=\sum_{t=1}^{k}b_{i,j}^{(t)}\beta_t$ where $a_{i,j}^{(l)}$ and $b_{i,j}^{(h)}$ are the (i,j)-entries of A_l and B_h, respectively, for $l=1,\dots,m$, $h=1,\dots,k$. Then $u=(f_{i,j})$, $v=(g_{i,j})$. If R is M-Armendariz and $[u\cdot v]_{i,j}=0$ for all i and j, then $A_iB_j=0$ for all i and j.

Proof Since $[u\cdot v]_{i,j}=0$ for all i and j and R is M-Armendariz , $a_{il}^{(s)}b_{lj}^{(t)}=0$ for all i and j, where $l=1,\dots,n$. Then $A_iB_j=0$ for all i and j. □

Lemma 2.3 ([13, Theorem 2.3]) *Let R be a reduced ring. If $AB=0$ in $S_{n,m}(R)$, then $[A\cdot B]_{i,j}=0$ for all i,j and all $2\leqslant m\leqslant n$.*

Theorem 2.4 *Let M be a monoid with $|M|\geqslant 2$. Then the following conditions are equivalent.*

(1) R is M-Armendariz and reduced;

(2) $S_{n,m}(R)$ is an M-Armendariz ring for all $2\leqslant m\leqslant n$.

Proof (1) ⇒ (2). Suppose that $u=\sum_{i=1}^{p}A_i\alpha_i$, $v=\sum_{j=1}^{p}B_j\beta_j\in S_{n,m}(R)[M]$ such that $uv=0$. We need to prove that $A_iB_j=0$ for all $1\leqslant i,\ j\leqslant p$. Let $f_{i,j}=\sum_{s=1}^{p}a_{i,j}^{(s)}\alpha_s$ and $g_{i,j}=\sum_{t=1}^{p}b_{i,j}^{(t)}\beta_t$ where $a_{i,j}^{(l)}$ and $b_{i,j}^{(l)}$ are the (i,j)-entries of A_l and B_l, respectively, for $l=1,\dots,p$. Then $u=(f_{i,j})$, $v=(g_{i,j})$. By Lemma 2.3, $[u\cdot v]_{i,j}=0$ for all i and j, then $A_iB_j=0$ for all $1\leqslant i,\ j\leqslant p$ by Lemma 2.2.

(2) ⇒ (1). Suppose that $S_{n,m}(R)$ is M-Armendariz. Note that R is isomorphic to the subring

$$\left\{\left.\begin{pmatrix}a&0&\cdots&0\\0&a&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&a\end{pmatrix}\right| a\in R.\right\}$$

of $S_{n,m}(R)$. Thus R is M-Armendariz, since each subring of an M-Armendariz ring is also M-Armendariz. By analogy with the proof of Lee and Wong [7], Lemma 2.3, we can show that R is reduced. □

Corollary 2.5 ([11, Proposition 1.7]) *Let M be a monoid with $|M|\geqslant 2$. Then the following conditions are equivalent.*

(1) R is M-Armendariz and reduced;

(2) $S_3=\left\{\left.\begin{pmatrix}a&b&c\\0&a&d\\0&0&a\end{pmatrix}\right| a,b,c,d\in R.\right\}$ is M-Armendariz.

Theorem 2.6 *Let M be a monoid with $|M|\geqslant 2$ and R be an M-Armendariz ring and reduced. Then $S_{n,m}(R)$ is a maximal M-Armendariz subring of $T_n(R)$ for all $2\leqslant m\leqslant n-1$.*

Proof Suppose that T is an M-Armendariz subring of $T_n(R)$ and T properly contains $S_{n,m}(R)$. Take $e\neq g\in M$, where e stands for the identity of M. Then there exists $A=(a_{i,j})\in T$ such that one of the following conditions holds:

1) $a_{k-1,l-1}\neq a_{k,l}$ for some $2\leqslant k\leqslant l\leqslant m$;

2) $a_{k,l}\neq a_{k+1,l+1}$ for some $m\leqslant k\leqslant l\leqslant n-1$.

Case 1 Suppose that 1) holds. We can assume without loss of generality that $a_{1,1+t}=a_{2,2+t}=$

$\cdots = a_{m-t,m}$ where $0 \leqslant t \leqslant l-k-1$, and $a_{1,l-k+1} = a_{2,l-k+2} = \cdots = a_{k-1,l-1}$. Let $V^0 = I_n$, and $A_1 = A - \sum_{t=0}^{l-k-1} a_{1,1+t}V^t - \sum_{t=0}^{k-1} a_{k-t,l}V^{l-k+t}$. Clearly $u = A_1 e - (a_{k-1,l-1} - a_{k,l})V^{l-1}g \in T[M]$ and $v = E_{l,n}e + E_{l-k+1,n}g \in T[M]$. One easily checks that $uv = 0$ but $(a_{k-1,l-1} - a_{k,l})V^{l-1}E_{l,n} = (a_{k-1,l-1} - a_{k,l})E_{1,n} \neq 0$. Hence T is not an M-Armendariz ring.

Case 2 Suppose that 2) holds. We can assume without loss of generality that $a_{m,m+t} = a_{m+1,m+t+1} = \cdots = a_{n-t,n}$ where $0 \leqslant t \leqslant l-k-1$, and $a_{k+1,l+1} = a_{k+2,l+2} = \cdots = a_{k-l+n,n}$. Let $A_1 = A - \sum_{t=0}^{l-k-1} a_{m,m+t}V^t - \sum_{t=0}^{n-l} a_{k,l+t}V^{l-k+t}$. Then $u = (a_{k+1,l+1} - a_{k,l})E_{1,k}e - E_{1,n-l+k}g$, $v = A_1 e + V^{n-k}g \in T[M]$. One checks that $uv = 0$, but $(a_{k+1,l+1} - a_{k,l})E_{1,k}V^{n-k} = (a_{k+1,l+1} - a_{k,l})E_{1,n} \neq 0$. Hence T is not an M-Armedariz ring. □

By using the same methods as in the proof of Theorem 2.6, we have the following Theorem 2.7.

Theorem 2.7 *Let M be a monoid with $|M| \geqslant 2$ and R be an M-Armendariz ring and reduced. Then*

$$F_2 = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in R. \right\}$$

is a maximal M-Armendariz subring of $T_2(R)$, and

$$F_3 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} | a, b, c, d. \in R \right\}$$

is a maximal M-Armendariz subring of $T_3(R)$.

3. Maximal general M-Armendariz subrings of $M_n(I)$

Definition 3.1 *A general ring I is called general reduced if it has no non-zero nilpotent elements.*

Definition 3.2 *Let M be a monoid. A general ring I is called general M-Armendariz if, whenever $(\sum_{i=1}^m a_i\alpha_i)(\sum_{j=1}^n b_j\beta_j) = 0$ in $I[M]$, $a_ib_j = 0$ for all i and j.*

Let $D_n(I) = \left\{ \begin{pmatrix} a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & \cdots & a_n \end{pmatrix} | a_i \in I \right\}$ for $n \geqslant 2$.

Theorem 3.3 *Let M be a monoid. If I is a general M-Armendariz ring, then $D_n(I)$ is a general M-Armendariz subring of $M_n(I)$ for $n \geqslant 2$.*

Proof Suppose that $f = \sum_{i=1}^m A_i\alpha_i$, $g = \sum_{j=1}^k B_j\beta_j \in D_n(I)[M]$, such that $fg = 0$. We need to prove that $A_iB_j = 0$ for all i and j.

Let

$$A_i = \begin{pmatrix} a_1^{(i)} & a_1^{(i)} & \cdots & a_1^{(i)} \\ a_2^{(i)} & a_2^{(i)} & \cdots & a_2^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ a_n^{(i)} & a_n^{(i)} & \cdots & a_n^{(i)} \end{pmatrix}, \quad B_j = \begin{pmatrix} b_1^{(j)} & b_1^{(j)} & \cdots & b_1^{(j)} \\ b_2^{(j)} & b_2^{(j)} & \cdots & b_2^{(j)} \\ \vdots & \vdots & \ddots & \vdots \\ b_n^{(j)} & b_n^{(j)} & \cdots & b_n^{(j)} \end{pmatrix},$$

where $a_s^{(i)}$, $b_s^{(j)} \in I$ for $1 \leqslant s \leqslant n$ and $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant k$, and let

$$f = \begin{pmatrix} f_1 & f_1 & \cdots & f_1 \\ f_2 & f_2 & \cdots & f_2 \\ \vdots & \vdots & \ddots & \vdots \\ f_n & f_n & \cdots & f_n \end{pmatrix}, \quad g = \begin{pmatrix} g_1 & g_1 & \cdots & g_1 \\ g_2 & g_2 & \cdots & g_2 \\ \vdots & \vdots & \ddots & \vdots \\ g_n & g_n & \cdots & g_n \end{pmatrix},$$

where $f_u = \sum_{i=1}^m a_u^{(i)} \alpha_i$, $g_v = \sum_{j=1}^k b_v^{(j)} \beta_j \in I$ for $1 \leqslant u, v \leqslant n$.

It follows from $fg = 0$ that

$$f_u[g_1 + g_2 + \cdots + g_n] = 0, \text{ for } 1 \leqslant u \leqslant n. \tag{3.1}$$

Because I is a general M-Armendariz ring, we have

$$a_u^{(i)}[b_1^{(j)} + b_2^{(j)} + \cdots + b_n^{(j)}] = 0 \text{ for } 1 \leqslant i, j \leqslant m \text{ and } 1 \leqslant u \leqslant n. \tag{3.2}$$

Hence we show that $A_i B_j = 0$ for all $1 \leqslant i,\ j \leqslant m$. $\square$

Theorem 3.4 *Let M be a monoid and $|M| \geqslant 2$. If I is a general M-Armendariz and reduced ring, and $|I| \geqslant 2$, then $D_n(I)$ is a maximal general M-Armendariz subring of $M_n(I)$ for $n \geqslant 2$.*

Proof Suppose that T is a general M-Armendariz subring of $M_n(I)$ and T properly contains $D_n(I)$, then there exists $A = (a_{i,j}) \in T \backslash D_n(I)$ where $1 \leqslant i,\ j \leqslant n$. It suffices to show that T is not general M-Armendariz. Take $e \neq g \in M$, where e stands for the identity of M. We will proceed with the following two cases.

Case 1 Suppose that $a_{11} = a_{12} = \cdots = a_{1,j-1} \neq a_{1,j}$ where $2 \leqslant j \leqslant n$. Then $a_{1,j-1} - a_{1,j} \neq 0$. Let

$$A_1 = A - \begin{pmatrix} a_{1,j} & a_{1,j} & \cdots & a_{1,j} \\ a_{2,j} & a_{2,j} & \cdots & a_{2,j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,j} & a_{n,j} & \cdots & a_{n,j} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} - a_{1,j} & \cdots & a_{1,j-1} - a_{1,j} & 0 & a_{1,j+1} - a_{1,j} & \cdots & a_{1,n} - a_{1,j} \\ a_{21} - a_{2,j} & \cdots & a_{2,j-1} - a_{2,j} & 0 & a_{2,j+1} - a_{2,j} & \cdots & a_{2,n} - a_{2,j} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} - a_{n,j} & \cdots & a_{n,j-1} - a_{n,j} & 0 & a_{n,j+1} - a_{n,j} & \cdots & a_{n,n} - a_{n,j} \end{pmatrix},$$

$$A_2 = A - \begin{pmatrix} a_{1,j-1} & a_{1,j-1} & \cdots & a_{1,j-1} \\ a_{2,j-1} & a_{2,j-1} & \cdots & a_{2,j-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,j-1} & a_{n,j-1} & \cdots & a_{n,j-1} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} - a_{1,j-1} & \cdots & a_{1,j-2} - a_{1,j-1} & 0 & a_{1,j} - a_{1,j-1} & \cdots & a_{1,n} - a_{1,j-1} \\ a_{21} - a_{2,j-1} & \cdots & a_{2,j-2} - a_{2,j-1} & 0 & a_{2,j} - a_{2,j-1} & \cdots & a_{2,n} - a_{2,j-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} - a_{n,j-1} & \cdots & a_{n,j-2} - a_{n,j-1} & 0 & a_{n,j} - a_{n,j-1} & \cdots & a_{n,n} - a_{n,j-1} \end{pmatrix}.$$

Then $A_1, A_2 \in T$.

Let $f = A_1e + A_2g$ be in $T[M]$, and let

$$B_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ a_{1,j-1} - a_{1,j} & a_{1,j-1} - a_{1,j} & \cdots & a_{1,j-1} - a_{1,j} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \ (j),$$

$$B_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ a_{1,j-1} - a_{1,j} & a_{1,j-1} - a_{1,j} & \cdots & a_{1,j-1} - a_{1,j} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \ (j-1).$$

Then $B_1, B_2 \in T$.

Let $g = B_1e + B_2g$ be in $T[M]$. Then $fg = 0$, but

$$A_1B_2 = \begin{pmatrix} (a_{1,j-1} - a_{1,j})(a_{1,j-1} - a_{1,j}) & \cdots & (a_{1,j-1} - a_{1,j})(a_{1,j-1} - a_{1,j}) \\ (a_{2,j-1} - a_{2,j})(a_{1,j-1} - a_{1,j}) & \cdots & (a_{2,j-1} - a_{2,j})(a_{1,j-1} - a_{1,j}) \\ \vdots & \ddots & \vdots \\ (a_{n,j-1} - a_{n,j})(a_{1,j-1} - a_{1,j}) & \cdots & (a_{n,j-1} - a_{n,j})(a_{1,j-1} - a_{1,j}) \end{pmatrix} \neq 0.$$

This is a contradiction.

Case 2 Suppose that $a_{t,1} = a_{t,2} = \cdots = a_{t,n}$ where $1 \leqslant t \leqslant i-1$, and $a_{i,1} = a_{i,2} = \cdots = a_{i,j-1} \neq a_{i,j}$ where $1 < i, j \leqslant n$. Then $a_{i,j-1} - a_{i,j} \neq 0$. Let

$$A_1 = A - \begin{pmatrix} a_{11} & a_{11} & \cdots & a_{11} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i-1,i-1} & a_{i-1,i-1} & \cdots & a_{i-1,i-1} \\ a_{i,j} & a_{i,j} & \cdots & a_{i,j} \\ a_{i+1,j} & a_{i+1,j} & \cdots & a_{i+1,j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,j} & a_{n,j} & \cdots & a_{n,j} \end{pmatrix}$$

$$= \begin{pmatrix} \cdots & 0 & 0 & 0 & \cdots & 0 \\ \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \cdots & 0 & 0 & 0 & \cdots & 0 \\ \\ \cdots & a_{i,j-1} - a_{i,j} & 0 & a_{i,j+1} - a_{i,j} & \cdots & a_{i,n} - a_{i,j} \\ \cdots & a_{i+1,j-1} - a_{i+1,j} & 0 & a_{i+1,j+1} - a_{i+1,j} & \cdots & a_{i+1,n} - a_{i+1,j} \\ \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \cdots & a_{n,j-1} - a_{n,j} & 0 & a_{n,j+1} - a_{n,j} & \cdots & a_{n,n} - a_{n,j} \end{pmatrix},$$

$$A_2 = A - \begin{pmatrix} a_{11} & a_{11} & \cdots & a_{11} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i-1,i-1} & a_{i-1,i-1} & \cdots & a_{i-1,i-1} \\ a_{i,j-1} & a_{i,j-1} & \cdots & a_{i,j-1} \\ a_{i+1,j-1} & a_{i+1,j-1} & \cdots & a_{i+1,j-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,j-1} & a_{n,j-1} & \cdots & a_{n,j-1} \end{pmatrix}$$

$$= \begin{pmatrix} \cdots & 0 & 0 & 0 & \cdots & 0 \\ \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \cdots & 0 & 0 & 0 & \cdots & 0 \\ \\ \cdots & a_{i,j-2} - a_{i,j-1} & 0 & a_{i,j} - a_{i,j-1} & \cdots & a_{i,n} - a_{i,j-1} \\ \cdots & a_{i+1,j-2} - a_{i+1,j-1} & 0 & a_{i+1,j} - a_{i+1,j-1} & \cdots & a_{i+1,n} - a_{i+1,j-1} \\ \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \cdots & a_{n,j-2} - a_{n,j-1} & 0 & a_{n,j} - a_{n,j-1} & \cdots & a_{n,n} - a_{n,j-1} \end{pmatrix}.$$

Then $A_1, A_2 \in T$.

Let $f = A_1 e + A_2 g$ be in $T[M]$, and let

$$B_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ a_{i,j-1} - a_{i,j} & a_{i,j-1} - a_{i,j} & \cdots & a_{i,j-1} - a_{i,j} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad (j)$$

$$B_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ a_{i,j-1} - a_{i,j} & a_{i,j-1} - a_{i,j} & \cdots & a_{i,j-1} - a_{i,j} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad (j-1).$$

Then $B_1, B_2 \in T$.

Let $g = B_1 e + B_2 g$ be in $T[M]$. Then $fg = 0$, but

$$A_1 B_2 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ (a_{i,j-1} - a_{i,j})(a_{i,j-1} - a_{i,j}) & \cdots & (a_{i,j-1} - a_{i,j})(a_{i,j-1} - a_{i,j}) \\ (a_{i+1,j-1} - a_{i+1,j})(a_{i,j-1} - a_{i,j}) & \cdots & (a_{i+1,j-1} - a_{i+1,j})(a_{i,j-1} - a_{i,j}) \\ \vdots & \ddots & \vdots \\ (a_{n,j-1} - a_{n,j})(a_{i,j-1} - a_{i,j}) & \cdots & (a_{n,j-1} - a_{n,j})(a_{i,j-1} - a_{i,j}) \end{pmatrix} \neq 0.$$

This is a contradiction.

Thus T is not general M-Armendariz. □

Let $D_n(I)^T = \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} | a_i \in I \right\}$ for $n \geqslant 2$.

By using the same methods as in the proofs of Theorems 3.3 and 3.4, we have the following Theorem 3.5

Theorem 3.5 *Let M be a monoid and $|M| \geqslant 2$. If I is a general M-Armendariz and reduced ring, and $|I| \geqslant 2$, then $D_n^T(I)$ is a maximal general M-Armendariz subring of $M_n(I)$ for $n \geqslant 2$.*

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