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Biharmonic Submanifolds in δ-Pinched Riemannian Manifolds

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Abstract We study biharmonic submanifolds in δ-pinched Riemannian manifolds, and obtain some sufficient conditions for biharmonic submanifolds to be minimal ones.

Keywords δ-pinched manifold; biharmonic submanifold; second fundamental form; mean curvature.

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1. Introduction and main results

The study of biharmonic maps between Riemannian manifolds, as a generalization of harmonic maps, was suggested by Eells and Sampson [1]. By integrating the square of the norm of the tension field one can consider the bienergy of a smooth map $\phi : M \to N$ and define its critical points biharmonic maps. The first variation formula for the bienergy, derived by Jiang [2], shows that any harmonic map is biharmonic.

During the last decade important progress has been made in the study of both the geometry and the analytic properties of biharmonic maps, see [3] for an account. In differential geometry, a special attention has been paid to the study of biharmonic submanifolds, i.e., submanifolds such that the isometric immersion map is a biharmonic map.

The Generalized Chen's Conjecture [4]: Biharmonic submanifolds of a manifold N with sectional curvature $K^N \leq 0$ are minimal, encouraged the study of biharmonic submanifolds in Euclidean space [4], spheres or other non-negatively curved spaces [3, 5, 6], manifolds with constant negative sectional curvature [7], and other manifolds [8–10].

Oniciuc [9] has proved that any biharmonic submanifold with constant mean curvature in a manifold with nonpositive sectional curvature is harmonic, i.e., minimal. The curvature condition in the above result can be weakened as $\mathrm{Ric}^N \leq 0$ in the case of codimension one [9].

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Those results suggest a natural question: "What is the situation for the ambient manifolds being with positive sectional curvature?" This paper is motivated by this question. We will answer it positively under the assumption that the squared norm of the second fundamental form is bounded from above or below by proving the following.

Theorem 1.1 *Let M^n be the compact (without boundary) biharmonic submanifold of δ-pinched Riemannian manifold N^{n+p}. Denoted by $S_H = S - \sum_{\alpha>n+1} \operatorname{tr} H_\alpha^2$ the squared norm of the second fundamental form of M^n along the direction of mean curvature vector(notice that $S_H = S$ for $p = 1$).*

(i) If $S_H \geq n$ or $S_H \leq n\delta$, then the mean curvature of M^n is constant.

(ii) If $S_H > n$ or $S_H < n\delta$, then M^n is minimal.

Our second purpose in this paper is to study the biharmonic submanifolds with parallel mean curvature vector in δ-pinched Riemannian manifolds, we obtain the following theorem.

Theorem 1.2 *Let M^n be the biharmonic submanifold with parallel mean curvature vector in δ-pinched Riemannian manifold N^{n+p}. If $S < n\delta - \frac{1}{2}np(1-\delta)$, then M^n is minimal.*

In particular, for $N = S^{n+p}(1)$ the unit sphere, Jiang [2] shows that if $S < n$, then M is minimal, so Theorem 1.2 generalizes Jiang's result. However, our methods are very different from [2]. It is important to note that the condition in Theorem 1.2 can not be improved to $S \leq n\delta - \frac{1}{2}np(1-\delta)$. Since the Clifford hypersurfaces

$$S^k\Big(\sqrt{\frac{1}{2}}\Big) \times S^{n-k}\Big(\sqrt{\frac{1}{2}}\Big), \quad 0 \leq k \leq n,\ k \neq n/2$$

in $S^{n+1}(1)$ are those with parallel second fundamental form and $S = n$, but all of them are not of minimal biharmonic submanifolds.

More generally, for what concerns biharmonic submanifolds with constant mean curvature in δ-pinched Riemannian manifolds, we also obtain an upper bound for its mean curvature using the method developed in the proof of the theorem 1.2.

Theorem 1.3 *Let M^n be the complete biharmonic submanifold with constant mean curvature H in δ-pinched Riemannian manifold N^{n+p}. Then*

$$H^2 \leq 1 + \frac{1}{2}p(1-\delta).$$

In other words, Theorem 1.3 tells us that there is no biharmonic submanifolds in δ-pinched Riemannian manifolds with constant mean curvature $H^2 > 1 + \frac{1}{2}p(1-\delta)$.

Our methods to prove all the Theorems 1.1–1.3 can be applied to more general cases, for example, if we replace the condition $0 < \delta \leq K^N \leq 1$ with $a(x) \leq K^N(x) \leq b(x)$ for smooth functions $a(x)$, $b(x)$ on N^{n+p}, the similar conclusions as in above three theorems also hold. It can be seen from the use of Lemmas 2.1, 2.2 and the estimates of the Laplacian of nH.

2. Preliminaries and lemmas

Let $\phi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds. The tension field of ϕ is given by $\tau(\phi) = \mathrm{tr}\nabla \mathrm{d}\phi$, and, for any compact domain $\Omega \subset M$, the bienergy is defined by $E_2(\phi) = \frac{1}{2}\int_\Omega |\tau(\phi)|^2 v_g$. Then we call biharmonic a smooth map ϕ if it is a critical point of the bienergy functional $E_2(\phi)$ for any compact domain $\Omega \subset M$. The first variation formula for the bienergy, derived in [2], shows that the Euler-Lagrange equation associated to E_2 is given by the vanishing of the bitension field $\tau_2(\phi) = -\triangle\tau(\phi) - \text{trace } R^N(\mathrm{d}\phi, \tau(\phi))\mathrm{d}\phi$.

Let M be a n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold N, and let h and $\overrightarrow{H}$ be the second fundamental form and the mean curvature vector field respectively. Let h^α_{ij}, $i, j = 1, \dots, n$, $\alpha = n+1, \dots, n+p$, be the coefficients of the second fundamental form h with respect to a local field of orthonormal frames $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{n+p}\}$ of N^{n+p} such that, restricted to M^n, the vectors $e_1, \dots, e_n$ are tangent to M^n and the remaining vectors $e_{n+1}, \dots, e_{n+p}$ are normal to M^n. Then the second fundamental form h, its squared length S, and the mean curvature H, of M^n are given respectively by

$$h(e_i, e_j) = \sum_\alpha h^\alpha_{ij} e_\alpha, \quad S = \sum_{i,j,\alpha} (h^\alpha_{ij})^2, \quad H = \frac{1}{n}\sqrt{\sum_\alpha (\mathrm{tr} H_\alpha)^2}. \tag{2.1}$$

Where $h^\alpha_{ij} = \langle A_\alpha e_i, e_j\rangle$ and A_α is the shape operator in the direction e_α, H_α denotes the matrix (h^α_{ij}). In the following, we denote K_{ABCD} the components of Riemannian curvature tensor of N.

If ϕ is an isometric immersion, then the following Lemma 2.1 due to Jiang [2], constitutes a useful tool in determining whether a submanifold of N^{n+p} is of biharmonic type (see also in [5] and [11] the other characterization results for biharmonic submanifolds in space forms).

Lemma 2.1 *Let $\phi : M^n \to N^{n+p}$ be an isometric immersion, then ϕ is biharmonic if and only if the following equations hold:*

$$\begin{cases} \displaystyle\sum_{\beta,j,k} (-2h^\beta_{jjk} h^\beta_{qk} - h^\beta_{jj} h^\beta_{qkk} + h^\beta_{jj} K_{kqk\beta}) = 0, & \forall\ q, \\ \displaystyle\sum_{j,k} h^\alpha_{jjkk} - \sum_{\beta,j,k,l} h^\beta_{jj} h^\alpha_{lk} h^\beta_{lk} + \sum_{\beta,j,k} h^\beta_{jj} K_{k\alpha\, k\beta} = 0, & \forall\ \alpha. \end{cases}$$

In order to prove our main theorems, we need the following proposition[12, pages 92–94]. The equality case in part (i) is proved by Fontenele[13, Proposition 3.4].

Lemma 2.2 *Let N be an $(n+p)$-dimensional Riemannian manifold. If $a \le K^N \le b$ at a point $x \in N$, then, at this point,*

(i) $|K_{ACBC}| \le \frac{1}{2}(b-a)$, for $A \ne B$;

(ii) $|K_{ABCD}| \le \frac{2}{3}(b-a)$, for A, B, C, D distinct with each other.

Equality in (i) implies that $K_{AC} = K_{BC}$, here K_{AC} denotes the sectional curvature $K^N(\pi)$ with $\pi = \mathrm{span}\{e_A, e_C\}$ for locally orthonormal frame $\{e_i\}_{i=1}^{n+p}$ around x.

3. Proofs of the main theorems

Proof of Theorem 1.1 We choose e_{n+1} such that $\vec{H} = He_{n+1}$, then

$$\mathrm{tr} H_{n+1} = nH,$$

$$\mathrm{tr}H_\alpha = 0, \quad \text{for} \quad n+2 \le \alpha \le n+p. \tag{3.1}$$

Since M^n is a biharmonic submanifold, it follows from the second equation in Lemma 2.1 for $\alpha = n+1$ that

$$\sum_{j,k} h^{n+1}_{jjkk} - S_H \sum_j h^{n+1}_{jj} + \sum_{j,k} h^{n+1}_{jj} K_{(n+1)k(n+1)k} = 0. \tag{3.2}$$

Meanwhile, because of (3.1), we also have $nH = \sum_j h^{n+1}_{jj}$ and

$$\sum_j \Big(\sum_k h^{n+1}_{jjkk}\Big) = \sum_j (\Delta\, h^{n+1}_{jj}) = \Delta\Big(\sum_j h^{n+1}_{jj}\Big) = \Delta(nH),$$

putting into (3.2) gives

$$\Delta(nH) - nHS_H + nH\sum_k K_{(n+1)k(n+1)k} = 0. \tag{3.3}$$

In view of (2.1), we know $H \ge 0$ (in fact, the mean curvature is defined to be the norm of the mean curvature vector), then (3.3) implies that

$$\Delta(nH) - nHS_H + n^2 H \ge 0, \tag{3.4}$$

$$\Delta(nH) - nHS_H + n^2\delta\, H \le 0. \tag{3.5}$$

Hence, as $S_H \ge n$ or $S_H \le n\delta$, it follows from (3.4) or (3.5) that $\Delta(nH) \ge 0$ or $\Delta(nH) \le 0$, respectively. Either of cases will leads to $H = \text{const}$ by the maximal principle. At this time, (3.4) and (3.5) become $nH(S_H - n) \le 0$ and $nH(S_H - n\delta) \ge 0$, which will imply that $H \equiv 0$ as $S_H > n$ or $S_H < n\delta$. We complete the proof of Theorem 1.1. □

Proof of Theorem 1.2 Multiplying the second equation in Lemma 2.1 by $\sum_i h^\alpha_{ii}$ and taking sum for α leads to

$$\sum_\alpha (\sum_j h^\alpha_{jj})(\sum_{i,k} h^\alpha_{iikk}) = \sum_{\alpha,\beta} (\sum_i h^\alpha_{ii})(\sum_j h^\beta_{jj})(\sum_{k,l} h^\alpha_{lk} h^\beta_{lk}) - \sum_{\alpha,\beta,k} (\sum_i h^\alpha_{ii})(\sum_j h^\beta_{jj}) K_{k\alpha\, k\beta}. \tag{3.6}$$

Since the mean curvature vector of M^n in N^{n+p} is parallel, then $\sum_j h^\alpha_{jjk} = 0$, $\forall \alpha, k$, and (3.6) becomes

$$\sum_{k,l} [\sum_\beta (\sum_i h^\beta_{ii}) h^\beta_{lk}]^2 - \sum_{\alpha,\beta} \big[(\sum_i h^\alpha_{ii})(\sum_j h^\beta_{jj})(\sum_k K_{\alpha k\beta k})\big] = 0. \tag{3.7}$$

From Cauchy inequality, we get

$$\sum_{k,l} [\sum_\beta (\sum_i h^\beta_{ii}) h^\beta_{lk}]^2 \le \sum_\beta (\sum_i h^\beta_{ii})^2 \sum_{\beta,k,l} (h^\beta_{lk})^2 = n^2 H^2 S. \tag{3.8}$$

Next, we want to estimate the second term on the left-hand side of (3.7). It is easy to see that

$$\begin{aligned} &\sum_{\alpha,\beta} \big[(\sum_i h^\alpha_{ii})(\sum_j h^\beta_{jj})(\sum_k K_{\alpha k\beta k})\big] \\ &= \sum_{\alpha\ne\beta} (\sum_i h^\alpha_{ii})(\sum_j h^\beta_{jj})(\sum_k K_{\alpha k\beta k}) + \sum_{\alpha,k} (\sum_i h^\alpha_{ii})^2 K_{k\alpha k\alpha} \end{aligned}$$

$$\geq \sum_{\alpha\neq\beta}(\sum_i h_{ii}^{\alpha})(\sum_j h_{jj}^{\beta})(\sum_k K_{\alpha k\beta k}) + n^3H^2\delta. \tag{3.9}$$

Set $A = \sum_{\alpha\neq\beta}(\sum_i h_{ii}^{\alpha})(\sum_i h_{jj}^{\beta})(\sum_k K_{\alpha\, k\beta\, k})$, using Cauchy-Schwarz inequality together with lemma 2.2, we have

$$\begin{aligned} A &\geq -\frac{1}{2}n(1-\delta)\sum_{\alpha\neq\beta}(|\sum_i h_{ii}^{\alpha}|)(|\sum_i h_{jj}^{\beta}|) = -\frac{1}{2}n(1-\delta)\sum_{\alpha\neq\beta}(|\sum_i h_{ii}^{\alpha}|)\sum_{\beta\neq\alpha}(|\sum_i h_{jj}^{\beta}|) \\ &\geq -\frac{1}{2}n(1-\delta)\sum_{\alpha}(|\sum_i h_{ii}^{\alpha}|)^2 \geq -\frac{1}{2}n^3p(1-\delta)H^2. \end{aligned} \tag{3.10}$$

So we have from (3.9) and (3.10) that

$$\sum_{\alpha,\beta}(\sum_i h_{ii}^{\alpha})(\sum_j h_{jj}^{\beta})(\sum_k K_{\alpha k\beta k}) \geq n^2H^2(n\delta - \frac{1}{2}np(1-\delta)). \tag{3.11}$$

Substituting (3.8) and (3.11) into (3.7), we finally arrive at

$$n^2H^2[S - (n\delta - \frac{1}{2}np(1-\delta))] \geq 0.$$

Hence, if $S < n\delta - \frac{1}{2}np(1-\delta)$, then $H = 0$, i.e., M^n is minimal, we complete the proof of Theorem 1.2. □

Proof of Theorem 1.3 Multiplying the second equation in Lemma 2.1 by $\sum_i h_{ii}^{\alpha}$ and taking sum for α, we get

$$\begin{aligned} \sum_{\alpha}(\sum_j h_{jj}^{\alpha})(\sum_{i,k} h_{iikk}^{\alpha}) &= \sum_{\alpha,\beta}(\sum_i h_{ii}^{\alpha})(\sum_j h_{jj}^{\beta})(\sum_{k,l} h_{lk}^{\alpha}h_{lk}^{\beta}) - \sum_{\alpha,\beta,k}(\sum_i h_{ii}^{\alpha})(\sum_j h_{jj}^{\beta})K_{k\alpha\, k\beta} \\ &:= B - C, \end{aligned} \tag{3.12}$$

where, B and C denote the first and second term on the right-hand side of (3.12), respectively. In the following, we shall estimate parts B, C and the left-hand side of (3.12). First, it is easy to see that

$$\begin{aligned} \frac{1}{2}\Delta\Big[\sum_{\alpha}\big(\sum_i h_{ii}^{\alpha}\big)^2\Big] &= \sum_{\alpha,k}\big(\sum_i h_{iik}^{\alpha}\big)^2 + \sum_{\alpha}\big(\sum_j h_{jj}^{\alpha}\big)\sum_{k,i} h_{iikk}^{\alpha} \\ &\geq \sum_{\alpha}\big(\sum_j h_{jj}^{\alpha}\big)\sum_{i,k} h_{iikk}^{\alpha}. \end{aligned} \tag{3.13}$$

Second,

$$\begin{aligned} B &= \sum_{k,l}\Big[\sum_{\beta}\big(\sum_j h_{jj}^{\beta}\big)h_{lk}^{\beta}\Big]^2 \geq \sum_{l}\Big[\sum_{\beta}\big(\sum_j h_{jj}^{\beta}\big)h_{ll}^{\beta}\Big]^2 \\ &\geq \frac{1}{n}\Big[\sum_{\beta}\big(\sum_j h_{jj}^{\beta}\big)\big(\sum_l h_{ll}^{\beta}\big)\Big]^2 \geq n^3H^4. \end{aligned} \tag{3.14}$$

Finally, similar to the estimate of part A in the proof of Theorem 1.2, we obtain

$$\begin{aligned} C =& \sum_{\alpha,k}(\sum_i h_{ii}^{\alpha})^2 K_{\alpha\, k\alpha\, k} + \sum_{\alpha,\beta,k,\beta\neq\alpha}(\sum_j h_{jj}^{\beta})(\sum_i h_{ii}^{\alpha})K_{\alpha\, k\beta\, k} \\ \leq& n^3H^2 + \frac{1}{2}n(1-\delta)\sum_{\alpha,\beta,\beta\neq\alpha}(|\sum_j h_{jj}^{\beta}|)(|\sum_i h_{ii}^{\alpha})|) \end{aligned}$$

$$\leq n^3H^2(1+\frac{1}{2}p(1-\delta)). \tag{3.15}$$

Substituting (3.13), (3.14) and (3.15) into (3.12), then we deduce

$$\Delta(n^2H^2) \geq n^3H^2\big(H^2-1-\frac{1}{2}p(1-\delta)\big).$$

Since we assume that the mean curvature H is constant, so we get $H^2 \leq 1+\frac{1}{2}p(1-\delta)$, this completes the proof of Theorem 1.3. □

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