Journal of Mathematical Research & Exposition
Sept., 2010, Vol. 30, No. 5, pp. 897–902
DOI:10.3770/j.issn:1000-341X.2010.05.018
Http://jmre.dlut.edu.cn

A Trace Theorem of Besov Spaces on a d-Set

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Abstract In the paper we give a trace theorem of Besov spaces $B_p^{\alpha,p}(\mathbb{R}^n)$ on a d-set. It is a kind of extension of related results of Jonsson and Wallin, which has important applications on PDE theory.

Keywords Besov spaces; trace; d-set.

Document code A
MR(2000) Subject Classification 46E35; 42B35; 46F05
Chinese Library Classification O174.2

1. Introduction

Trace problem has many important applications in PDE theory [1]. The purpose of the paper is to study the trace theorem of Besov spaces $B_p^{s,q}$ for $p \leq 1$. It seems that in many references, most of them care about the problem when $p \geq 1$ (see [2, 3]). In [4], a trace theorem of Besov spaces for $p \leq 1$ was given on a Lipschitz domain without proof. In this paper, we mainly discuss the trace problem of Besov spaces for $p \leq 1$ on a d-set.

As we know, there are many examples known as d-sets, for example, a Lipschitz domain with a bounded Lipschitz boundary in $\mathbb{R}^n$ with $d = n-1$ or a bounded hyperplane of $\mathbb{R}^n$ with $d < n$. For further examples, we refer to [2] and [5]. The d-set is defined as follows

Definition 1.1 *Let Γ be a closed, non-empty subset in $\mathbb{R}^n$, and $0 < d < n$. Then Γ is called a d-set if there exists a Borel measure μ in $\mathbb{R}^n$ with the following properties:*

1) $\mathrm{supp}\mu = \Gamma$;

2) for any $Q \in \Gamma$ and any r with $0 < r < r_0$, there exist c_1, $c_2 > 0$ such that

$$c_1 r^d \leq \mu(B(Q,r) \cap \Gamma) \leq c_2 r^d,$$

where $B(Q,r)$ denotes the ball with center $Q \in \Gamma$ and radius r, and $r_0 > 0$ denotes the diameter of Γ.

Remark 1.1 Such definition can be found in [5, pp 1-5]. We also use "$A \sim B$" to denote "$c_1 B \leq A \leq c_2 B$" in the following.

Actually the trace theorem of Besov spaces on a d-set has been studied by several people. In [2, p. 103], Jonsson and Wallin proved that if Γ is a d-set, then $B_p^{\alpha,p}(\mathbb{R}^n)|_\Gamma = B_p^{\beta,p}(\Gamma)$ for

Received August 20, 2008; Accepted September 15, 2009
Supported by the Ji'nan University's Young Scholar's Foundation (Grant No. 51208036).
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$1 \le p \le \infty$, $0 < \alpha < 1$ and $0 < \beta = \alpha - \frac{n-d}{p} < 1$. In [5, p. 139], Triebel also gave the trace theorem that if Γ is a d-set, then $B_p^{\frac{n-d}{p},q}(\mathbb{R}^n)|_\Gamma = L^p(\Gamma)$ for $\frac{d}{n} < p < \infty$, $0 < q \le \min(1,p)$. One can consider if we can obtain similar results for larger range to p and α.

The purpose of the paper is to prove

Theorem 1.1 *Let Γ be a d-set in $\mathbb{R}^n$ $(0 < d < n)$, $\frac{d}{n} < p \le \infty$ and $\beta = \alpha - \frac{n-d}{p}$, $\frac{n-d}{p} < \alpha < 1 + \frac{n-d}{p}$. Then*

$$B_p^{\alpha,p}(\mathbb{R}^n)|_\Gamma = B_p^{\beta,p}(\Gamma).$$

Throughout the paper, the letter "C" will denote (possibly different) constants that are independent of the essential variables.

2. The proof of Theorem 1.1

In the section, we prove the main theorem. Firstly we introduce the Besov spaces $B_p^{\alpha,p}(\mathbb{R}^n)$ and $B_p^{\alpha,p}(\Gamma)$. Take $\varphi_0 \in \mathcal{S}$ and $\psi(X) = \varphi_0(X) - 2^{-n}\varphi_0(X/2)$ satisfying

$$\int_{\mathbb{R}^n} \varphi_0(X)\mathrm{d}X = 1 \quad \text{and} \quad \int_{\mathbb{R}^n} X^\alpha \psi(X)\mathrm{d}X = 0, \quad \text{for any} \quad |\alpha| \ge 0. \tag{1}$$

Definition 2.1 *Let $\varphi_0 \in \mathcal{S}$ satisfy (2.1) and $\psi(X) = \varphi_0(X) - 2^{-n}\varphi_0(X/2)$. Then for $s \in \mathbb{R}$ and $0 < p \le \infty$, $0 < q \le \infty$*

$$\begin{aligned} B_p^{s,q}(\mathbb{R}^n) \sim &\{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B_p^{s,q}(\mathbb{R}^n)} \\ &= (\int_0^1 (t^{-\alpha}\|\psi_t * f\|_{L^p(\mathbb{R}^n)})^q \frac{\mathrm{d}t}{t})^{1/q} + \|\varphi_0 * f\|_{L^p(\mathbb{R}^n)} < \infty\}, \end{aligned}$$

where $\psi_t(X) = t^{-n}\psi(X/t)$.

Definition 2.2 *Let Γ be a d-set $(0 < d < n)$. For $\frac{d}{n} < p \le \infty$ and $0 \le s < 1$, the Besov space on Γ is defined as*

$$\begin{aligned} B_p^{s,p}(\Gamma) = &\{f \in L^p(\Gamma) : \|f\|_{L^p(\Gamma)} + \\ &(\int\int_{\{X,Y\in\Gamma:|X-Y|<1\}} \frac{|f(X)-f(Y)|^p}{|X-Y|^{d+sp}} \mathrm{d}\mu(X)\mathrm{d}\mu(Y))^{1/p} < +\infty\}. \end{aligned}$$

Next we give the following lemma [3, p. 58].

Lemma 2.1 *Let $0 < p \le \infty$, $s \in \mathbb{R}$ and $\sigma \in \mathbb{R}$, for any $f \in \mathcal{S}'(\mathbb{R}^n)$, $G_\sigma(f) = \mathcal{F}^{-1}(1 + |X|^2)^{\sigma/2}\mathcal{F}(f)$. Then $\|G_\sigma(f)\|_{B_p^{s-\sigma,p}(\mathbb{R}^n)}$ is equivalent to the quasi-norm of $B_p^{s,p}(\mathbb{R}^n)$.*

The following lemma is easily verified by using the definition of d-set,

Lemma 2.2 *Let Γ be a d-set $(0 < d < n)$. Then for $\epsilon > 0$, $0 < r < 1$, and $p > \frac{d}{n}$*

$$\int_\Gamma \frac{r^\epsilon}{(r+|X|)^{np+\epsilon}}\mathrm{d}\mu(X) \sim r^{np-d}.$$

Denote by Tr the trace operator, initially defined on $\mathcal{S}(\mathbb{R}^n)$ as the restriction to Γ, since $\mathcal{S}(\mathbb{R}^n)$ is dense in $B_p^{s,p}(\mathbb{R}^n)$. We also use $\mathcal{E}$ to denote the extension operator that extends functions from Γ to $\mathbb{R}^n$.

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1 For the theorem about $p \geq 1$, readers can refer to Theorem 1 in Chapter V in [2]. We only need to deal with the case $p < 1$.

Step 1. Firstly we should prove that

$$B_p^{\beta,p}(\Gamma) \subset B_p^{\alpha,p}(\mathbb{R}^n)|_\Gamma.$$

Since $B_p^{\beta,p}(\Gamma) \subset L^p(\Gamma)$, this is verified if

$$L^p(\Gamma) \subset B_p^{\alpha,p}(\mathbb{R}^n)|_\Gamma.$$

That means we need to prove for $f \in L^p(\Gamma)$, there exists $\mathcal{E}$ such that

$$\|\mathcal{E}(f)\|_{B_p^{\alpha,p}(\mathbb{R}^n)} \leq C\|f\|_{L^p(\Gamma)}.$$

In fact, such extension operator $\mathcal{E}$ has been established in [5, p. 140]. We only need some little modifications of the corresponding part in [5] to fit our case. Thus we omit its proof here.

Step 2. Next we prove that

$$B_p^{\alpha,p}(\mathbb{R}^n)|_\Gamma \subset B_p^{\beta,p}(\Gamma).$$

With Lemma 2.1, it suffices to prove for $g \in \mathcal{S}'(\mathbb{R}^n)$

$$\|Tr(G_{-\alpha}(g))(\cdot)\|_{B_p^{\beta,p}(\Gamma)} \leq C\|g(\cdot)\|_{B_p^{0,p}(\mathbb{R}^n)}.$$

Set $S(\cdot) \in \mathcal{S}(\mathbb{R}^n)$ satisfying

$$\int S(X)\mathrm{d}X \neq 0 \quad \text{and} \quad \hat{S}(\xi) \in \mathcal{D}(\mathbb{R}^n),$$

and supp$\hat{S}(\xi) \subset B(0,1)$. Set $S_k(X) = 2^{kn}S(2^kX)$, $D_k(X) = S_k - S_{k-1}$ and $D_0 = S$. We also assume that

$$\int D_k(X)X^\beta \mathrm{d}X = 0(|\beta| \geq 0).$$

Set

$$\tilde{G}_{-\alpha} = \sum_{k=0}^{\infty} 2^{-k\alpha} D_k.$$

Using the Fourier analysis, we can easily obtain that when $|\xi| < 1$, $\hat{\tilde{G}}_{-\alpha}(\xi) \sim 1$; when $|\xi| \geq 1$, $\hat{\tilde{G}}_{-\alpha}(\xi) \sim |\xi|^{-\alpha}$. Thus we can substitute the standard Bessel potential operator $G_{-\alpha}$ with $\tilde{G}_{-\alpha}$ in the process of proof. Next we use the so-called φ-transform to denote distribution function [6], for $g \in \mathcal{S}'\backslash\mathcal{P}(\mathbb{R}^n)$,

$$g(\cdot) = \sum_{l(Q)=1} \langle g, \phi_Q\rangle \tilde{\phi}_Q + \sum_{m=1}^{\infty} \sum_{l(Q)=2^{-m}} \langle g, \psi_Q\rangle \tilde{\psi}_Q,$$

where

$$Q = Q_{k,\mu} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \mu_i \leq 2^m x_i < \mu_i + 1, \mu \in \mathbb{Z}^n, i = 1, \ldots, n\},$$

and $\phi_Q, \tilde{\phi}_Q, \psi_Q, \tilde{\psi}_Q \in \mathcal{S}(\mathbb{R}^n)$ satisfy

$$|\partial^\gamma \phi_Q|, |\partial^\gamma \tilde{\phi}_Q|, |\partial^\gamma \psi_Q|, |\partial^\gamma \tilde{\psi}_Q| \leq C_{\gamma,L} \frac{|Q|^{-\frac{1}{2}-\frac{\gamma}{n}}}{(1 + l(Q)^{-1}|X - X_Q|)^{L+\gamma}},$$

where X_Q is the center of cube Q, $L, |\gamma| > 0$ and $\psi_Q, \tilde{\psi}_Q$ have vanishing moments up to some needed order, while $\phi_Q, \tilde{\phi}_Q$ may not have vanishing conditions. Now we need to deal with the term $\|\tilde{G}_{-\alpha}(g)\|_{L^p(\Gamma)}$.

$$\|\tilde{G}_{-\alpha}(g)\|^p_{L^p(\Gamma)} = \int_\Gamma |\sum_{k=0}^\infty 2^{-k\alpha} D_k * g(X)|^p \mathrm{d}\mu(X)$$
$$\leq \int_\Gamma |\sum_{k=0}^\infty 2^{-k\alpha} D_k * \{\sum_{l(Q)=1} \langle g, \phi_Q\rangle \tilde{\phi}_Q + \sum_{m=1}^\infty \sum_{l(Q)=2^{-m}} \langle g, \psi_Q\rangle \tilde{\psi}_Q\}|^p \mathrm{d}\mu(X).$$

Assume that $g_1 = \sum_{l(Q)=1}\langle g, \phi_Q\rangle\tilde{\phi}_Q$ and $g_2 = \sum_{m=1}^\infty \sum_{l(Q)=2^{-m}} \langle g, \psi_Q\rangle\tilde{\psi}_Q$. Notice that for some $\epsilon > 0$ and $a \wedge b = \min\{a, b\}$, $l(Q) \sim 2^{-m}$,

$$|D_0 * \tilde{\phi}_Q(\cdot)| \leq C\frac{1}{(1+|X-X_Q|)^{n+\epsilon}}; \tag{2}$$
$$|D_0 * \tilde{\psi}_Q(\cdot)| \leq C|Q|^{1/2-\epsilon/n}\frac{1}{(1+|X-X_Q|)^{n+\epsilon}}; \tag{3}$$
$$|D_k * \tilde{\phi}_Q(\cdot)| \leq C2^{k\epsilon}\frac{1}{(1+|X-X_Q|)^{n+\epsilon}}, \text{ for } \ k>0; \tag{4}$$
$$|D_k * \tilde{\psi}_Q(\cdot)| \leq C|Q|^{1/2}2^{-|k-m|\epsilon}\frac{2^{-(k\wedge m)\epsilon}}{(2^{-(k\wedge m)}+|X-X_Q|)^{n+\epsilon}}, \text{ for } \ k>0. \tag{5}$$

Let us deal with the term $\int_\Gamma |\sum_{k>0} 2^{-k\alpha} D_k * g_2|^p \mathrm{d}\mu(X)$. The other terms are similar and easy. By using $(a+b)^p \leq a^p + b^p (p \leq 1)$ and (2.4), we have

$$\int_\Gamma |\sum_{k>0} 2^{-k\alpha} D_k * g_2|^p \mathrm{d}\mu(X)$$
$$\leq C\int_\Gamma \sum_{k>0}\sum_{m=1}^\infty \sum_{l(Q)=2^{-m}} 2^{-kp\alpha}2^{-mnp/2}2^{-|k-m|p\epsilon}\frac{2^{-(k\wedge m)p\epsilon}}{(2^{-(k\wedge m)}+|X-X_Q|)^{(n+\epsilon)p}}|\langle g, \psi_Q\rangle|^p \mathrm{d}\mu(X)$$
$$\leq C\sum_{k>0}\sum_{m=1}^\infty \sum_{l(Q)=2^{-m}} 2^{-kp\alpha}2^{(k\wedge m)(np-d)}2^{-mnp/2}2^{-|k-m|p\epsilon}|\langle g, \psi_Q\rangle|^p$$
$$\leq C(\sum_{k>0}\sum_{m>k}^\infty \sum_{l(Q)=2^{-m}} 2^{-kp(\alpha-(n-d/p)-\epsilon)}2^{-mn(p(1+\epsilon/n)-1)}2^{mnp(1/2-1/p)}|\langle g, \psi_Q\rangle|^p +$$
$$\sum_{k>0}\sum_{m\leq k}\sum_{l(Q)=2^{-m}} 2^{m(n-d)-kp\alpha}2^{(m-k)p\epsilon}2^{mnp(1/2-1/p)}|\langle g, \psi_Q\rangle|^p).$$

Note that $\alpha > \frac{n-d}{p}$ and $p > \frac{d}{n}$ and take $\epsilon > \frac{n}{p} - n$. Then we have

$$\int_\Gamma |\sum_{k>0} 2^{-k\alpha} D_k * g_2|^p \mathrm{d}\mu(X) \leq C\sum_{m=1}^\infty \sum_{l(Q)=2^{-m}} 2^{mnp(1/2-1/p)}|\langle g, \psi_Q\rangle|^p.$$

Here we have just used Theorem 7.1 in [7]. It follows $\|\tilde{G}_{-\alpha}(g)\|^p_{L^p(\Gamma)} \leq C\|g\|_{B^{0,p}_p(\mathbb{R}^n)}$. Finally we prove

$$(\int\int_{X,Y\in\Gamma} \frac{|\tilde{G}_{-\alpha}(g)(X) - \tilde{G}_{-\alpha}(g)(Y)|^p}{|X-Y|^{d+\beta p}} \mathrm{d}\mu(X)\mathrm{d}\mu(Y))^{1/p} \leq C\|g\|_{B^{0,p}_p(\mathbb{R}^n)}.$$

We have

$$
\begin{aligned}
&\int\int_{X,Y\in\Gamma}\frac{|\tilde{G}_{-\alpha}(g)(X)-\tilde{G}_{-\alpha}(g)(Y)|^p}{|X-Y|^{d+\beta p}}\mathrm{d}\mu(X)\mathrm{d}\mu(Y)\\
&\le\int\int_{X,Y\in\Gamma}\sum_{k=0}^{\infty}2^{-kp\alpha}\frac{|D_k(g)(X)-D_k(g)(Y)|^p}{|X-Y|^{d+\beta p}}\mathrm{d}\mu(X)\mathrm{d}\mu(Y)\\
&\le\sum_{k=0}^{\infty}2^{-kp\alpha}\int\int_{X,Y\in\Gamma}\frac{|\int_{\mathbb{R}^n}(D_k(X,Z)-D_k(Y,Z))g(Z)dZ|^p}{|X-Y|^{d+\beta p}}\mathrm{d}\mu(X)\mathrm{d}\mu(Y)\\
&\le\sum_{k=0}^{\infty}2^{-kp\alpha}\int\int_{\{X,Y\in\Gamma:|X-Y|\ge\frac{1}{2}2^{-k}\}}+\int\int_{\{X,Y\in\Gamma:|X-Y|<\frac{1}{2}2^{-k}\}}\cdots.
\end{aligned}
$$

Note that for the first term,

$$
\int_{\{X\in\Gamma:|X-Y|\ge\frac{1}{2}2^{-k}\}}\frac{1}{|X-Y|^{d+\beta p}}\mathrm{d}\mu(X)=\int_{\{Y\in\Gamma:|X-Y|\ge\frac{1}{2}2^{-k}\}}\frac{1}{|X-Y|^{d+\beta p}}\mathrm{d}\mu(Y)\sim C2^{k\beta p},
$$

then the following step is almost the same as with the term $\|\tilde{G}_{-\alpha}(g)\|_{L^p(\Gamma)}$. Now we deal with the second term. Notice that

$$
|D_k(X,Z)-D_k(Y,Z)|\le C\frac{|X-Y|^{\epsilon}}{(2^{-k}+|X-Z|)^{\epsilon}}\Big(\frac{2^{-k\epsilon}}{(2^{-k}+|X-Z|)^{n+\epsilon}}+\frac{2^{-k\epsilon}}{(2^{-k}+|Y-Z|)^{n+\epsilon}}\Big),
$$

$$
\text{for }\ |X-Y|\le\frac{1}{2}(2^{-k}+|X-Z|).
$$

Take $\beta p<\epsilon=\beta p+\epsilon_0<1$, then we have

$$
\begin{aligned}
&\int_{\{X\in\Gamma:|X-Y|<\frac{1}{2}2^{-k}\}}\frac{1}{|X-Y|^{d-\epsilon_0}}2^{k\beta p+k\epsilon_0}\mathrm{d}\mu(X)\\
&=\int_{\{Y\in\Gamma:|X-Y|<\frac{1}{2}2^{-k}\}}\frac{1}{|X-Y|^{d-\epsilon_0}}2^{k\beta p+k\epsilon_0}\mathrm{d}\mu(Y)\sim C2^{k\beta p}.
\end{aligned}
$$

Then we can obtain

$$
\int\int_{X,Y\in\Gamma}\frac{|\tilde{G}_{-\alpha}(g)(X)-\tilde{G}_{-\alpha}(g)(Y)|^p}{|X-Y|^{d+\beta p}}\mathrm{d}\mu(X)\mathrm{d}\mu(Y)\le C\|g\|_{B_p^{0,p}(\mathbb{R}^n)}.
$$

Thus we have proved $B_p^{\alpha,p}(\mathbb{R}^n)|_\Gamma\subset B_p^{\beta,p}(\Gamma)$. The proof is completed. □

Remark 2.1 Notice that in our setting in the main theorem, Γ is a d-set with radius $r_0>0$. In fact from the proof of our main theorem, one finds that r_0 can be infinite. In this case we need the moment conditions on the kernel of D_k $(k>0)$. That means $p>d/n$ which connects with the moment conditions of the kernel D_k. If $r_0<\infty$, Γ is a compact set when $0<d<n$, which means Γ is porous (see related definition in [9] or [10]). In that case, related results can be found in [8] where similar trace theorem holds in the case $0<p<\infty$. Hence this is the difference between our results and the works of Triebel [8–10].

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