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On Generalized E-Monotonicity in Banach Spaces

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Abstract Several new kinds of generalized E-preinvexity and generalized invariant E-monotonicity are introduced in the setting of Banach spaces. The relations between E-preinvexity, E-prequasiinvexity, (pseudo, quasi) E-invexity and invariant (pseudo, quasi) E-monotonicity are studied, which can be viewed as an extension of some known results.

Keywords E-preinvexity; E-invexity; invariant E-monotonicity; relations.

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1. Introduction

Convexity is a common assumption made in mathematical programming. There have been increasing attempts to weaken the convexity of objective functions, see for example [1–9] and references therein. An interesting generalization for convexity is E-convexity, which was introduced and studied by Youness [1–3] and Yang [4]. They studied characterizations of efficient solutions and optimality criteria for a class of E-convex programming problems. Later, E-quasiconvexity was introduced in [5] and some basic properties for E-convex and E-quasiconvex functions were developed there. Very recently, Fulga and Preda [6] introduced E-invex sets and E-preinvex and E-prequasiinvex functions as an extension of the E-convex sets and E-convex and E-quasiconvex functions, respectively.

A concept related to the convexity is the monotonicity of mappings. In 1990, Karamardian and Schaible [7] studied the relations between the convexity of a real-valued function and the monotonicity of its gradient mapping. Yang et al. [8] investigated the relations between invexity and generalized invariant monotonicity in R^n. In this paper, we will introduce several new notions of generalized invexity and generalized invariant monotonicity, which are called E-quasiinvexity, E-pseudoinvexity and invariant pseudo (quasi) E-monotonicity, and study their relations in Banach spaces. The results here can be viewed as an extension and improvement of corresponding results in [7, 8].

2. Invariant E-monotonicity

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Throughout this paper, let X be a real Banach space and M a nonempty subset of X. Let $\eta : X \times X \to X$ and $E : X \to X$ be single-valued mappings and $f : X \to R$ a function.

The set M is said to be invex with respect to η (see [9]) if for any $x, y \in M$ and any $\lambda \in [0,1]$ one has $x + \lambda\eta(y, x) \in M$, and E-invex with respect to η (see [6]) if for any $x, y \in M$ and any $\lambda \in [0,1]$ one has $E(x) + \lambda\eta(E(y), E(x)) \in M$. For the sake of brevity, $E(x)$ will be written as Ex for any $x \in M$.

The function f is said to be Gâteaux differentiable at $x_0 \in M$ if there exists a linear function $f'(x_0) : X \to R$ such that

$$\langle f'(x_0), v\rangle = \lim_{t\downarrow 0} \frac{f(x_0 + tv) - f(x_0)}{t}, \quad \forall v \in X,$$

where $f'(x_0)$ is called the Gâteaux derivative of f at x_0. The function f is said to be Gâteaux differentiable on M if it is Gâteaux differentiable at every x in M.

Definition 2.1 *Let M be an E-invex set with respect to η and $f : X \to R$ be a Gâteaux differentiable function on M. The function f is said to be*

(i) ([6]) E-preinvex on M with respect to η if

$$f(Ey + \lambda\eta(Ex, Ey)) \le \lambda f(Ex) + (1 - \lambda) f(Ey), \quad \forall x, y \in M, \forall \lambda \in [0,1];$$

(ii) E-invex on M with respect to η if

$$\langle f'(Ey), \eta(Ex, Ey)\rangle \le f(Ex) - f(Ey), \quad \forall x, y \in M.$$

Definition 2.2 *Let M be an E-invex set with respect to η and $f : X \to R$ be a Gâteaux differentiable function on M. The operator f' is said to be invariant E-monotone on M with respect to η if*

$$\langle f'(Ey), \eta(Ex, Ey)\rangle + \langle f'(Ex), \eta(Ey, Ex)\rangle \le 0, \quad \forall x, y \in M.$$

When $E = I$, the identity mapping, Definitions 2.1 and 2.2 reduce to the concepts of preinvexity, invexity and invariant monotonicity in [8], respectively.

Motivated by the previous works on this issue, in this paper, we will study the closed relations between E-preinvexity, E-invexity and invariant E-monotonicity. For this end, we first recall the following two assumptions, which are taken from [8, 9] and used in many papers. Let K be a nonempty subset of X.

Assumption A$'$ *Let the set K be invex with respect to η and $f : K \to R$ be a function. Assume that $f(y + \eta(x, y)) \le f(x)$ for all $x, y \in K$.*

Assumption C$'$ *Let $\eta : X \times X \to X$ be a mapping. Assume that for any $x, y \in K$ and any $\lambda \in [0,1]$ one has*

$$\eta(y, y + \lambda\eta(x, y)) = -\lambda\eta(x, y) \text{ and } \eta(x, y + \lambda\eta(x, y)) = (1 - \lambda)\eta(x, y).$$

Yang et al. [8] showed that if η satisfies Assumption C$'$, then

$$\eta(y + \lambda_1\eta(x, y), y + \lambda_2\eta(x, y)) = (\lambda_1 - \lambda_2)\eta(x, y), \ \forall x, y \in K, \lambda_1, \lambda_2 \in [0,1]. \tag{1}$$

The following example shows that the converse implication is not necessarily true, that is, the equality (1) is a proper generalization of Assumption C′.

Example 2.1 Let $K = [-2, 2]$ and

$$\eta(x,y) = \begin{cases} x - y, & xy \geq 0, \\ -\frac{y}{2}, & x > 0,\ y < 0, \\ 2 - y, & x < 0,\ y > 0. \end{cases}$$

We can verify that η satisfies the equality (1). But Assumption C′ does not hold for $x > 0$, $y < 0$ and $\lambda \in (0, 1]$.

Now, we introduce two similar assumptions, which are used in the sequel.

Assumption A *Let $f : X \to R$ be a function and $\eta : X \times X \to X$ and $E : X \to X$ be single-valued mappings. Assume that $f(Ey + \eta(Ex, Ey)) \leq f(Ex)$, $\forall x, y \in M$.*

Assumption C *Let $\eta : X \times X \to X$ and $E : X \to X$ be two single-valued mappings. Assume that for any $x, y \in M$ and any $\lambda_1, \lambda_2 \in [0, 1]$ one has*

$$\eta(Ey + \lambda_1 \eta(Ex, Ey), Ey + \lambda_2 \eta(Ex, Ey)) = (\lambda_1 - \lambda_2)\eta(Ex, Ey).$$

Theorem 2.1 *Let M be an E-invex set with respect to η, $E(M)$ an invex set with respect to η and $f : X \to R$ Gâteaux differential on M. Then*

(i) E-preinvexity of f implies E-invexity of f on M with respect to η;

(ii) E-invexity of f implies invariant E-monotonicity of f' on M with respect to η;

(iii) If Assumptions A and C are both satisfied, then invariant E-monotonicity of f' implies E-preinvexity of f on M with respect to η.

Proof (a) Let f be E-preinvex on M. Then for any $x, y \in M$ and any $\lambda \in (0, 1]$ one has

$$\frac{f(Ey + \lambda\eta(Ex, Ey)) - f(Ey)}{\lambda} \leq f(Ex) - f(Ey).$$

Taking the limit for the above inequality as $\lambda \downarrow 0$, we get

$$\langle f'(Ey), \eta(Ex, Ey)\rangle \leq f(Ex) - f(Ey), \tag{2}$$

which shows that f is E-invex on M.

(b) Let f be E-invex on M. Then the inequality (2) holds for any $x, y \in M$ and then

$$\langle f'(Ey), \eta(Ex, Ey)\rangle + \langle f'(Ex), \eta(Ey, Ex)\rangle \leq 0,$$

which indicates that f' is invariant E-monotone on M.

(c) Let f' be invariant E-monotone on M and Assumptions A and C hold. Assume to the contrary that f is not E-preinvex on M. Then there exist $x^0, y^0 \in M$ and $\lambda^0 \in (0, 1)$ such that $f(Ez^0) > \lambda^0 f(Ex^0) + (1 - \lambda^0) f(Ey^0)$, where $Ez^0 = Ey^0 + \lambda^0\eta(Ex^0, Ey^0)$. By Assumption A, we have $f(Ez^0) > \lambda^0 f(Ey^0 + \eta(Ex^0, Ey^0)) + (1 - \lambda^0) f(Ey^0)$, that is,

$$\lambda^0(f(Ez^0) - f(Ey^0 + \eta(Ex^0, Ey^0))) + (1 - \lambda^0)(f(Ez^0) - f(Ey^0)) > 0.$$

By the mean-value theorem, we get

$$\langle f'(Ez^1), \lambda^0(\lambda^0 - 1)\eta(Ex^0, Ey^0)\rangle + \langle f'(Ez^2), (1-\lambda^0)\lambda^0\eta(Ex^0, Ey^0)\rangle > 0, \qquad (3)$$

where $Ez^1 = Ey^0 + \lambda^1\eta(Ex^0, Ey^0), Ez^2 = Ey^0 + \lambda^2\eta(Ex^0, Ey^0)$ and $0 < \lambda^2 < \lambda^0 < \lambda^1 < 1$. By Assumption C, it follows from (3) that

$$\langle f'(Ez^1), \eta(Ez^2, Ez^1)\rangle + \langle f'(Ez^2), \eta(Ez^1, Ez^2)\rangle > 0,$$

which contradicts the invariant E-monotonicity of f'. Therefore, the assertion (iii) holds. □

3. Generalized E-invexity and generalized E-monotonicity

In this section, we will introduce several new kinds of generalized E-invexity and generalized E-monotonicity, and establish the relationships between generalized E-invexity of the function f and generalized E-monotonicity of its Gâteaux differential f'.

Definition 3.1 *Let M be an E-invex set with respect to η. A Gâteaux differential function $f : X \to R$ is said to be*

(i) ([6]) E-prequasiinvex on M with respect to η if

$$f(Ey + \lambda\eta(Ex, Ey)) \le \max\{f(Ex), f(Ey)\}, \quad \forall x, y \in M, \forall\lambda \in [0, 1];$$

(ii) E-quasiinvex on M with respect to η if

$$f(Ex) \le f(Ey) \Rightarrow \langle f'(Ey), \eta(Ex, Ey)\rangle \le 0, \quad \forall x, y \in M;$$

(iii) E-pseudoinvex on M with respect to η if

$$\langle f'(Ey), \eta(Ex, Ey)\rangle \ge 0 \Rightarrow f(Ex) \ge f(Ey), \quad \forall x, y \in M.$$

Definition 3.2 *Let M be an E-invex set with respect to η and $f : X \to R$ a Gâteaux differential function. The operator f' is said to be*

(i) invariant quasi E-monotone on M with respect to η if

$$\langle f'(Ey), \eta(Ex, Ey)\rangle > 0 \Rightarrow \langle f'(Ex), \eta(Ey, Ex)\rangle \le 0, \quad \forall x, y \in M;$$

(ii) invariant pseudo E-monotone on M with respect to η if

$$\langle f'(Ey), \eta(Ex, Ey)\rangle \ge 0 \Rightarrow \langle f'(Ex), \eta(Ey, Ex)\rangle \le 0, \quad \forall x, y \in M.$$

In the following, we study the relations between E-prequasiinvexity, (pseudo) quasi E-invexity and invariant (pseudo) quasi E-monotonicity.

Theorem 3.1 *Let M be an E-invex set with respect to η, $E(M)$ an invex set with respect to η and $f : X \to R$ a Gâteaux differential function. Then*

(i) E-prequasiinvexity of f implies E-quasiinvexity of f on M with respect to η;

(ii) E-quasiinvexity of f implies invariant quasi E-monotonicity of f' on M with respect to η;

(iii) If Assumptions A and C are satisfied, then invariant quasi E-monotonicity of f' implies E-prequasiinvexity of f on M with respect to η.

Proof (a) Let f be E-prequasiinvex on M. For any $x, y \in M$ and any $\lambda \in (0,1]$, assume without loss of generality that $f(Ex) \leq f(Ey)$. Then $f(Ey + \lambda\eta(Ex, Ey)) \leq f(Ey)$, that is, $\frac{f(Ey+\lambda\eta(Ex,Ey))-f(Ey)}{\lambda} \leq 0$. Taking the limit for the last inequality as $\lambda \downarrow 0$, we get $\langle f'(Ey), \eta(Ex, Ey)\rangle \leq 0$, which indicates that f is E-quasiinvex on M.

(b) Let f be E-quasiinvex on M. Assume to the contrary that f' is not invariant quasi E-monotone on M. Then there exist $x^0, y^0 \in M$ such that

$$\langle f'(Ey^0), \eta(Ex^0, Ey^0)\rangle > 0 \Rightarrow \langle f'(Ex^0), \eta(Ey^0, Ex^0)\rangle > 0.$$

By the definition of E-quasiinvexity, we get $f(Ex^0) > f(Ey^0)$ and $f(Ey^0) > f(Ex^0)$, which is a contradiction. Therefore, the assertion (ii) holds.

(c) Let f' be invariant quasi E-monotone on M and Assumptions A and C hold. Assume to the contrary that f is not E-prequasiinvex. Then there exist $x^0, y^0 \in M$ and $\lambda^0 \in (0,1)$ such that

$$f(Ey^0 + \lambda^0\eta(Ex^0, Ey^0)) > \max\{f(Ex^0), f(Ey^0)\}.$$

By the mean-value theorem and Assumption A, there exist $\lambda^1 \in (0, \lambda^0)$ and $\lambda^2 \in (\lambda^0, 1)$ such that

$$0 < f(Ez^0) - f(Ey^0) = \langle f'(Ez^1), \lambda^0\eta(Ex^0, Ey^0)\rangle$$

and

$$0 < f(Ez^0) - f(Ey^0 + \eta(Ex^0, Ey^0)) = \langle f'(Ez^2), (\lambda^0 - 1)\eta(Ex^0, Ey^0)\rangle,$$

where $Ez^0 = Ey^0 + \lambda^0\eta(Ex^0, Ey^0)$, $Ez^1 = Ey^0 + \lambda^1\eta(Ex^0, Ey^0)$ and $Ez^2 = Ey^0 + \lambda^2\eta(Ex^0, Ey^0)$. From Assumption C, it follows that $\langle f'(Ez^1), \eta(Ez^2, Ez^1)\rangle > 0$ and $\langle f'(Ez^2), \eta(Ez^1, Ez^2)\rangle > 0$ which is a contradiction to the invariant quasi E-monotonicity of f'. So the assertion (iii) holds. □

Theorem 3.2 *Let M be an E-invex set with respect to η, $E(M)$ an invex set with respect to η and $f : X \to R$ a Gâteaux differential function. If Assumption A holds and f is E-pseudoinvex on M with respect to η, then f is E-prequasiinvex on M with respect to η.*

Proof Assume to the contrary that f is not E-prequasiinvex with respect to η. Then there exist $x^0, y^0 \in M$ and $\lambda^0 \in (0,1)$ such that

$$f(Ey^0 + \lambda^0\eta(Ex^0, Ey^0)) > \max\{f(Ex^0), f(Ey^0)\}.$$

Suppose without loss of generality that $f(Ex^0) \leq f(Ey^0)$. By Assumption A, we have

$$f(Ey^0 + \lambda^0\eta(Ex^0, Ey^0)) > f(Ey^0) \geq f(Ex^0) \geq f(Ey^0 + \eta(Ex^0, Ey^0)),$$

which indicates that there exists $\overline{\lambda} \in (0,1)$ such that

$$f(E\overline{y}) = \max_{\lambda\in[0,1]} f(Ey^0 + \lambda\eta(Ex^0, Ey^0)) > f(Ex^0),$$

where $E\overline{y} = Ey^0 + \overline{\lambda}\eta(Ex^0, Ey^0)$. Consequently, $f'(E\overline{y}) = 0$ and then $\langle f'(E\overline{y}), \eta(Ex^0, E\overline{y})\rangle = 0$. It follows from the pseudoinvexity of f that $f(Ex^0) \geq f(E\overline{y})$, which is a contradiction. Thus, the assertion of the theorem is true.

Theorem 3.3 *Let M be an E-invex set with respect to η, $E(M)$ an invex set with respect* *to η and $f : X \to R$ a Gâteaux differential function. If Assumptions A and C hold, then f is E-pseudoinvex on M with respect to η if and only if f' is invariant pseudo E-monotone on M with respect to η.*

Proof Let f be E-pseudoinvex on M with respect to η. Assume to the contrary that f' is not invariant pseudo E-monotone on M with respect to η. Then there exist $x^0, y^0 \in M$ such that

$$\langle f'(Ey^0), \eta(Ex^0, Ey^0)\rangle \geq 0 \Rightarrow \langle f'(Ex^0), \eta(Ey^0, Ex^0)\rangle > 0. \tag{4}$$

By the E-pseudoinvexity of f, we have

$$\langle f'(Ey^0), \eta(Ex^0, Ey^0)\rangle \geq 0 \Rightarrow f(Ex^0) \geq f(Ey^0). \tag{5}$$

According to Theorems 3.2 and 3.1(i), it follows from (5) that $\langle f'(Ex^0), \eta(Ey^0, Ex^0)\rangle \leq 0$, which contradicts the implication (4).

Conversely, let f' be invariant pseudo E-monotone on M with respect to η. For any $x, y \in M$ with

$$\langle f'(Ey), \eta(Ex, Ey)\rangle \geq 0, \tag{6}$$

we want to show that $f(Ex) \geq f(Ey)$. Assume to the contrary that $f(Ex) < f(Ey)$. By the mean-value theorem, there exists $\lambda^0 \in (0, 1)$ such that

$$f(Ey + \eta(Ex, Ey)) - f(Ey) = \langle f'(Ey + \lambda^0\eta(Ex, Ey)), \eta(Ex, Ey)\rangle.$$

By Assumptions A and C, it follows that

$$\langle f'(Ey + \lambda^0\eta(Ex, Ey)), \eta(Ey, Ey + \lambda^0\eta(Ex, Ey))\rangle > 0.$$

By the invariant pseudo E-monotonicity of f', we can deduce that $\langle f'(Ey), \eta(Ex, Ey)\rangle < 0$, which contradicts the inequality (6). Hence, f is E-pseudoinvex on M with respect to η. $\square$

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