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The Applications of Perturbations on Accretive Mappings to Nonlinear Elliptic Systems Related to Generalized (p, q)-Laplacian

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Abstract Using perturbation results on the sums of ranges of nonlinear accretive mappings of Calvert and Gupta, we present some abstract results for the existence of the solutions of nonlinear Neumann elliptic systems which is related to the so-called generalized (p, q)-Laplacian in this paper. The systems discussed in this paper and the method used extend and complement some of the previous work.

Keywords accretive mapping; generalized (p, q)-Laplacian; nonlinear elliptic systems.

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1. Introduction

Since p-Laplacian operator $-\Delta_p$ occurs in a variety of physical phenomena, many mathematicians do researches from different angles on it. Some significant work has been done by us too [1–6].

Later, in 2005, we extend the elliptic boundary value problem involving the p-Laplacian operator we studied before to the following general form:

$$\begin{aligned} &-\mathrm{div}(\alpha(\mathrm{grad}u)) + |u|^{p-2}u + g(x, u(x)) = f(x), && \text{a.e. in} \quad \Omega \\ &-\langle \vartheta, \alpha(\mathrm{grad}u)\rangle \in \beta_x(u(x)), && \text{a.e. on} \quad \Gamma \end{aligned} \tag{1.1}$$

where $\alpha : R^N \to R^N$ is a given monotone function, and there exist positive constants k_1, k_2 and k_3 such that for $\forall \xi,\ \xi' \in R^N$, the following conditions are satisfied:

(i) $|\alpha(\xi)| \le k_1|\xi|^{p-1}$;

(ii) $|\alpha(\xi) - \alpha(\xi')| \le k_2||\xi|^{p-2}\xi - |\xi'|^{p-2}\xi'|$;

(iii) $\langle \alpha(\xi), \xi\rangle \ge k_3|\xi|^p$.

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We note that if $\alpha(\xi) = |\xi|^{p-2}\xi$, for $\forall \xi \in R^N$, then (1.1) is reduced to the case involving the p-Laplacian operator. We proved in [7] that (1.1) had a solution in $L^2(\Omega)$, where $\frac{2N}{N+1} < p < +\infty$ and $N \geq 1$. And, in [8], we showed that (1.1) had a solution in $L^p(\Omega)$, where $2 \leq p < +\infty$.

Can (1.1) be extended to the case of nonlinear elliptic systems? In this paper, we will find a sufficient condition for the existence of solution in $L^p(\Omega) \times L^q(\Omega)$ of the following system by using perturbations of accretive mappings:

$$\begin{aligned} &-\mathrm{div}(\alpha_1(\mathrm{grad} u)) + \varepsilon_1 |u|^{p-2}u + g(x, u(x), v(x)) = f_1(x), && \text{a.e. in } \Omega \\ &-\mathrm{div}(\alpha_2(\mathrm{grad} v)) + \varepsilon_2 |v|^{q-2}v + g(x, v(x), u(x)) = f_2(x), && \text{a.e. in } \Omega \\ &-\langle \vartheta, \alpha_1(\mathrm{grad}(u)) \rangle \in \beta_x(u(x)), && \text{a.e. on } \Gamma \\ &-\langle \vartheta, \alpha_2(\mathrm{grad}(v)) \rangle \in \beta_x(v(x)), && \text{a.e. on } \Gamma \end{aligned} \tag{1.2}$$

Necessary details of (1.2) will be provided in Section 3.

2. Preliminaries

2.1 Perturbations for m-accretive mappings

Let X be a real Banach space with a strictly convex dual space X'. We shall use " $\rightarrow$ " and "$w - \lim$" to denote strong and weak convergences, respectively. For any subset G of X, we denote by $\mathrm{int}\, G$ its interior and $\overline{G}$ its closure, respectively. A mapping $T : X \rightarrow X'$ is said to be hemi-continuous on X if $w - \lim_{t\rightarrow 0} T(x + ty) = Tx$ for any $x, y \in X$.

Let J denote the normalized duality mapping from X into $2^{X'}$ defined by

$$J(x) = \{f \in X' : (x, f) = \|x\| \cdot \|f\|,\ \|f\| = \|x\|\}, \quad \forall x \in X$$

where $(\cdot, \cdot)$ denotes the generalized duality pairing between X and X'. Since X' is strictly convex, J is a single–valued mapping.

A multi–valued mapping $A : X \rightarrow 2^X$ is said to be accretive if $(v_1 - v_2, J(u_1 - u_2)) \geq 0$, for any $u_i \in D(A)$ and $v_i \in Au_i$, $i = 1, 2$. The accretive mapping A is said to be m-accretive if $R(I + \lambda A) = X$ for some $\lambda > 0$. We say that $A : X \rightarrow 2^X$ is boundedly-inversely-compact if, for any pair of bounded subsets G and G' of X, the subset $G \bigcap A^{-1}(G')$ is relatively compact in X.

A multi-valued operator $B : X \rightarrow 2^{X'}$ is said to be monotone if its graph $G(B)$ is a monotone subset of $X \times X'$ in the sense that $(u_1 - u_2, w_1 - w_2) \geq 0$, for any $[u_i, w_i] \in G(B)$, $i = 1, 2$. The monotone operator B is said to be maximal monotone if $G(B)$ is maximal among all monotone subsets of $X \times X'$ in the sense of inclusion.

Definition 2.1 ([9]) *The normalized duality mapping $J : X \rightarrow X'$ is said to satisfy Condition (I) if there exists a function $\eta : X \rightarrow [0, +\infty)$ such that for $u, v \in X$,*

$$\|Ju - Jv\| \leq \eta(u - v). \tag{I}$$

Lemma 2.1 ([9]) *Let Ω be a bounded domain in R^N. Then the normalized duality mapping $J_p : L^p(\Omega) \rightarrow L^{p'}(\Omega)$ satisfies Condition (I), for $1 < p < +\infty$.*

Definition 2.2 ([9]) *Let $A : X \rightarrow 2^X$ be an accretive mapping and $J : X \rightarrow X'$ be the* *normalized duality mapping. We say that A satisfies Condition $(*)$ if, for any $f \in R(A)$ and $a \in D(A)$, there exists a constant $C(a,f)$ such that, for any $u \in D(A)$, $v \in Au$,*

$$(v-f, J(u-a)) \geq C(a,f). \tag{*}$$

Lemma 2.2 ([9]) *Let Ω be a bounded domain in R^N and $g : \Omega \times R \to R$ be a function satisfying Carathéodory's conditions such that*

(i) $g(x,\cdot)$ is monotonically increasing on R;

(ii) the mapping $u \in L^p(\Omega) \to g(x,u(x)) \in L^p(\Omega)$, $1 < p < +\infty$, is well defined.

Then, the mapping $B : L^p(\Omega) \to L^p(\Omega)$ defined by $(Bu)(x) = g(x,u(x))$, for any $x \in \Omega$, satisfies Condition $(*)$.

Theorem 2.1 ([9]) *Let X be a real Banach space with a strictly convex dual X'. Let $J : X \to X'$ be the normalized duality mapping on X satisfying Condition (I). Let A, $C_1 : X \to 2^X$ be accretive mappings such that*

(i) either both A and C_1 satisfy Condition $(*)$, or $D(A) \subset D(C_1)$ and C_1 satisfies Condition $(*)$;

(ii) $A + C_1$ is m-accretive and boundedly-inversely-compact.

If $C_2 : X \to X$ is a bounded continuous mapping such that, for any $y \in X$, there is a constant $C(y)$ satisfying $(C_2(u+y), Ju) \geq -C(y)$ for any $u \in X$, then:

(a) $\overline{[R(A) + R(C_1)]} \subset \overline{R(A + C_1 + C_2)}$;

(b) $\operatorname{int}[R(A) + R(C_1)] \subset \operatorname{int} R(A + C_1 + C_2)$.

2.2 Basic results for product space

Our discussion is based on some results for product space, which can be found in [10].

The product space of Banach spaces X_1 and X_2, which is denoted by $X_1 \times X_2$, is a set of all (ordered) pair of (x_1, x_2) of elements x_1 in X_1 and x_2 in X_2. $X_1 \times X_2$ is a vector space if the linear operation is defined by

$$k_1(x_1, y_1) + k_2(x_2, y_2) = (k_1x_1 + k_2x_2, k_1y_1 + k_2y_2).$$

Furthermore, $X_1 \times X_2$ becomes a normed space if the norm is defined by

$$\|(x_1, x_2)\| = (\|x_1\|^2 + \|x_2\|^2)^{\frac{1}{2}}.$$

The above norm ensures that $(X_1 \times X_2)' = X_1' \times X_2'$, where X_1' and X_2' are dual spaces of X_1 and X_2, respectively.

$(X_1 \times X_2)' = X_1' \times X_2'$ means that: (i) each element $(f,g) \in X_1' \times X_2'$ defines an element $F \in (X_1 \times X_2)'$ by $((x_1,x_2),F) = (x_1,f) + (x_2,g)$ and, conversely, each $F \in (X_1 \times X_2)'$ is expressed in this form by a unique $(f,g) \in X_1' \times X_2'$; (ii) the norm of the above $F \in (X_1 \times X_2)'$ is exactly equal to $\|(f,g)\| = (\|f\|^2 + \|g\|^2)^{\frac{1}{2}}$.

It is easily seen that $X_1 \times X_2$ is a Banach space since both X_1 and X_2 are Banach spaces.

3. Main results

3.1 Explanation of nonlinear elliptic system (1.2)

In this paper, unless otherwise stated, we shall assume that $\frac{2N}{N+1} < p < +\infty$ and $\frac{2N}{N+1} < q < +\infty$, where $N \geq 1$. We use $\|\cdot\|_p$ and $\|\cdot\|_q$ to denote the norm of spaces $L^p(\Omega)$ and $L^q(\Omega)$, respectively.

In system(1.2), Ω is a bounded conical domain of a Euclidean space R^N with its boundary $\Gamma \in C^1$ ([1]). We shall assume that Green's Formula is available. $f_1(x) \in L^p(\Omega)$ and $f_2(x) \in L^q(\Omega)$ are given functions. ε_1 and ε_2 are non-negative constants, and ϑ denotes the exterior normal derivative of Γ.

Suppose $\alpha_1 : R^N \to R^N$ is a given monotone function, and there exist positive constants k_1, k_2 and k_3 such that for $\forall \xi,\ \xi' \in R^N$, the following conditions are satisfied:

(i) $|\alpha_1(\xi)| \leq k_1 |\xi|^{p-1}$;

(ii) $|\alpha_1(\xi) - \alpha_1(\xi')| \leq k_2 ||\xi|^{p-2}\xi - |\xi'|^{p-2}\xi'|$;

(iii) $\langle \alpha_1(\xi), \xi \rangle \geq k_3 |\xi|^p$.

Moreover, suppose $\alpha_2 : R^N \to R^N$ is another given monotone function, and there exist positive constants k_1', k_2' and k_3' such that for $\forall \xi,\ \xi' \in R^N$, the following conditions are satisfied:

(iv) $|\alpha_2(\xi)| \leq k_1' |\xi|^{q-1}$;

(v) $|\alpha_2(\xi) - \alpha_2(\xi')| \leq k_2' ||\xi|^{q-2}\xi - |\xi'|^{q-2}\xi'|$;

(vi) $\langle \alpha_2(\xi), \xi \rangle \geq k_3' |\xi|^q$.

Let $\varphi : \Gamma \times R \to R$ be a given function such that, for each $x \in \Gamma$, $\varphi_x = \varphi(x, \cdot) : R \to R$ is a proper, convex and lower-semi-continuous function with $\varphi_x(0) = 0$. Let β_x be the subdifferential of φ_x, i.e., $\beta_x \equiv \partial \varphi_x$. Suppose that $0 \in \beta_x(0)$ and for each $t \in R$, the function $x \in \Gamma \to (I + \lambda \beta_x)^{-1}(t) \in R$ is measurable for $\lambda > 0$.

Suppose that $g : \Omega \times R \times R \to R$ is a given function satisfying Carathéodory's conditions such that for any $1 < p < +\infty$, the mapping $u(x) \in L^p(\Omega) \to g(x, u(x), v(x)) \in L^p(\Omega)$ is well-defined for all fixed $v(x) \in L^q(\Omega)$ and for any $1 < q < +\infty$, the mapping $v(x) \in L^q(\Omega) \to g(x, v(x), u(x)) \in L^q(\Omega)$ is well-defined for all fixed $u(x) \in L^p(\Omega)$. We shall also assume that there exists a function $0 \leq T(x) \in L^{\min(p,q)}(\Omega)$ such that $g(x, s, t)t \geq 0$, for $|t| \geq T(x)$, $x \in \Omega$ and for fixed number $s \in R$; and $g(x, s, t)s \geq 0$, for $|s| \geq T(x)$, $x \in \Omega$ and for fixed number $t \in R$.

We note that (1.2) is an extension of (1.1). Moreover, $-\mathrm{div}(\alpha_1(\mathrm{grad}\cdot))$ and $-\mathrm{div}(\alpha_2(\mathrm{grad}\cdot))$ are generalized p-Laplacian operator and generalized q-Laplacian operator, respectively.

3.2 Discussion of system (1.2)

We'll use Theorem 2.1 to discuss the existence of solution of system (1.2) as we have done before.

Lemma 3.1 ([11]) *Let X be a Banach space and $J : X \to X'$ be the normalized duality* *mapping. Then, X is strictly convex if and only if*

$$x^* \in Jx, y^* \in Jy, x \neq y \Longrightarrow (x-y, x^*-y^*) > 0.$$

By using Lemma 3.1, we can easily get the following result:

Proposition 3.1 *The dual space $(L^p(\Omega)\times L^q(\Omega))'$ of $L^p(\Omega)\times L^q(\Omega)$ is strictly convex.*

Definition 3.1 *Define $J : L^p(\Omega)\times L^q(\Omega) \to (L^p(\Omega)\times L^q(\Omega))'$ by $J(u,v) = (J_p u, J_q v)$, for $(u,v)\in L^p(\Omega)\times L^q(\Omega)$, where J_p and J_q are normalized duality mappings on $L^p(\Omega)$ and $L^q(\Omega)$, respectively.*

Proposition 3.2 *The mapping $J : L^p(\Omega)\times L^q(\Omega) \to (L^p(\Omega)\times L^q(\Omega))'$ defined in Definition 3.1 is the normalized duality mapping on $L^p(\Omega)\times L^q(\Omega)$ and satisfies Condition (I).*

Proof Let $(u,v)\in L^p(\Omega)\times L^q(\Omega)$, then

$$\begin{aligned}((u,v), J(u,v)) &= ((u,v),(J_p u, J_q v)) = (u, J_p u) + (v, J_q v)\\ &= \|u\|_p^2 + \|v\|_q^2 = \|(u,v)\|^2,\end{aligned}$$

and moreover,

$$\|J(u,v)\|^2 = \|J_p u\|^2 + \|J_q v\|^2 = \|u\|_p^2 + \|v\|_q^2 = \|(u,v)\|^2,$$

which imply that J is the normalized duality mapping from $L^p(\Omega)\times L^q(\Omega)$ to $(L^p(\Omega)\times L^q(\Omega))'$.

From Lemma 2.1, we know that both J_p and J_q satisfy condition (I). That means there exists a function $\eta_1 : L^p(\Omega) \to [0,+\infty)$ such that

$$\|J_p u - J_p v\| \le \eta_1(u-v), \quad \forall u, v \in L^p(\Omega);$$

and there exists $\eta_2 : L^q(\Omega) \to [0,+\infty)$ such that

$$\|J_q w - J_q z\| \le \eta_2(w-z), \quad \forall w, z \in L^q(\Omega).$$

Define $\eta : L^p(\Omega)\times L^q(\Omega) \to [0,+\infty)$ as follows:

$$\eta(a,b) = (\eta_1^2(a) + \eta_2^2(b))^{\frac{1}{2}}, \quad \forall (a,b)\in L^p(\Omega)\times L^q(\Omega).$$

Then for any $(u,w),(v,z)\in L^p(\Omega)\times L^q(\Omega)$, we have

$$\|J(u,w) - J(v,z)\| = \|(J_p u - J_p v, J_q w - J_q z)\| = (\|J_p u - J_p v\|^2 + \|J_q w - J_q z\|^2)^{\frac{1}{2}} \le \eta((u-v, w-z)).$$

This completes the proof. $\Box$

Lemma 3.2 ([7]) *Define the mapping $B_p : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$ by*

$$(w, B_p u) = \int_\Omega \langle \alpha(\mathrm{grad}u), \mathrm{grad}w\rangle \mathrm{d}x + \varepsilon_1 \int_\Omega |u(x)|^{p-2} u(x) w(x) \mathrm{d}x$$

for any $u, w \in W^{1,p}(\Omega)$. Then, B_p is everywhere defined, monotone, hemi-continuous and coercive.

Similarly, the mapping $B_q : W^{1,q}(\Omega) \to (W^{1,q}(\Omega))'$ defined by

$$(w, B_q v) = \int_\Omega \langle \alpha(\mathrm{grad}v), \mathrm{grad}w\rangle \mathrm{d}x + \varepsilon_2 \int_\Omega |v(x)|^{q-2} v(x) w(x) \mathrm{d}x$$

for any $v, w \in W^{1,q}(\Omega)$, is also everywhere defined, monotone, hemi-continuous and coercive.

Here $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the Euclidean inner-product and Euclidean norm in R^N.

Lemma 3.3 ([7]) *The mapping $\Phi_p : W^{1,p}(\Omega) \to R$ defined by $\Phi_p(u) = \int_\Gamma \varphi_x(u|_\Gamma(x))\mathrm{d}\Gamma(x)$, for any $u \in W^{1,p}(\Omega)$, is proper, convex and lower-semi-continuous on $W^{1,p}(\Omega)$.*

Similarly, the mapping $\Phi_q : W^{1,q}(\Omega) \to R$ defined by $\Phi_q(v) = \int_\Gamma \varphi_x(v|_\Gamma(x))\mathrm{d}\Gamma(x)$, for any $v \in W^{1,q}(\Omega)$, is also a proper, convex and lower-semi-continuous on $W^{1,q}(\Omega)$.

Similar to the proof of the corresponding result in [7], we have the following Lemma:

Lemma 3.4 *Define the mapping $A_p : L^p(\Omega) \to 2^{L^p(\Omega)}$ as follows:*

$$D(A_p) = \{u \in L^p(\Omega) | \text{there exists an } \ f \in L^p(\Omega) \ \text{ such that } \ f \in B_p u + \partial\Phi_p(u)\}.$$

For $u \in D(A_p)$, we set $A_p u = \{f \in L^p(\Omega) | f \in B_p u + \partial\Phi_p(u)\}$. T hen A_p is m-accretive.

Define the mapping $A_q : L^q(\Omega) \to 2^{L^q(\Omega)}$ as follows:

$$D(A_q) = \{v \in L^q(\Omega) | \text{there exists a } \ g \in L^q(\Omega) \ \text{ such that } \ g \in B_q v + \partial\Phi_q(v)\}.$$

For $v \in D(A_q)$, we set $A_q v = \{g \in L^q(\Omega) | g \in B_q v + \partial\Phi_q(v)\}$. Then A_q is also m-accretive.

Definition 3.2 *Define a mapping $A_{p,q} : L^p(\Omega) \times L^q(\Omega) \to 2^{L^p(\Omega) \times L^q(\Omega)}$ as follows:*

For $(u, v) \in L^p(\Omega) \times L^q(\Omega)$, we set $A_{p,q}(u, v) = (A_p u, A_q v)$.

Proposition 3.3 *The mapping $A_{p,q} : L^p(\Omega) \times L^q(\Omega) \to 2^{L^p(\Omega) \times L^q(\Omega)}$ is m-accretive.*

Proof To see that $A_{p,q}$ is accretive, let $u_i \in L^p(\Omega)$, $v_i \in L^q(\Omega)$, $i = 1, 2$, then from Lemma 3.4, we have

$$\begin{aligned}&(A_{p,q}(u_1, v_1) - A_{p,q}(u_2, v_2), J((u_1, v_1) - (u_2, v_2)))\\&\quad = (A_p u_1 - A_p u_2, J_p(u_1 - u_2)) + (A_q v_1 - A_q v_2, J_q(v_1 - v_2)) \geq 0.\end{aligned}$$

Now, let $u^* \in L^p(\Omega)$, $v^* \in L^q(\Omega)$, then it follows from Lemma 3.4 that there exist $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$ such that $u^* = u + \lambda A_p u$ and $v^* = v + \lambda A_q v$. Therefore,

$$(u^*, v^*) = (u, v) + \lambda(A_p u, A_q v) = (u, v) + \lambda A_{p,q}(u, v),$$

which implies that $R(I + \lambda A_{p,q}) = L^p(\Omega) \times L^q(\Omega)$. This completes the proof. □

Similar to the proof of the corresponding result in [7], we have the following Lemma:

Lemma 3.5 *Both the mappings $A_p : L^p(\Omega) \to 2^{L^p(\Omega)}$ and $A_q : L^q(\Omega) \to 2^{L^q(\Omega)}$ have a compact resolvent, for $\frac{2N}{N+1} < p, q \leq 2$ and $N \geq 1$.*

Proposition 3.4 *The mapping $A_{p,q} : L^p(\Omega) \times L^q(\Omega) \to 2^{L^p(\Omega) \times L^q(\Omega)}$ has a compact resolvent, for $\frac{2N}{N+1} < p$, $q \leq 2$ and $N \geq 1$.*

Proof Since $A_{p,q}$ is m-accretive, it suffices to prove that if $(u, v) + \lambda A_{p,q}(u, v) = (f, h)$ $(\lambda > 0)$ and $\{(f, h)\}$ is bounded in $L^p(\Omega) \times L^q(\Omega)$, then $\{(u, v)\}$ is relatively compact in $L^p(\Omega) \times L^q(\Omega)$.

In fact, from $(u, v) + \lambda A_{p,q}(u, v) = (f, h)$,we have $u + \lambda A_p u = f$ and $v + \lambda A_q v = h$. Since $\{(f, h)\}$ is bounded in $L^p(\Omega) \times L^q(\Omega)$, then $\{f\}$ is bounded in $L^p(\Omega)$ and $\{h\}$ is bounded in

$L^q(\Omega)$. Lemma 3.5 implies that $\{u\}$ is relatively compact in $L^p(\Omega)$ and $\{v\}$ is relatively compact in $L^q(\Omega)$.

Therefore, $\{(u, v)\}$ is relatively compact in $L^p(\Omega) \times L^q(\Omega)$. This completes the proof. □

Definition 3.3 *Define* $g_+(x) = \liminf_{s,t\to+\infty} g(x, s, t)$ *and* $g_-(x) = \limsup_{s,t\to-\infty} g(x, s, t)$. *Further, define a function* $g_1 : \Omega \times R \times R \to R$ *by*

$$g_1(x, s, t) = \begin{cases} (\inf_{a\geq s, b\geq t} g(x, a, b)) \bigwedge (s - T(x)) \bigwedge (t - T(x)), & \forall s, t \geq T(x) \\ (\sup_{a\leq s, b\leq t} g(x, a, b)) \bigvee (s + T(x)) \bigvee (t + T(x)), & \forall s, t \leq -T(x) \\ 0, & \textit{for the rests and } t \end{cases}$$

We note that for each $x \in \Omega$, $g_1(x, s, t)$ is increasing in t if $s \in R$ is fixed and is also increasing in s if $t \in R$ is fixed. Moreover, $\lim_{s,t\to\pm\infty} g_1(x, s, t) = g_\pm(x)$ for $x \in \Omega$. And, if we define $g_2(x, s, t) = g(x, s, t) - g_1(x, s, t)$, then $g_2(x, s, t)s \geq 0$ for $|s| \geq T(x), x \in \Omega$ and for fixed $t \in R$; and $g_2(x, s, t)t \geq 0$ for $|t| \geq T(x)$, $x \in \Omega$ and for fixed $s \in R$.

Lemma 3.6 *The mapping* $g_1 : \Omega \times R \times R \to R$ *satisfies Carathéodory's condition and the functions* $g_\pm(x)$ *are measurable on* Ω.

Proof We use Q to denote the set of rational numbers in the following proof. Now, for s, $t \in R$, $g_1(\cdot, s, t)$ is measurable on Ω since

$$\begin{aligned} &\{x|g_1(x, s, t) < \alpha\} \\ &\quad = \{x|s \leq T(x)\} \bigcup \{x|t \leq T(x)\} \bigcup \{x|0 < t - T(x) < \alpha\} \bigcup \{x|0 < s - T(x) < \alpha\} \\ &\quad \bigcup \{x|\exists r_1, r_2 \in Q, r_1 > s, r_2 > t, g_1(x, r_1, r_2) < \alpha\}, \end{aligned}$$

when $\alpha \geq 0$, and,

$$\begin{aligned} \{x|g_1(x, s, t) < \alpha\} = \{x|s + T(x) < \alpha\} \bigcap \{x|t + T(x) < \alpha\} \\ \bigcap \{x|\exists r_1, r_2 \in Q, r_1 < s, r_2 < t, g_1(x, r_1, r_2) < \alpha\}, \end{aligned}$$

when $\alpha \leq 0$.

Next, let $x \in \Omega$ be such that $g(x, \cdot, \cdot)$ is continuous on $R \times R$. We'll show that

(i) For fixed $s \in R$, $\forall t \in R$ such that $t > T(x)$, $t_n \uparrow t$, we have $\lim_{n\to\infty} g_1(x, s, t_n) = g_1(x, s, t)$;

(ii) For fixed $s \in R$, $\forall t \in R$ such that $t \geq T(x)$, $t_n \downarrow t$, we have $\lim_{n\to\infty} g_1(x, s, t_n) = g_1(x, s, t)$.

In fact, for (i), we notice that $g(x, s, t)$ satisfies Carathéodory's condition and $t_n \uparrow t$, then

$$\lim_{n\to\infty} g(x, s, t_n) = g(x, s, t).$$

Therefore, for $\forall\varepsilon > 0$, there exists N_1 such that for fixed $s \in R$,

$$g(x, s, b) > g(x, s, t) - \frac{\varepsilon}{2},$$

when $t_{N_1} \leq b \leq t$ and $n \geq N_1$.

From the definition of $g_1(x,s,t)$, we can also know that there exists N_2 such that for $n \geq N_2$,

$$|g_1(x,s,t_n) - g_1(x,s,t) \bigwedge \inf_{t_n \leq b \leq t} g(x,s,b)| \leq \frac{\varepsilon}{2}.$$

Therefore, if $n \geq \max(N_1, N_2)$, then

$$g_1(x,s,t_n) > g_1(x,s,t) - \varepsilon.$$

That is,

$$0 < g_1(x,s,t) - g_1(x,s,t_n) < \varepsilon,$$

when $n \geq \max(N_1, N_2)$, which implies that (i) is true.

For (ii), noticing the fact that $g_1(x,s,t)$ is increasing in t if $s \in R$ is fixed and $x \in \Omega$, we can see from $t_n \downarrow t$ that for fixed $s \in R$ and $x \in \Omega$,

$$g_1(x,s,t) \leq g_1(x,s,t_n).$$

From the definition of $g_1(x,s,t)$, we can also know that there exists N such that for $\forall \varepsilon > 0$, when $n \geq N$,

$$|g_1(x,s,t_n) - g_1(x,s,t) \bigwedge \inf_{t_n \leq b \leq t} g(x,s,b)| < \varepsilon.$$

Therefore,

$$g_1(x,s,t) \leq g_1(x,s,t_n) < g_1(x,s,t) + \varepsilon,$$

for $n \geq N$, which implies that (ii) is true.

Similarly, we can show that for fixed $s \in R$, $\forall t \in R$ such that $t < -T(x)$, $g_1(x,s,t)$ is still continuous for t. In the same way, $g_1(x,s,t)$ is continuous for $s > T(x)$ or $s < -T(x)$. Combining the previous results that for each $x \in \Omega$, $g_1(x,s,t)$ is increasing in t for each fixed $s \in R$, and is also increasing in s for each fixed $t \in R$, we know that $g_1(x,s,t)$ is continuous for $(s,t) \in R \times R$.

Hence g_1 satisfies Caratheodory's conditions. The measurable of $g_\pm(x)$ on Ω is obvious from its definition. This completes the proof. $\square$

Based the assumption of $g(x,s,t)$ and Lemma 3.6, using similar proving method of Proposition 3.5 in [9], we have the following two results:

Lemma 3.7 $C_1^{(1)} : L^p(\Omega) \to L^p(\Omega)$ *defined by* $C_1^{(1)}(u,v) = g_1(x,u(x),v(x))$ *for any* $u \in L^p(\Omega)$, *for fixed* $v(x) \in L^q(\Omega)$ *and* $x \in \Omega$, *is bounded, continuous and* m*-accretive. Also* $C_1^{(2)} : L^q(\Omega) \to L^q(\Omega)$ *defined by* $C_1^{(2)}(v,u) = g_1(x,v(x),u(x))$ *for any* $v \in L^q(\Omega)$, *for fixed* $u(x) \in L^p(\Omega)$ *and* $x \in \Omega$, *is bounded, continuous and* m*-accretive.*

Lemma 3.8 *The mapping* $C_2^{(1)} : L^p(\Omega) \to L^p(\Omega)$ *defined by* $C_2^{(1)}(u,v) = g_2(x,u(x),v(x)) = g(x,u(x),v(x)) - g_1(x,u(x),v(x))$ *satisfies the condition*

$$(C_2^{(1)}(u+y,v), J_p u) \geq -C(y), \tag{3.1}$$

for any $u,y \in L^p(\Omega)$, *where* $C(y)$ *is a constant depending on* y. *The mapping* $C_2^{(2)} : L^q(\Omega) \to L^q(\Omega)$ *defined by* $C_2^{(2)}(v,u) = g_2(x,v(x),u(x)) = g(x,v(x),u(x)) - g_1(x,v(x),u(x))$ *also satisfies* (3.1), *i.e.,*

$$(C_2^{(2)}(v+y,u), J_q v) \geq -C'(y)$$

for any $v, y \in L^q(\Omega)$, where $C'(y)$ is a constant depending on y.

Proposition 3.5 *The mapping $C_1 : L^p(\Omega) \times L^q(\Omega) \to L^p(\Omega) \times L^q(\Omega)$ defined by*

$$C_1(u, v) = (C_1^{(1)}(u, v), C_1^{(2)}(v, u))$$

for $(u, v) \in L^p(\Omega) \times L^q(\Omega)$, is bounded, continuous and m-accretive.

The mapping $C_2 : L^p(\Omega) \times L^q(\Omega) \to L^p(\Omega) \times L^q(\Omega)$ defined by

$$C_2(u, v) = (C_2^{(1)}(u, v), C_2^{(2)}(v, u))$$

for $(u, v) \in L^p(\Omega) \times L^q(\Omega)$, satisfies the condition

$$(C_2((u, v) + (w, y)), J(u, v)) \geq -C(w, y)$$

for any $u, w \in L^p(\Omega)$, $v, y \in L^q(\Omega)$, where $C(w, y)$ is a constant depending on w and y, and J is the normalized duality mapping from $L^p(\Omega) \times L^q(\Omega)$ to $L^{p'}(\Omega) \times L^{q'}(\Omega)$ defined in Proposition 3.1.

Proof The result follows from Lemmas 3.7 and 3.8.

Proposition 3.6 *The mapping $C_1 : L^p(\Omega) \times L^q(\Omega) \to L^p(\Omega) \times L^q(\Omega)$ defined in Proposition 3.5 satisfies Condition $(*)$.*

Proof By Lemma 2.2, we can know that both $C_1^{(1)} : L^p(\Omega) \to L^p(\Omega)$ and $C_1^{(2)} : L^q(\Omega) \to L^q(\Omega)$ satisfy Condition $(*)$, then it is not difficult to check that $C_1 : L^p(\Omega) \times L^q(\Omega) \to L^p(\Omega) \times L^q(\Omega)$ satisfies Condition $(*)$ in view of the definition of C_1. This completes the proof. □

Remark 3.1 ([7]) If $\beta_x \equiv 0$, $\forall x \in \Gamma$, then $\partial\Phi_p(u) \equiv 0$, $\forall u \in W^{1,p}(\Omega)$ and $\partial\Phi_q(v) \equiv 0$, $\forall v \in W^{1,q}(\Omega)$.

Lemma 3.9 *If $\beta_x \equiv 0$, $\forall x \in \Gamma$, then we have*

(i) $\{f \in L^p(\Omega) | \int_\Omega f \mathrm{d}x = 0\} \subset R(A_p)$, *for* $\frac{2N}{N+1} < p < +\infty$ *and* $N \geq 1$, *and*

(ii) $\{f \in L^q(\Omega) | \int_\Omega f \mathrm{d}x = 0\} \subset R(A_q)$, *for* $\frac{2N}{N+1} < q < +\infty$ *and* $N \geq 1$.

Proposition 3.7 *If $\beta_x \equiv 0$, $\forall x \in \Gamma$, then we have*

$$\{(f_1, f_2) \in L^p(\Omega) \times L^q(\Omega) | \int_\Omega f_1(x)\mathrm{d}x = 0 = \int_\Omega f_2(x)\mathrm{d}x\} \subset R(A_{p,q}).$$

Proof From the fact that $(f_1, f_2) \in L^p(\Omega) \times L^q(\Omega)$ with $\int_\Omega f_1(x)\mathrm{d}x = 0 = \int_\Omega f_2(x)\mathrm{d}x$, and Lemma 3.9, we have $f_1(x) \in R(A_p)$ and $f_2(x) \in R(A_q)$. Then $(f_1, f_2) \in R(A_{p,q})$ from the definition of $A_{p,q}$. This completes the proof. □

Definition 3.4 ([9]) *For $t \in R$ and $x \in \Gamma$, let $\beta_x^0(t) \in \beta_x(t)$ be the element with least absolute value if $\beta_x(t) \neq \emptyset$ and $\beta_x^0(t) = \pm\infty$, where $t > 0$ or < 0, respectively, in case $\beta_x(t) = \emptyset$. Finally, let $\beta_\pm(x) = \lim_{t \to \pm\infty} \beta_x^0(t)$ (in the extended sense) for $x \in \Gamma$. Then, $\beta_\pm(x)$ define measurable functions on Γ.*

Lemma 3.10 *If $f_1(x) \in L^p(\Omega)$ satisfies*

$$\int_\Gamma \beta_-(x)\mathrm{d}\Gamma(x) < \int_\Omega f_1(x)\mathrm{d}x < \int_\Gamma \beta_+(x)\mathrm{d}\Gamma(x). \tag{3.2}$$

Then, $f_1(x) \in intR(A_p)$, for $\frac{2N}{N+1} < p < +\infty$ and $N \geq 1$.

Similarly, if $f_2(x) \in L^q(\Omega)$ satisfies (3.2), then, $f_2(x) \in intR(A_q)$, for $\frac{2N}{N+1} < q < +\infty$ and $N \geq 1$.

From Lemma 3.10, we can easily get the following result:

Proposition 3.8 *Let $f_1(x) \in L^p(\Omega)$, $f_2(x) \in L^q(\Omega)$ satisfy (3.2). Then, we have $(f_1, f_2) \in \operatorname{int} R(A_{p,q})$.*

Proposition 3.9 *Let $(f_1, f_2) \in L^p(\Omega) \times L^q(\Omega)$, $(u, v) \in L^p(\Omega) \times L^q(\Omega)$ and $(f_1, f_2) \in A_{p,q}(u, v)$. Then, the following hold*

(a) $-\mathrm{div}(\alpha_1(\mathrm{grad}u)) + \varepsilon_1|u|^{p-2}u = f_1(x)$, a.e., $x \in \Omega$; $-\langle \vartheta, \alpha_1(\mathrm{grad}u)\rangle \in \beta_x(u(x))$, a.e., $x \in \Gamma$;

(b) $-\mathrm{div}(\alpha_2(\mathrm{grad}v)) + \varepsilon_2|v|^{q-2}v = f_2(x)$, a.e., $x \in \Omega$; $-\langle \vartheta, \alpha_2(\mathrm{grad}v)\rangle \in \beta_x(v(x))$, a.e., $x \in \Gamma$.

Proof The proof is similar to that of Proposition 2.2 in [7].

Theorem 3.1 *Let $(f_1, f_2) \in L^p(\Omega) \times L^q(\Omega)$ satisfy*

$$\int_\Gamma \beta_-(x)\mathrm{d}\Gamma(x) + \int_\Omega g_-(x)\mathrm{d}x < \int_\Omega f_1(x)\mathrm{d}x < \int_\Gamma \beta_+(x)\mathrm{d}\Gamma(x) + \int_\Omega g_+(x)\mathrm{d}x. \tag{3.3}$$

and

$$\int_\Gamma \beta_-(x)\mathrm{d}\Gamma(x) + \int_\Omega g_-(x)\mathrm{d}x < \int_\Omega f_2(x)\mathrm{d}x < \int_\Gamma \beta_+(x)\mathrm{d}\Gamma(x) + \int_\Omega g_+(x)\mathrm{d}x. \tag{3.4}$$

Then, system (1.2) has a solution in $L^p(\Omega) \times L^q(\Omega)$.

Proof Let $A_{p,q}$ be the m-accretive mapping as in Definition 3.2 and $C_i : L^p(\Omega) \times L^q(\Omega) \to L^p(\Omega) \times L^q(\Omega)$ be as in Proposition 3.5, i.e., $(C_i(u, v))(x) = (g_i(x, u(x), v(x)), g_i(x, v(x), u(x)))$ for $x \in \Omega$, and $i = 1, 2$. We need to prove that $A_{p,q} + C_1$ is boundedly-inversely-compact. In fact, we only need to show that if $(w, y) \in A_{p,q}(u, v) + C_1(u, v)$ with $\{(w, y)\}$ and $\{(u, v)\}$ being bounded in $L^p(\Omega) \times L^q(\Omega)$, then $\{(u, v)\}$ is relatively compact in $L^p(\Omega) \times L^q(\Omega)$.

For this, we need to discuss the following two cases:

(i) $\frac{2N}{N+1} < p, q \leq 2$, for $N \geq 1$, from the above we can see that $(w, y) - C_1(u, v) \in A_{p,q}(u, v)$ with $\{(w, y) - C_1(u, v)\}$ and $\{(u, v)\}$ bounded in $L^p(\Omega) \times L^q(\Omega)$ which gives that $\{(u, v)\}$ is relatively compact in $L^p(\Omega) \times L^q(\Omega)$ since $A_{p,q}$ is m-accretive and has a compact resolvent from Proposition 3.4.

(ii) If $p, q \geq 2$, or $\frac{2N}{N+1} < p \leq 2$ and $q \geq 2$, or $\frac{2N}{N+1} < q \leq 2$ and $p > 2$, we know from $(w, y) \in A_{p,q}(u, v) + C_1(u, v)$ that $w \in A_p u + C_1^{(1)}(u, v)$ and $y \in A_q u + C_1^{(2)}(v, u)$. Since $\{(u, v)\}$ is bounded in $L^p(\Omega) \times L^q(\Omega)$, then $\{u\}$ is bounded in $L^p(\Omega)$ and $\{v\}$ bounded in $L^q(\Omega)$. Similar to the proof of the corresponding result in [7], we know that $\{u\}$ is relatively compact in

$L^p(\Omega)$ and $\{v\}$ is relatively compact in $L^q(\Omega)$, which imply that $\{(u, v)\}$ is relatively compact in $L^p(\Omega) \times L^q(\Omega)$.

Notice the facts of Propositions 3.1–3.3, 3.5 and 3.6, it is easy to show that all the conditions of Theorem 2.1 required are satisfied. Further, from Propositions 3.7 and 3.8, we have $(f_1, f_2) \in \mathrm{int}[R(A_{p,q}) + R(C_1)]$. Therefore, Proposition 3.9 implies that the Theorem 3.1 holds. This completes the proof. □

Remark 3.2 System (1.2) can be further extended to the case of containing finite such equations and the corresponding boundaries.

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