Journal of Mathematical Research & Exposition
Sept., 2010, Vol. 30, No. 5, pp. 929–935
DOI:10.3770/j.issn:1000-341X.2010.05.022
Http://jmre.dlut.edu.cn

Dynamics of Two Extensive Classes of Recursive Sequences

Bin ZHAO[1,2]

1. *Department of Mathematics, Northwest Agriculture & Forestry University, Shaanxi* 712100, *P. R. China;*

2. *Department of Mathematics, Northwest University, Shaanxi* 710127, *P. R. China*

Abstract We investigate the dynamics of two extensive classes of recursive sequences:

$$x_{n+1}=\frac{c\sum\limits_{j=0}^{k}\sum\limits_{(i_0,i_1,\ldots,i_{2j})\in A_{2j}}x_{n-i_0}x_{n-i_1}\cdots x_{n-i_{2j}}+f(x_{n-i_0},x_{n-i_1},\ldots,x_{n-i_{2k}})}{c\sum\limits_{j=1}^{k}\sum\limits_{(i_0,i_1,\ldots,i_{2j-1})\in A_{2j-1}}x_{n-i_0}x_{n-i_1}\cdots x_{n-i_{2j-1}}+c+f(x_{n-i_0},x_{n-i_1},\ldots,x_{n-i_{2k}})},$$

and

$$x_{n+1}=\frac{c\sum\limits_{j=1}^{k}\sum\limits_{(i_0,i_1,\ldots,i_{2j-1})\in A_{2j-1}}x_{n-i_0}x_{n-i_1}\cdots x_{n-i_{2j-1}}+c+f(x_{n-i_0},x_{n-i_1},\ldots,x_{n-i_{2k}})}{c\sum\limits_{j=0}^{k}\sum\limits_{(i_0,i_1,\ldots,i_{2j})\in A_{2j}}x_{n-i_0}x_{n-i_1}\cdots x_{n-i_{2j}}+f(x_{n-i_0},x_{n-i_1},\ldots,x_{n-i_{2k}})}.$$

We prove that their unique positive equilibrium $\overline{x}=1$ is globally asymptotically stable. And a new access is presented to study the theory of recursive sequences.

Keywords recursive sequence; equilibrium; dynamics.

Document code A

MR(2000) Subject Classification 39A11

Chinese Library Classification O175.7

1. Introduction

Our aim in this paper is to study the following two extensive classes of recursive sequences:

$$x_{n+1}=\frac{c\sum\limits_{j=0}^{k}\sum\limits_{(i_0,i_1,\ldots,i_{2j})\in A_{2j}}x_{n-i_0}x_{n-i_1}\cdots x_{n-i_{2j}}+f(x_{n-i_0},x_{n-i_1},\ldots,x_{n-i_{2k}})}{c\sum\limits_{j=1}^{k}\sum\limits_{(i_0,i_1,\ldots,i_{2j-1})\in A_{2j-1}}x_{n-i_0}x_{n-i_1}\cdots x_{n-i_{2j-1}}+c+f(x_{n-i_0},x_{n-i_1},\ldots,x_{n-i_{2k}})},\quad (1)$$

Received August 23, 2008; Accepted January 5, 2009
Supported by the National Natural Science Foundation of China (Grant No. 10771169).
E-mail address: zhaobin835@nwsuaf.edu.cn

and

$$x_{n+1}=\frac{c\sum_{j=1}^{k}\sum_{(i_0,i_1,\ldots,i_{2j-1})\in A_{2j-1}}x_{n-i_0}x_{n-i_1}\cdots x_{n-i_{2j-1}}+c+f(x_{n-i_0},x_{n-i_1},\ldots,x_{n-i_{2k}})}{c\sum_{j=0}^{k}\sum_{(i_0,i_1,\ldots,i_{2j})\in A_{2j}}x_{n-i_0}x_{n-i_1}\cdots x_{n-i_{2j}}+f(x_{n-i_0},x_{n-i_1},\ldots,x_{n-i_{2k}})},$$

$$n=0,1,\ldots, \tag{2}$$

where $c\in(0,\infty)$, $k\in\{1,2,\ldots\}$ and the initial conditions $x_{-i_{2k}},x_{-i_{2k}+1},\ldots,x_0\in(0,\infty)$, $f\in C((0,\infty)^{2k+1},[0,\infty))$, $A_j=\{(i_0,i_1,\ldots,i_j)|i_0<i_1<\cdots<i_j,\{i_0,i_1,\ldots,i_j\}\subset\{t_0,t_1,\ldots,t_{2k}\}\}$, $0\le t_0<t_1<\cdots<t_{2k}$, and $\{t_j\}_{j=0}^{2k}$ are constants.

Our research is motivated by [1–5], where the dynamics of some recursive sequences was studied.

Hamza and Khalaf-Allah [1] investigated the global asymptotic stability of the following rational recursive sequence:

$$x_{n+1}=\frac{A\prod_{i=l}^{k}x_{n-2i-1}}{B+C\prod_{i=l}^{k-1}x_{n-2i}},\quad n=0,1,\ldots, \tag{E1}$$

where A, B, C are nonnegative real numbers and l, k are nonnegative integers, $l<k$.

Nesemann [2] utilized the strong negative feedback property of [6] to study the following recursive sequence:

$$x_{n+1}=\frac{x_{n-1}+x_nx_{n-2}}{x_nx_{n-1}+x_{n-2}},\quad n=0,1,\ldots, \tag{E2}$$

where the initial values $x_{-2},x_{-1},x_0\in(0,\infty)$.

Papaschinopoulos and Schinas [3] investigated the global asymptotic stability of the following nonlinear recursive sequence:

$$x_{n+1}=\frac{\sum_{i\in\mathbf{Z}_k-\{j-1,j\}}x_{n-i}+x_{n-j}x_{n-j+1}+1}{\sum_{i\in\mathbf{Z}_k}x_{n-i}},\quad n=0,1,\ldots, \tag{E3}$$

where $k\in\{1,2,3,\ldots\}$, $\{j,j-1\}\subset\mathbf{Z}_k\equiv\{0,1,\ldots,k\}$ and the initial values $x_{-k},x_{-k+1},\ldots,x_0\in(0,\infty)$.

Li [4, 5] studied the global asymptotic stability of the following two nonlinear recursive sequences:

$$x_{n+1}=\frac{x_{n-1}x_{n-2}x_{n-3}+x_{n-1}+x_{n-2}+x_{n-3}+a}{x_{n-1}x_{n-2}+x_{n-1}x_{n-3}+x_{n-2}x_{n-3}+1+a},\quad n=0,1,\ldots \tag{E4}$$

and

$$x_{n+1}=\frac{x_nx_{n-1}x_{n-3}+x_n+x_{n-1}+x_{n-3}+a}{x_nx_{n-1}+x_nx_{n-3}+x_{n-1}x_{n-3}+1+a},\quad n=0,1,\ldots, \tag{E5}$$

where $a\in[0,+\infty)$ and the initial values $x_{-3},x_{-2},x_{-1},x_0\in(0,\infty)$.

In this paper, by using an effective method which is different from the usual methods we derive the global asymptotic stability of the positive equilibrium of the two extensive classes of recursive sequences, whereas it is extremely difficult to use the method in the literature [1], to obtain the global asymptotic stability of the positive equilibrium of the two extensive classes of recursive sequences.

To prove the global asymptotic stability of the positive equilibrium of the above two extensive classes of recursive sequences, we construct the following class of nonlinear recursive sequence (3):

First, let $k \geq 1$ and $i_0, i_1, \ldots, i_{2k} \in \{0, 1, \ldots\}$ with $i_0 < i_1 < \cdots < i_{2k}$. Let $F_0(x_{n-i_0}) = x_{n-i_0}$ and $G_0(x_{n-i_0}) = 1$, and for any $1 \leq j \leq k$, let

$$\begin{aligned} F_j(x_{n-i_0}, \ldots, x_{n-i_{2j}}) =& (x_{n-i_{2j}} x_{n-i_{2j-1}} + 1) F_{j-1}(x_{n-i_0}, \ldots, x_{n-i_{2j-2}}) + \\ & (x_{n-i_{2j}} + x_{n-i_{2j-1}}) G_{j-1}(x_{n-i_0}, \ldots, x_{n-i_{2j-2}}) \end{aligned}$$

and

$$\begin{aligned} G_j(x_{n-i_0}, \ldots, x_{n-i_{2j}}) =& (x_{n-i_{2j}} x_{n-i_{2j-1}} + 1) G_{j-1}(x_{n-i_0}, \ldots, x_{n-i_{2j-2}}) + \\ & (x_{n-i_{2j}} + x_{n-i_{2j-1}}) F_{j-1}(x_{n-i_0}, \ldots, x_{n-i_{2j-2}}). \end{aligned}$$

Then we can find out that the class of recursive sequence (1) and the following Eq.(3) give rise to substantially the equivalent form.

$$x_{n+1} = \frac{cF_k(x_{n-i_0}, x_{n-i_1}, \ldots, x_{n-i_{2k}}) + f(x_{n-i_0}, x_{n-i_1}, \ldots, x_{n-i_{2k}})}{cG_k(x_{n-i_0}, x_{n-i_1}, \ldots, x_{n-i_{2k}}) + f(x_{n-i_0}, x_{n-i_1}, \ldots, x_{n-i_{2k}})}, \quad n = 0, 1, \ldots, \tag{3}$$

where $f \in C((0, \infty)^{2k+1}, [0, \infty))$ and the initial conditions $x_{-i_{2k}}, x_{-i_{2k}+1}, \ldots, x_0 \in (0, \infty)$.

It is easy to see that the positive equilibrium $\overline{x}$ of Eq.(3) satisfies

$$\begin{aligned} \overline{x} =& \frac{cF_k \overbrace{(\overline{x}, \overline{x}, \ldots, \overline{x})}^{2k+1 \text{ elements}} + f \overbrace{(\overline{x}, \overline{x}, \ldots, \overline{x})}^{2k+1 \text{ elements}}}{cG_k \underbrace{(\overline{x}, \overline{x}, \ldots, \overline{x})}_{2k+1 \text{ elements}} + f \underbrace{(\overline{x}, \overline{x}, \ldots, \overline{x})}_{2k+1 \text{ elements}}} \\ =& \frac{c(\overline{x}^2 + 1)F_{k-1} \overbrace{(\overline{x}, \overline{x}, \ldots, \overline{x})}^{2k-1 \text{ elements}} + 2c\overline{x}G_{k-1} \overbrace{(\overline{x}, \overline{x}, \ldots, \overline{x})}^{2k-1 \text{ elements}} + f \overbrace{(\overline{x}, \overline{x}, \ldots, \overline{x})}^{2k+1 \text{ elements}}}{c(\overline{x}^2 + 1)G_{k-1} \underbrace{(\overline{x}, \overline{x}, \ldots, \overline{x})}_{2k-1 \text{ elements}} + 2c\overline{x}F_{k-1} \underbrace{(\overline{x}, \overline{x}, \ldots, \overline{x})}_{2k-1 \text{ elements}} + f \underbrace{(\overline{x}, \overline{x}, \cdots, \overline{x})}_{2k+1 \text{ elements}}}. \end{aligned}$$

Thus, we have

$$(\overline{x} - 1)[c(\overline{x}^2 + \overline{x})G_{k-1} \overbrace{(\overline{x}, \overline{x}, \ldots, \overline{x})}^{2k-1 \text{ elements}} + c(\overline{x} + 1)F_{k-1} \overbrace{(\overline{x}, \overline{x}, \ldots, \overline{x})}^{2k-1 \text{ elements}} + f \overbrace{(\overline{x}, \overline{x}, \ldots, \overline{x})}^{2k+1 \text{ elements}}] = 0,$$

from which one can see that Eq.(3) has the unique positive equilibrium $\overline{x} = 1$.

Remark Let $k = 1$, $c = 1$, $f(x_{n-i_0}, x_{n-i_1}, \ldots, x_{n-i_{2k}}) = a$. Then Eq.(3) is Eq.(E4) when $(i_0, i_1, i_2) = (1, 2, 3)$ and is Eq.(E5) when $(i_0, i_1, i_2) = (0, 1, 3)$.

2. Properties of positive solutions of the two extensive classes of recursive sequences.

In this section we shall study the properties of positive solutions of the two extensive classes of recursive sequences (1) and (2).

If we change Eq.(3) to the form:

$$x_{n+1} = \frac{cG_k(x_{n-i_0}, x_{n-i_1}, \ldots, x_{n-i_{2k}}) + f(x_{n-i_0}, x_{n-i_1}, \ldots, x_{n-i_{2k}})}{cF_k(x_{n-i_0}, x_{n-i_1}, \ldots, x_{n-i_{2k}}) + f(x_{n-i_0}, x_{n-i_1}, \ldots, x_{n-i_{2k}})}, \quad n = 0, 1, \ldots . \tag{4}$$

Then it can be seen that Eq.(4) and the class of recursive sequence (2) give rise to substantially the equivalent form.

Thus, we only need to discuss Eq.(3) (because Eq.(4) and Eq.(3) give rise to substantially same dynamic behaviors, and the proof is similar).

Since

$$\begin{aligned}
&F_k(x_{n-i_0}, x_{n-i_1}, \ldots, x_{n-i_{2k}}) - G_k(x_{n-i_0}, x_{n-i_1}, \ldots, x_{n-i_{2k}}) \\
&\quad = (x_{n-i_{2k}} - 1)(x_{n-i_{2k-1}} - 1)[F_{k-1}(x_{n-i_0}, \ldots, x_{n-i_{2k-2}}) - G_{k-1}(x_{n-i_0}, \ldots, x_{n-i_{2k-2}})] \\
&\quad = \cdots \\
&\quad = (x_{n-i_{2k}} - 1)(x_{n-i_{2k-1}} - 1) \cdots (x_{n-i_2} - 1)(x_{n-i_1} - 1)[F_0(x_{n-i_0}) - G_0(x_{n-i_0})] \\
&\quad = (x_{n-i_0} - 1)(x_{n-i_1} - 1) \cdots (x_{n-i_{2k}} - 1),
\end{aligned}$$

it follows from Eq.(3) that for any $n \geq 0$,

$$x_{n+1} - 1 = \frac{c(x_{n-i_0} - 1)(x_{n-i_1} - 1) \cdots (x_{n-i_{2k}} - 1)}{cG_k(x_{n-i_0}, x_{n-i_1}, \ldots, x_{n-i_{2k}}) + f(x_{n-i_0}, x_{n-i_1}, \ldots, x_{n-i_{2k}})}. \tag{5}$$

Definition 1 *Let $\{x_n\}_{n=-i_{2k}}^{\infty}$ be a solution of Eq.(3) and $\{a_n\}_{n=-i_{2k}}^{\infty}$ be a sequence with $a_n \in \{-1, 0, 1\}$ for every $n \geq -i_{2k}$. $\{a_n\}_{n=-i_{2k}}^{\infty}$ is called itinerary of $\{x_n\}_{n=-i_{2k}}^{\infty}$ if $a_n = -1$ when $x_n < 1$, $a_n = 0$ when $x_n = 1$ and $a_n = 1$ when $x_n > 1$.*

From Eq.(5), we get

Proposition 1 *Let $\{a_n\}_{n=-i_{2k}}^{\infty}$ be an itinerary of a solution $\{x_n\}_{n=-i_{2k}}^{\infty}$ of Eq.(3). Then $a_{n+1} = a_{n-i_0} a_{n-i_1} \cdots a_{n-i_{2k}}$ for any $n \geq 0$.*

Proposition 2 *Let $\{x_n\}_{n=-i_{2k}}^{\infty}$ be a solution of Eq.(3). Then $x_n \neq 1$ for any $n \geq 1 \iff \prod_{j=0}^{i_{2k}} (x_{-j} - 1) \neq 0$.*

Proof. Let $\{a_n\}_{n=-i_{2k}}^{\infty}$ be an itinerary of the solution $\{x_n\}_{n=-i_{2k}}^{\infty}$. Then it follows from Proposition 1 that $x_n \neq 1$ for any $n \geq 1 \iff a_n \neq 0$ for any $n \geq 1 \iff \prod_{j=0}^{i_{2k}} a_{-j} \neq 0 \iff \prod_{j=0}^{i_{2k}} (x_{-j} - 1) \neq 0$. $\square$

Proposition 3 *If $\gcd(i_s + 1, i_{2k} + 1) = 1$ for some $s \in \{0, 1, \ldots, 2k-1\}$, then a positive solution $\{x_n\}_{n=-i_{2k}}^{\infty}$ of Eq.(3) is eventually equal to $1 \iff x_p = 1$ for some $p \geq -i_{2k}$.*

Proof $\Longrightarrow$ is obvious.

$\Longleftarrow$. If $x_p = 1$ for some $p \geq -i_{2k}$, then $a_p = 0$, where $\{a_n\}_{n=-i_{2k}}^{\infty}$ is the itinerary of $\{x_n\}_{n=-i_{2k}}^{\infty}$. By Proposition 1, we have $a_{j(i_{2k}+1)+p} = a_{j(i_s+1)+p} = 0$ for any $j \geq 0$. Since $\gcd(i_s + 1, i_{2k} + 1) = 1$, we see that for any $t \in \{0, 1, \ldots, i_{2k}\}$, there exist $j_t \in \{1, 2, \ldots, i_{2k} + 1\}$ and $m_t \in \{0, 1, \ldots, i_s + 1\}$ such that

$$j_t(i_s + 1) = m_t(i_{2k} + 1) + t,$$

from which it follows by Proposition 1 that

$$a_{(i_s+1)(i_{2k}+1)+t+p} = 0.$$

Again by Proposition 1, we have $a_n = 0$ for any $n \geq (i_s+1)(i_{2k}+1)+p$, which implies $x_n = 1$ for any $n \geq (i_s+1)(i_{2k}+1)+p$. □

Example 1 Consider the recursive sequence:

$$x_{n+1} = \frac{x_{n-i_0}x_{n-i_1}x_{n-3} + x_{n-i_0} + x_{n-i_1} + x_{n-3} + b}{x_{n-i_0}x_{n-i_1} + x_{n-i_0}x_{n-3} + x_{n-i_1}x_{n-3} + 1 + b}, \quad n = 0, 1, \ldots, \tag{6}$$

where $b \in [0, +\infty)$, $0 \leq i_0 < i_1 < 3$ and the initial values x_{-3}, x_{-2}, x_{-1}, $x_0 \in (0, \infty)$. Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(6). Then

1) $x_n \neq 1$ for any $n \geq 1 \Longleftrightarrow \prod_{j=-3}^{0}(x_j - 1) \neq 0$;

2) $\{x_n\}_{n=-3}^{\infty}$ is eventually equal to $1 \Longleftrightarrow x_p = 1$ for some $p \geq -3$.

Proof 1) The conclusion follows from Proposition 2.

2) The conclusion follows from Proposition 3 since either $\gcd(i_0+1, 4) = 1$ or $\gcd(i_1+1, 4) = 1$. □

3. Global asymptotic stability of the two extensive classes of recursive sequences

In this section we shall study the global asymptotic stability of the two extensive classes of recursive sequences (1) and (2). To do this, we need the following lemmas.

Lemma 1 *Let* $(y_0, y_1, \ldots, y_{i_{2k}}) \in R_+^{i_{2k}+1} - \{(1, 1, \ldots, 1)\}$, $f \in C((0, \infty)^{2k+1}, [0, \infty))$ *and* $M = \max\{y_j, \frac{1}{y_j} | 0 \leq j \leq i_{2k}\}$. *Then*

$$\frac{1}{M} < \frac{cF_k(y_{i_0}, y_{i_1}, \ldots, y_{i_{2k}}) + f(y_{i_0}, y_{i_1}, \ldots, y_{i_{2k}})}{cG_k(y_{i_0}, y_{i_1}, \ldots, y_{i_{2k}}) + f(y_{i_0}, y_{i_1}, \ldots, y_{i_{2k}})} < M. \tag{7}$$

Proof Since $(y_0, y_1, \ldots, y_{i_{2k}}) \in R_+^{i_{2k}+1} - \{(1, 1, \ldots, 1)\}$ and $M = \max\{y_j, \frac{1}{y_j} | 0 \leq j \leq i_{2k}\}$, we have $M > 1$ and either $M \geq a > \frac{1}{M}$ or $M > a \geq \frac{1}{M}$ for any $a \in \{y_j, \frac{1}{y_j} | 0 \leq j \leq i_{2k}\}$.

It is easy to verify that

$$\begin{aligned} F_1(y_{i_0}, y_{i_1}, y_{i_2}) &= (y_{i_1}y_{i_2} + 1)y_{i_0} + (y_{i_1} + y_{i_2}) < (y_{i_1}y_{i_2} + 1)M + (y_{i_1} + y_{i_2})y_{i_0}M \\ &= G_1(y_{i_0}, y_{i_1}, y_{i_2})M, \end{aligned}$$

and

$$\begin{aligned} F_1(y_{i_0}, y_{i_1}, y_{i_2})M &= [(y_{i_1}y_{i_2} + 1)y_{i_0} + (y_{i_1} + y_{i_2})]M > (y_{i_1}y_{i_2} + 1) + (y_{i_1} + y_{i_2})y_{i_0} \\ &= G_1(y_{i_0}, y_{i_1}, y_{i_2}). \end{aligned}$$

From which we have

$$\begin{aligned} F_2(y_{i_0}, y_{i_1}, y_{i_2}, y_{i_3}, y_{i_4}) &= (y_{i_3}y_{i_4} + 1)F_1(y_{i_0}, y_{i_1}, y_{i_2}) + (y_{i_3} + y_{i_4})G_1(y_{i_0}, y_{i_1}, y_{i_2}) \\ &< (y_{i_3}y_{i_4} + 1)G_1(y_{i_0}, y_{i_1}, y_{i_2})M + (y_{i_3} + y_{i_4})F_1(y_{i_0}, y_{i_1}, y_{i_2})M \end{aligned}$$

$$= G_2(y_{i_0}, y_{i_1}, y_{i_2}, y_{i_3}, y_{i_4})M,$$

and

$$\begin{aligned} F_2(y_{i_0}, y_{i_1}, y_{i_2}, y_{i_3}, y_{i_4})M &= [(y_{i_3}y_{i_4}+1)F_1(y_{i_0}, y_{i_1}, y_{i_2}) + (y_{i_3}+y_{i_4})G_1(y_{i_0}, y_{i_1}, y_{i_2})]M \\ &> (y_{i_3}y_{i_4}+1)G_1(y_{i_0}, y_{i_1}, y_{i_2}) + (y_{i_3}+y_{i_4})F_1(y_{i_0}, y_{i_1}, y_{i_2}) \\ &= G_2(y_{i_0}, y_{i_1}, y_{i_2}, y_{i_3}, y_{i_4}). \end{aligned}$$

By induction we have that for any $1 \le j \le k$,

$$F_j(y_{i_0}, y_{i_1}, \dots, y_{i_{2j}}) < G_j(y_{i_0}, y_{i_1}, \dots, y_{i_{2j}})M$$

and

$$F_j(y_{i_0}, y_{i_1}, \dots, y_{i_{2j}})M > G_j(y_{i_0}, y_{i_1}, \dots, y_{i_{2j}}).$$

Thus

$$\frac{1}{M} < \frac{F_k(y_{i_0}, y_{i_1}, \dots, y_{i_{2k}})}{G_k(y_{i_0}, y_{i_1}, \dots, y_{i_{2k}})} < M.$$

Therefore, we get

$$\frac{1}{M} < \frac{cF_k(y_{i_0}, y_{i_1}, \dots, y_{i_{2k}}) + f(y_{i_0}, y_{i_1}, \dots, y_{i_{2k}})}{cG_k(y_{i_0}, y_{i_1}, \dots, y_{i_{2k}}) + f(y_{i_0}, y_{i_1}, \dots, y_{i_{2k}})} < M. \quad \square$$

Let n be a positive integer and ρ denote the part-metric on R_+^n (see [7]) which is defined by

$$\rho(x, y) = -\log\min\{\frac{x_i}{y_i}, \frac{y_i}{x_i} | 1 \le i \le n\} \quad \text{for} \quad x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in R_+^n.$$

It was shown by Thompson [7] that (R_+^n, ρ) is a complete metric space. Krause and Nussbaum [8] proved that the distances indicated by the part-metric and by the Euclidean norm are equivalent on R_+^n.

Lemma 2 ([9]) *Let $T : R_+^n \longrightarrow R_+^n$ be a continuous mapping with unique fixed point $x^* \in R_+^n$. Suppose that there exists some $l \ge 1$ such that for the part-metric ρ,*

$$\rho(T^l x, x^*) < \rho(x, x^*) \quad \text{for all} \quad x \ne x^*.$$

Then x^* is globally asymptotically stable.

Theorem 1 *The unique equilibrium $\overline{x} = 1$ of the two extensive classes of recursive sequences (1) and (2) is globally asymptotically stable.*

Proof Let $\{x_n\}_{n=-i_{2k}}^{\infty}$ be a solution of Eq.(3) with initial conditions $x_{-i_{2k}}, x_{-i_{2k}+1}, \dots, x_0 \in R_+^{i_{2k}+1}$ such that $\{x_n\}_{n=-i_{2k}}^{\infty}$ is not eventually equal to 1 since otherwise there is nothing to show. Denote by $T : R_+^{i_{2k}+1} \to R_+^{i_{2k}+1}$ the mapping

$$\begin{aligned} &T(x_{n-i_{2k}}, x_{n-i_{2k}+1}, \dots, x_n) \\ &\quad = (x_{n-i_{2k}+1}, x_{n-i_{2k}+2}, \dots, x_n, \frac{cF_k(x_{n-i_0}, x_{n-i_1}, \dots, x_{n-i_{2k}}) + f(x_{n-i_0}, x_{n-i_1}, \dots, x_{n-i_{2k}})}{cG_k(x_{n-i_0}, x_{n-i_1}, \dots, x_{n-i_{2k}}) + f(x_{n-i_0}, x_{n-i_1}, \dots, x_{n-i_{2k}})}). \end{aligned}$$

Then solution $\{x_n\}_{n=-i_{2k}}^{\infty}$ of Eq.(3) is represented by the first component of the solution $\{y_n\}_{n=0}^{\infty}$ of the system $y_{n+1} = Ty_n$ with initial condition $y_0 = (x_{-i_{2k}}, x_{-i_{2k}+1}, \dots, x_0)$. It follows from

Lemma 1 that for all $n \geq 0$ the following inequalities hold:

$$x_{n+1} > \min\{x_n, x_{n-1}, \ldots, x_{n-i_{2k}}, \frac{1}{x_n}, \frac{1}{x_{n-1}}, \ldots, \frac{1}{x_{n-i_{2k}}}\},$$

$$x_{n+1} < \max\{x_n, x_{n-1}, \ldots, x_{n-i_{2k}}, \frac{1}{x_n}, \frac{1}{x_{n-1}}, \ldots, \frac{1}{x_{n-i_{2k}}}\}.$$

Thus, for $x^* = (1, 1, \ldots, 1)$ and the part-metric ρ we have

$$\rho(T^{i_{2k}+1}(y_n), x^*) < \rho(y_n, x^*)$$

for all $n \geq 0$. It follows from Lemma 2 that the positive equilibrium $\overline{x} = 1$ of Eq.(3) is globally asymptotically stable. Therefore, the positive equilibrium $\overline{x} = 1$ of the two extensive classes of recursive sequences (1) and (2) is globally asymptotically stable. □

References

[1] HAMZA A E, KHALAF-ALLAH R. *On the recursive sequence* $x_{n+1} = \frac{A\prod_{i=l}^{k} x_{n-2i-1}}{B+C\prod_{i=l}^{k-1} x_{n-2i}}$ [J]. Comput. Math. Appl., 2008, **56**(7): 1726–1731.

[2] NESEMANN T. *Positive nonlinear difference equations: some results and applications* [J]. Nonlinear Anal., 2001, **47**(7): 4707–4717.

[3] PAPASCHINOPOULOS G, SCHINAS C J. *Global asymptotic stability and oscillation of a family of difference equations* [J]. J. Math. Anal. Appl., 2004, **294**(2): 614–620.

[4] LI Xianyi. *Global behavior for a fourth-order rational difference equation* [J]. J. Math. Anal. Appl., 2005, **312**(2): 555–563.

[5] LI Xianyi. *Qualitative properties for a fourth-order rational difference equation* [J]. J. Math. Anal. Appl., 2005, **311**(1): 103–111.

[6] AMLEH A M, KRUSE N, LADAS G. *On a class of difference equations with strong negative feedback* [J]. J. Differ. Equations Appl., 1999, **5**(6): 497–515.

[7] THOMPSON A C. *On certain contraction mappings in a partially ordered vector space* [J]. Proc. Amer. Math. Soc., 1963, **14**: 438–443.

[8] KRAUSE U, NUSSBAUM R D. *A limit set trichotomy for self-mappings of normal cones in Banach spaces* [J]. Nonlinear Anal., 1993, **20**(7): 855–870.

[9] KRUSE N, NESEMANN T. *Global asymptotic stability in some discrete dynamical systems* [J]. J. Math. Anal. Appl., 1999, **235**(1): 151–158.