Journal of Mathematical Research & Exposition
Sept., 2010, Vol. 30, No. 5, pp. 936–938
DOI:10.3770/j.issn:1000-341X.2010.05.023
Http://jmre.dlut.edu.cn

Some Characterizations of Algebras of Finite Representation Type

Shu Gui WANG
Department of Mathematics, Huaihua College, Hunan 418008*, P. R. China*

Abstract Let A be a finite dimensional, connected, basic algebra over an algebraically closed field. We prove that A is of finite representation type if and only if there is a natural number m such that $\mathrm{rad}^m(\mathrm{End}(M)) = 0$, for any indecomposable A-modules M. This gives a partial answer to one of problems posed by Skowroński.

Keywords connected basic algebra; algebras of finite representation type; Auslander-Reiten component.

Document code A
MR(2000) Subject Classification 16G10; 16G20
Chinese Library Classification O153.3

1. Introduction

Let A be a finite dimensional algebra over an algebraically closed field K. mod-A denotes the category of all finitely generated left A-modules. If there are only finite many indecomposable modules (up to isomorphisms) in mod-A, then A is said to be of finite representation type. Associated to the algebra A, a quiver Γ_A which now is called the Auslander-Reiten quiver [4] of A can be defined as follows: the vertices of Γ are given by the set Γ_0 of isomorphism classes $[X]$ of indecomposable modules X; there are m arrows from $[X]$ to $[Y]$ in Γ if the $\dim_K \mathrm{Irr}(X,Y) = m$; the translation in Γ is the Auslander-Reiten translation $\tau_A = DTr$. Where $\mathrm{Irr}(X,Y) = \frac{\mathrm{rad}(X,Y)}{\mathrm{rad}^2(X,Y)}$. A connected component of Γ will be called an Auslander-Reiten component of A. The Auslander-Reiten quiver of A and its components are important research objects and have a lot of important homological nature in the representation theory of the algebra A, see, for example, [4] or [5]. In the note, we will give some characterizations of algebras of finite representation type in terms of components, one of which gives an affirmative answer to the problem 11 posed by Skowroński in [5].

Before we prove the theorem, we will state a lemma. For the completeness, we give a proof here.

Lemma 1 *Let $F : K\langle T_1, T_2\rangle$-mod$\longrightarrow$ A-mod be an embedding functor (i.e., an exact functor which respects isomorphism classes and carries indecomposable modules to indecomposable*

Received October 29, 2008; Accepted July 17, 2009
Supported by the Education Department Foundation of Hunan Province (Grant No. 04C469).
E-mail address: wsg1009@163.com

modules). Then F is faithful.

Proof Let F be such an embedding functor. Then $F = M \otimes -$, where M is an $A - K\langle T_1, T_2\rangle$-bimodule which is a free right $K\langle T_1, T_2\rangle$-module of finite rank and $\otimes$ is the tensor product over the algebra $K\langle T_1, T_2\rangle$ (see [2,3]). Assume that $f : N \longrightarrow E$ is a non-zero morphism in $K\langle T_1, T_2\rangle$-mod, it follows that there is an exact sequence

$$0 \longrightarrow \ker(f) \xrightarrow{i} N \xrightarrow{f} E \xrightarrow{j} \mathrm{cok}(f) \longrightarrow 0.$$

Applying $F = M \otimes -$ to the exact sequence, we get an exact sequence

$$0 \to M \otimes \ker(f) \xrightarrow{1\otimes i} M \otimes N \xrightarrow{1\otimes f} M \otimes E \xrightarrow{1\otimes j} M \otimes \mathrm{cok}(f) \longrightarrow 0.$$

Suppose $F(f) = 1 \otimes f = 0$, then $1 \otimes i : M \otimes \ker(f) \longrightarrow M \otimes N$ is an isomorphism. Then $i : \ker(f) \longrightarrow N$ is an isomorphism, which implies that f is zero, leading to a contradiction. $\square$

2. The main result

Theorem 2 *Let A be a finite dimensional algebra over an algebraically closed field K. Then the following statements are equivalent.*

(1) A is of finite representation type;

(2) Each component of Γ_A is generalizing standard [5], and Γ_A has finitely many vertices lying on oriented cycles;

(3) A has finitely many generalizing standard connected components;

(4) There is a positive integer m such that $\mathrm{rad}^m(\mathrm{End}_A(M)) = 0$ for any modules M in ind A.

Proof The implications from (1) to (2), (1) to (3) and (1) to (4) are trivial. We will prove all the inverse implications.

(2) $\Longrightarrow$ (1). Suppose A is representation-infinite. Then, by the famous tame-and-wild theorem by Drozd [2,3], A is either of tame type (not finite type) or of wild type. If A is of tame representation type, then A has infinitely many indecomposable modules lying on stable tubes, which contradicts the condition (2). Then we have that A is of wild type. Then we have an embedding functor $F : K\langle X_1, X_2\rangle$-mod$\longrightarrow A$-mod. Let $A_1 := kQ$ be the Kronecker algebra. Then there is a faithful and full functor $F_1 : A_1$-mod$\longrightarrow K\langle X_1, X_2, X_3, X_4\rangle$-mod. But we have a faithful and full functor $G : K\langle X_1, X_2, X_3, X_4\rangle$-mod$\longrightarrow K\langle X_1, X_2\rangle$-mod. So the composition of the three functors $FGF_1 : A_1$-mod $\longrightarrow A$-mod is a faithful functor since F is faithful by Lemma 1. It is denoted by H for simplicity. Let $M(a)$ be a regular module of regular length 2, for any $a \in \mathbb{P}^1(\mathbb{K})$, where $\mathbb{P}^1(\mathbb{K})$ is the 1-dimension projective line. Then there exists a non-zero and non-isomorphism $f(a) : M(a) \longrightarrow M(a)$. It follows that $H(f(a))$ is not zero and is not an isomorphism. Since each component of A is generalizing standard, $H(f(a))$ is a sum of compositions of irreducible maps. It follows that $H(M(a))$ lies on an oriented cycle in Γ_A for any $a \in \mathbb{P}^1(\mathbb{K})$, which contradicts the condition (2).

(3) $\Longrightarrow$ (1). Let $\Gamma_1, \ldots, \Gamma_t$ be all connected components of A. Suppose A is representation-

infinite. Then, by Drozd tame-and-wild theorem [2, 3], A is either of tame type (not finite type) or of wild type. If A is of tame representation type, then A has infinite many connected components [2], leading to a contradiction. Then A is of wild type. As in the proof of part (2) $\Longrightarrow$ (1), we get that the functor $H : A_1\text{-mod} \longrightarrow A\text{-mod}$ is faithful, where A_1 is assumed to be a tubular algebra since

$$0 \neq \mathrm{Hom}_{A_1}(A_1, D(A_1)) = \mathrm{rad}^{\infty}(A_1, D(A_1)) = (\mathrm{rad}^{\infty}(A_1, D(A_1)))^2 = \cdots .$$

Let $f : M \to N$, where M (or N) is indecomposable projective (or injective, respectively) A_1-module, such that $f = f_t \cdots f_2 f_1$, where $f_i : M_i \to M_{i+1}$ and $M_2, \ldots, M_t$ are indecomposable modules lying on distinct tubular families and $M_1 = M$, $M_{t+1} = N$. Thus $f_i \in \mathrm{rad}^{\infty}(M, N)$, $\forall i$. It follows that $H(f_i) \in \mathrm{rad}^{\infty}(H(M), H(N))$, $\forall i$. Then $H(f) = H(f_t) \cdots H(f_2)H(f_1)$ is not zero by Lemma 1. Then there are $0 < t_1 < t_2 < t + 1$ such that $H(M_{t_1})$, $H(M_{t_2})$ are in the same connected component, say, Γ_m, which implies that $\mathrm{rad}^{\infty}(H(M_{t_1}), H(M_{t_2})) = 0$, by the generalizing standardness of Γ_m. It follows that $H(f) = 0$. This is a contradiction.

(4) $\Longrightarrow$ (1). Suppose A is representation-infinite. Then, by Drozd tame-and-wild theorem, A is either of tame type (not finite type) or of wild type. If A is of tame representation type, then A has infinite many homogeneous tubes [2]. We choose one, say $T(a)$, and the regular simple modules in $T(a)$ are denoted by $S(a)$. For any m, let $E(a)[m]$ be the indecomposable regular module in $T(a)$ of regular length m. Then there exists a morphism from $E(a)[m]$ to itself with $f(a)^m = 0$ and $f(a)^{m-1} \neq 0$. This contradicts the condition (2). Therefore A is of wild type. As in the proof of part (2) $\Longrightarrow$ (1), we have got an embedding functor $H : A_1\text{-mod} \longrightarrow A\text{-mod}$ which is faithful, where A_1 is assumed to be the Kronecker algebra. Using a similar argument as above, we obtain a regular A_1-module $E[a][m]$ such that there exists a non-zero morphism f from $E[a][m]$ to itself with $f^m = 0$ but $f^{m-1} \neq 0$, for any m and $a \in \mathbb{P}^1(\mathbb{K})$. It follows from $H(f)^m = H(f^m) = 0$ and $H(f)^{m-1} = H(f^{m-1})$ that A has infinitely many indecomposable modules $H(E[a][m])$, denoted by $M(a)_m$, with property that $(\mathrm{rad}(\mathrm{End} M_m))^{m-1} \neq 0$, $\forall m$. It is a contradiction. Then A is representation-finite. $\square$

Remark We point out that the equivalence between (1) and (4) gives an affirmative answer to the problem 11 posed by Skowroński in [5] for algebras over an algebraically closed field.

References

[1] AUSLANDER M, REITEN I. *Representation theory of Artin algebras (III)* [J]. Comm. Algebra, 1975, **3**: 239–294.

[2] CRAWLEY-BOEVEY W W. *On tame algebras and bocses* [J]. Proc. London Math. Soc. (3), 1988, **56**(3): 451–483.

[3] DROZD JU A. *Tame and Wild Matrix Problems* [M]. Springer, Berlin-New York, 1980.

[4] RINGEL C M. *Tame Algebras and Integral Quadratic Forms* [M]. Springer-Verlag, Berlin, 1984.

[5] SKOWROŃSKI A. *Cycles in module categories* [M]. in: Finite Dimensional Algebras and Related Topics, NATO ASI Series C, Vol. 424, Kluwer Acad. Publ., 1994, 309-345.