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Congruences for a Restricted m-ary Overpartition Function

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Abstract We discuss a family of restricted m-ary overpartition functions $\bar{b}_{m,j}(n)$, which is the number of m-ary overpartitions of n with at most $i+j$ copies of the non-overlined part m^i allowed, and obtain a family of congruences for $\bar{b}_{m,lm-1}(n)$.

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1. Introduction

An overpartition of n is a non-increasing sequence of positive integers whose sum is n in which the first occurrence of an integer may be overlined. According to Corteel and Lovejoy [1], overpartitions were discussed by MacMahon and have proven useful in several combinatorial studies of basic hypergeometric series [2–5].

We denote by $\bar{p}(n)$ the number of overpartitions of n. Since the overlined parts form a partition into distinct parts and the non-overlined parts form an ordinary partition, we have the generating function

$$\sum_{n=0}^{\infty}\bar{p}(n)q^n=\prod_{n=1}^{\infty}\frac{1+q^n}{1-q^n}.$$

Let $m\geq 2$ be an integer. An m-ary partition of a positive integer n is a non-increasing sequence of non-negative integral powers of m whose sum is n. In 2005, Rødseth and Sellers [6] considered m-ary overpartition of n, which is a non-increasing sequence of non-negative integral powers of m whose sum is n, and where the first occurrence of a power of m may be overlined. They obtained a congruence property (Theorem 1.1) which is a lifting to general m of the well-known Churchhouse congruences [7] for the binary partition function. They also considered the number of restricted m-ary overpartitions of n, where the largest part is at most m^{k-1}.

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In this note, we consider another family of restricted m-ary overpartition function $\bar{b}_{m,j}(n)$, which is the number of m-ary overpartitions of n with at most $i+j$ copies of the non-overlined part m^i allowed. For example, for $m=2$, $j=1$, we find

$$\sum_{n=0}^{\infty} \bar{b}_{2,1}(n)q^n = 1 + 2q + 3q^2 + 4q^3 + 6q^4 + \cdots,$$

where the 6 restricted binary overpartitions of 4 are

$$4, \bar{4}, 2+2, \bar{2}+2, 2+\bar{1}+1, \bar{2}+\bar{1}+1.$$

It is not difficult to see that $m-1$ is the smallest integer j which guarantees that $\bar{b}_{m,j}(n)$ is positive for all nonnegative integers n. This makes the study of this specific function especially attractive.

Theorem 1.1 *For all $n \geq 0$, $m \geq 4$, $3 \leq k \leq m-1$, and $1 \leq t \leq m-k+1$, we have*

$$\begin{aligned}\sum_{n=0}^{\infty} &\bar{b}_{m,m-1}(m^{k+t}n + 2m^{k+t-1} + \cdots + 2m^k)q^n \\ &= (2^{k+1} + 2^k - 2)4^{t-1}(1+q)B_{m,m+k+t-1}(q),\end{aligned}$$

where $B_{m,j}(q)$ is the generating function $\sum_{n\geq 0} \bar{b}_{m,j}(n)q^n$.

Remark 1.2 Theorem 1.1 implies

$$\begin{aligned}&\bar{b}_{m,m-1}(m^{k+t}n + 2m^{k+t-1} + \cdots + 2m^k) \\ &\quad = (2^{k+1} + 2^k - 2)4^{t-1}(\bar{b}_{m,m+k+t-1}(n) + \bar{b}_{m,m+k+t-1}(n-1)).\end{aligned}$$

Of course, it also implies the following congruence for $n \geq 0$, $m \geq 4$, $3 \leq k \leq m-1$, and $1 \leq t \leq m-k+1$:

$$\bar{b}_{m,m-1}(m^{k+t}n + 2m^{k+t-1} + \cdots + 2m^k) \equiv 0 \ (\mathrm{mod}(2^{k+1} + 2^k - 2)4^{t-1}).$$

We prove Theorem 1.1 by using generating function dissections. In Section 2 below we give two preliminary lemmas. In Section 3 we complete the proof of Theorem 1.1. Finally, in Section 4 we state a theorem which deals with function $\bar{b}_{m,lm-1}(n)$.

2. Preliminary lemmas

Denote by $\bar{b}_m(n)$ the number of m-ary overpartitions of n, and put $\bar{b}_m(0) = 1$. It is clear that the generating function for $\bar{b}_m(n)$ is given by

$$\sum_{n\geq 0} \bar{b}_m(n)q^n = \prod_{i\geq 0}\Big((1+q^{m^i})\sum_{k\geq 0} q^{m^i k}\Big) = \prod_{i\geq 0} \frac{1+q^{m^i}}{1-q^{m^i}}.$$

We see that the generating function for $\bar{b}_{m,j}(n)$ can be written as

$$B_{m,j}(q) = \sum_{n\geq 0} \bar{b}_{m,j}(n)q^n = \prod_{i\geq 0}\Big((1+q^{m^i})\sum_{k=0}^{i+j} q^{m^i k}\Big)$$

$$= (1+q)(1+q+\cdots+q^j)B_{m,j+1}(q^m). \tag{1}$$

Lemma 2.1 *For all $n \geq 0$, $m \geq 2$, and $1 \leq k \leq m-1$, we have*

$$\bar{b}_{m,m-1}(m^k n) = \bar{b}_{m,m+k-1}(n) + (2^k + 2^{k-1} - 2)\bar{b}_{m,m+k-1}(n-1).$$

Proof We prove this lemma by induction on k. We first consider the case $k = 1$. We have from (1) the following

$$\begin{aligned} B_{m,m-1}(q) &= (1+q)(1+q+\cdots+q^{m-1})B_{m,m}(q^m) \\ &= (1+2q+\cdots+2q^{m-1}+q^m)B_{m,m}(q^m). \end{aligned}$$

Then the coefficient of q^{mn} on the left-hand side of (2) is simply $\bar{b}_{m,m-1}(mn)$. We see that the terms in $(1+2q+\cdots+2q^{m-1}+q^m)$ that contribute to a term of the form q^{mn} on the right-hand side of (2) are 1 and q^m, because $B_{m,m}(q^m)$ is a power series in q^m. Therefore, the coefficient of q^{mn} on the right-hand side of (2) is $\bar{b}_{m,m}(n) + \bar{b}_{m,m}(n-1)$.

Now, we assume the lemma is true for some k satisfying $1 \leq k < m-1$. This means we are assuming that

$$\bar{b}_{m,m-1}(m^k n) = \bar{b}_{m,m+k-1}(n) + (2^k + 2^{k-1} - 2)\bar{b}_{m,m+k-1}(n-1)$$

or that

$$\sum_{n\geq 0} \bar{b}_{m,m-1}(m^k n)q^n = (1 + (2^k + 2^{k-1} - 2)q)B_{m,m+k-1}(q).$$

Then we have

$$\begin{aligned} &\bar{b}_{m,m-1}(m^{k+1}n) = [q^{mn}](1 + (2^k + 2^{k-1} - 2)q)B_{m,m+k-1}(q) \\ &\quad = [q^{mn}](1 + (2^k + 2^{k-1} - 2)q)(1+q)(1+q+\cdots+q^{m+k-1})B_{m,m+k}(q^m) \\ &\quad = [q^{mn}](1 + (2^k + 2^{k-1} - 1)q + (2^k + 2^{k-1} - 2)q^2)\cdot(1+q+\cdots+q^{m+k-1})B_{m,m+k}(q^m) \\ &\quad = [q^{mn}](1 + q^m + (2^k + 2^{k-1} - 1)q\cdot q^{m-1} + \\ &\qquad (2^k + 2^{k-1} - 2)q^2\cdot q^{m-2})B_{m,m+k}(q^m) \text{ (because of } 1 \leq k < m-1) \\ &\quad = [q^{mn}](1 + (2^{k+1} + 2^k - 2)q^m)B_{m,m+k}(q^m) \\ &\quad = [q^n](1 + (2^{k+1} + 2^k - 2)q)B_{m,m+k}(q) \\ &\quad = \bar{b}_{m,m+k}(n) + (2^{k+1} + 2^k - 2)\bar{b}_{m,m+k}(n-1). \end{aligned}$$

This completes the proof of Lemma 2.1. □

Lemma 2.2 *For all $n \geq 0$, $m \geq 4$, $3 \leq k \leq m-1$, we have*

$$\bar{b}_{m,m-1}(m^{k+1}n + 2m^k) = (2^{k+1} + 2^k - 2)(\bar{b}_{m,m+k}(n) + \bar{b}_{m,m+k}(n-1)).$$

Proof By Lemma 2.1, we have

$$\begin{aligned} &\bar{b}_{m,m-1}(m^{k+1}n + 2m^k) = [q^{mn+2}](1 + (2^k + 2^{k-1} - 2)q)B_{m,m+k-1}(q) \\ &\quad = [q^{mn+2}](1 + (2^k + 2^{k-1} - 2)q)(1+q)(1+q+\cdots+q^{m+k-1})B_{m,m+k}(q^m) \\ &\quad = [q^{mn+2}](2^{k+1} + 2^k - 2)q^2(1+q^m)B_{m,m+k}(q^m) \text{ (because of } 3 \leq k \leq m-1) \end{aligned}$$

$$= [q^n](2^{k+1} + 2^k - 2)(1+q)B_{m,m+k}(q)$$
$$= (2^{k+1} + 2^k - 2)(\bar{b}_{m,m+k}(n) + \bar{b}_{m,m+k}(n-1)).$$

This completes the proof of Lemma 2.2. □

3. Proof of Theorem 1.1

We prove Theorem 1.1 by induction on t. The case $t = 1$ is handled in Lemma 2.2. We now assume

$$\sum_{n=0}^{\infty} \bar{b}_{m,m-1}(m^{k+t-1}n + 2m^{k+t-2} + \cdots + 2m^k)q^n$$
$$= (2^{k+1} + 2^k - 2)4^{t-2}(1+q)B_{m,m+k+t-2}(q)$$

for $2 \leq t \leq m - k + 2$, i.e.,

$$\bar{b}_{m,m-1}(m^{k+t-1}n + 2m^{k+t-2} + \cdots + 2m^k)$$
$$= (2^{k+1} + 2^k - 2)4^{t-2}(\bar{b}_{m,m+k+t-2}(n) + \bar{b}_{m,m+k+t-2}(n-1)).$$

Then we have

$$\bar{b}_{m,m-1}(m^{k+t}n + 2m^{k+t-1} + \cdots + 2m^k)$$
$$= [q^{mn+2}](2^{k+1} + 2^k - 2)4^{t-2}(1+q)B_{m,m+k+t-2}(q)$$
$$= [q^{mn+2}](2^{k+1} + 2^k - 2)4^{t-2}(1+q)(1+q) \cdot (1 + q + \cdots + q^{m+k+t-2})B_{m,m+k+t-1}(q^m)$$
$$= [q^{mn+2}](2^{k+1} + 2^k - 2)4^{t-1}q^2(1+q^m)B_{m,m+k+t-1}(q^m)$$
$$= [q^n](2^{k+1} + 2^k - 2)4^{t-1}(1+q)B_{m,m+k+t-1}(q)$$
$$= (2^{k+1} + 2^k - 2)4^{t-1}(\bar{b}_{m,m+k+t-1}(n) + \bar{b}_{m,m+k+t-1}(n-1)).$$

This completes the proof of Theorem 1.1. □

4. More congruences for $\bar{b}_{m,j}(n)$

In this section, we consider the congruence properties for $\bar{b}_{m,j}(n)$ with $j = lm-1$. Proceeding in a way similar to, but a little bit more complicated than, the proof of Theorem 1.1, we can prove the following result which is an extension of Theorem 1.1.

Theorem 4.1 *For all $n \geq 0$, $m \geq 4$, $1 \leq l \leq m/2 - 1$, $l + 2 \leq k \leq m - l$, and $1 \leq t \leq m - k + 1$, we have*

$$\sum_{n \geq 0} \bar{b}_{m,lm-1}(m^{k+t}n + (l+1)m^{k+t-1} + \cdots + (l+1)m^k)q^n$$
$$= 2^t(l+1)^{t-1} \prod_{\substack{0 \leq r \leq k \\ r \neq k-1}} (2l)^r(1 + q + \cdots + q^l)B_{m,m+k+t-1}(q).$$

Remark 4.2 Theorem 4.1 implies the following congruence:

$$\bar{b}_{m,lm-1}(m^{k+t}n+(l+1)m^{k+t-1}+\cdots+(l+1)m^k)\equiv 0 \left(\bmod 2^t(l+1)^{t-1}\prod_{\substack{0\le r\le k\\ r\ne k-1}}(2l)^r\right).$$

We now sketch a proof of Theorem 4.1. We can first prove

Lemma 4.3 *For all $n\ge 0$, $m\ge 2$, and $l\ge 1$, we have*

$$\bar{b}_{m,lm-1}(mn)=\bar{b}_{m,lm}(n)+2\bar{b}_{m,lm}(n-1)+\cdots+2\bar{b}_{m,lm}(n-l+1)+\bar{b}_{m,lm}(n-l).$$

By using Lemma 4.3 and induction on k, we can prove

Lemma 4.4 *For all $n\ge 0$, $m\ge 3$, $1\le l\le m-2$, and $2\le k\le m-l$, we have*

$$\begin{aligned}\bar{b}_{m,lm-1}(m^kn)=&\bar{b}_{m,lm+k-1}(n)+2\prod_{\substack{0\le r\le k-1\\ r\ne k-2}}(2l)^r(\bar{b}_{m,lm+k-1}(n-1)+\cdots+\\ &\bar{b}_{m,lm+k-1}(n-l)).\end{aligned}$$

By using Lemma 4.4, we can prove

Lemma 4.5 *For all $n\ge 0$, $m\ge 4$, $1\le l\le m/2-1$, and $l+2\le k\le m-l$, we have*

$$\sum_{n\ge 0}\bar{b}_{m,lm-1}(m^{k+1}n+(l+1)m^k)=2\prod_{\substack{0\le r\le k\\ r\ne k-1}}(2l)^r(1+q+\cdots+q^l)B_{m,m+k}(q).$$

Finally, by using Lemma 4.5 and induction on t, we can prove Theorem 4.1.

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