Journal of Mathematical Research & Exposition
Sept., 2010, Vol. 30, No. 5, pp. 944–946
DOI:10.3770/j.issn:1000-341X.2010.05.025
Http://jmre.dlut.edu.cn

A Note on the 3-Edge-Connected Supereulerian Graphs

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Abstract For two integers $l > 0$ and $k \geq 0$, define $C(l, k)$ to be the family of 2-edge connected graphs such that a graph $G \in C(l, k)$ if and only if for every bond $S \subseteq E(G)$ with $|S| \leq 3$, each component of $G - S$ has order at least $(|V(G)| - k)/l$. In this note we prove that if a 3-edge-connected simple graph G is in $C(10, 3)$, then G is supereulerian if and only if G cannot be contracted to the Petersen graph. Our result extends an earlier result in [Supereulerian graphs and Petersen graph. JCMCC 1991, 9: 79-89] by Chen.

Keywords supereulerian; collapsible; reduction; 3-edge-connected.

Document code A
MR(2000) Subject Classification 05C45
Chinese Library Classification O157.5

1. Introduction

We follow Bondy and Murty [1] for undefined notations and terminology. For a graph G, let $\kappa'(G)$ denote the edge-connectivity of G. If G has a spanning eulerian subgraph, then it is called supereulerian. We write SL for all supereulerian graphs. In particular, K_1 is both eulerian and supereulerian. Let P denote the Petersen graph.

Let $l > 0$, $k \geq 0$ be two integers and let $C(l, k)$ denote a class of 2-connected graphs of order n such that a graph $G \in C(l, k)$ if and only if for every bond $S \subseteq E(G)$ with $|S| \leq 3$, each component of $G - S$ has order at least $(n - k)/l$. For 3-edge-connected graphs, Chen presented the following result.

Theorem 1 ([6, Theorem 2]) *Let G be a 3-edge-connected graph. If $G \in C(10, 0)$, then G is supereulerian if and only if G cannot be contracted to P.*

In this note we shall prove the following result which extends Theorem 1.

Theorem 2 *Let G be a simple graph. If $G \in C(10, 3)$ with $\kappa'(G) \geq 3$, then G is supereulerian if and only if G cannot be contracted to P.*

In the proof of Theorem 2 we need Catlin's reduction method.

A graph H is called collapsible if for every even set $X \subseteq V(H)$, there is a spanning connected subgraph H_X of H such that X is the set of vertices with odd degree in H_X. In [2], Catlin showed

Received August 19, 2008; Accepted June 30, 2009
Supported by the Science Foundation of Chongqing Education Committee (Grant No. KJ100725).
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that every graph G has a unique collection of pairwise disjoint maximal collapsible subgraphs $H_1, H_2, \ldots, H_c$. The reduction of G is obtained from G by contracting each H_i into a single vertex v_{H_i} (H_i is called the preimage of v_{H_i}) and by deleting any resulting loops ($1 \le i \le c$). A graph is reduced if it is the reduction of some graph, and K_1 is the only collapsible reduced graph. Let $F(G)$ be the minimum number of edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. The following theorems are useful in the proof of our main result.

Theorem 3 ([2, Theorem 8]) *Let G be a connected graph. Each of the following holds:*

(i) Let G' be the reduction of G. Then $G \in SL$ if and only if $G' \in SL$.

(ii) If G is simple and nontrivial collapsible graph, then $|V(G)| \ge 3$.

Theorem 4 *Let G be a connected reduced graph. Each of the following holds:*

(i) [4, Theorem 1.5] If $F(G) \le 2$, then $G \in \{K_1, K_2, K_{2,t}\ (t \ge 1)\}$.

(ii) [3] $F(G) = 2|V(G)| - |E(G)| - 2$.

Theorem 5 ([5]) *Let G be a connected graph with $|V(G)| \le 13$ and $\delta(G) \ge 3$. Then either G is a supereulerian graph with 12 vertices and with an odd cycle, or the reduction of G is in $\{K_1, K_2, K_{1,2}, K_{1,3}, P\}$.*

2. The proof of main result

Proof of Theorem 2 Let G' be the reduction of G. Denote $D_i = \{v | d(v) = i, v \in V(G')\}$ and $d_i = |D_i|$, for each interger $i \ge 1$. By (i) of Theorem 3, it suffices to show $G' \notin SL$ if and only if $G' = P$. Suppose that $G' \notin SL$.

Claim 1 $d_1 = d_2 = 0$ and $F(G') \ge 3$.

Since $\kappa'(G) \ge 3$, we have $\kappa'(G') \ge 3$ and $\delta(G') \ge 3$. Therefore $d_1 = d_2 = 0$.

If $F(G') \le 2$, then by (i) of Theorem 4, $G' \in \{K_1, K_2, K_{2,t}(t \ge 1)\}$. As $G' \notin SL$, $G' \ne K_1$, so $G' \in \{K_2, K_{2,t}(t \ge 1)\}$, contrary to $\kappa'(G') \ge 3$. Hence Claim 1 holds.

Claim 2 $d_3 \ge 10$.

By (ii) of Theorem 4 we have $F(G') = 2|V(G')| - |E(G')| - 2$. As $|V(G')| = \sum_{j \ge 1} d_j$ and $2|E(G')| = \sum_{j \ge 1} j d_j$. Therefore

$$d_3 = 4 + 2F(G^{'}) + \sum_{j \ge 5} (j-4) d_j. \tag{1}$$

By (1), if $d_3 \le 9$, then $2F(G') = d_3 - 4 - \sum_{j \ge 5}(j-4)d_j \le 5 - \sum_{j \ge 5}(j-4)d_j < 6$, hence $F(G') < 3$, contrary to Claim 1. Hence $d_3 \ge 10$.

If $d_3 = 10$, let $H_1, H_2, \ldots, H_{10}$ be the preimages of those vertices in D_3. Let $V_4' = \{v | d(v) > 3, v \in V(G')\}$ and $|V(G)| = n$. Since $G \in C(10, 3)$, we have

$$n \ge \sum_{i=1}^{10} |V(H_i)| + |V_4'| \ge n - 3 + |V_4'|.$$

So $|V_4'| \leq 3$ and $|V(G')| \leq 13$, then by Theorem 5, by $G' \notin SL$ and by $\delta(G') \geq 3$, we have $G' = P$.

If $d_3 \geq 11$, let $c = d_3$, and $H_1, H_2, \ldots, H_c$ be the preimages of those vertices in D_3. Therefore,

$$|V(G)| \geq \sum_{i=1}^{c} |V(H_i)| \geq 11 \frac{|V(G)| - 3}{10},$$

hence $|V(G)| \leq 33$.

Case 1 $14 \leq |V(G)| \leq 33$.

For each $i = 1, 2, \ldots, c$, by $G \in C(10, 3)$, $|V(H_i)| \geq \frac{|V(G)|-3}{10} > 1$, then H_i is nontrivial collapsible, so by (ii) of Theorem 3 $|V(H_i)| \geq 3$. Thus $|V(G')| \leq 11$. Hence by $d_3 \geq 11$, we have $|V(G')| = d_3 = 11$, a contradiction obtained.

Case 2 $|V(G)| \leq 13$.

Since $\kappa'(G) \geq 3$, $\delta(G) \geq 3$. By Theorem 5, either $G \in SL$ or $G = P$. If $G \in SL$, by (i) of Theorem 3, $G' \in SL$, contrary to the assumption, hence $G = P$ and $G' = P$.

This completes the proof of Theorem 2. □

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