Journal of Mathematical Research & Exposition
Sept., 2010, Vol. 30, No. 5, pp. 947–950
DOI:10.3770/j.issn:1000-341X.2010.05.026
Http://jmre.dlut.edu.cn

A Note on Maximal Spectral Distance

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Abstract Two equivalent conditions are given for the non-commutative unital C^*-subalgebras of $\mathcal{B}(\mathcal{H})$ algebras $\mathcal{A}$ and $\mathcal{B}$ to be quantum C^* independent.

Keywords maximal spectral distance; quantum C^* independent.

Document code A
MR(2000) Subject Classification 47A13; 14E20; 46E25; 20C20
Chinese Library Classification O177.2

1. Introduction

The distance between normal matrices is an interesting topic and many mathematicians obtained many beautiful results [2, 3, 7, 9, 11].

Let A and B be any normal $n \times n$ matrices. If their eigenvalues are denoted by $\lambda_1, \dots, \lambda_n$ and $\mu_1, \dots, \mu_n$, then the norm distance of A and B, (i.e., $\|A - B\|$), given in terms of their eigenvalues is the following:

$$\min_{\sigma \in S_n} \max_{1 \le i \le n} |\lambda_i - \mu_{\sigma(i)}| \le \|A - B\| \le \max_{\sigma \in S_n} \max_{1 \le i \le n} |\lambda_i - \mu_{\sigma(i)}|,$$

where $A - B$ is normal and S_n denotes the set of permutations of n indices. The left side was proved by Bhatia [1] and the right side was showed by Sunder [12]. The special case of A, B being Hermitian was obtained by Weyl in [13].

Holbrook, Omladic and Semel in [8] introduced spectral function

$$F(\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n) = \max_{U \in \mathcal{U}} \|A - UBU^*\|$$

and gave a clear study of the function, then sufficient conditions are given for the equality of the spectral function and the maximal spectral distance.

The global property of the maximal spectral distance is considered in this paper, that is, if $\mathcal{A}$ and $\mathcal{B}$ are mutually commutative non-commutative C^*-subalgebras of $\mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$, whether for all normal operators $A \in \mathcal{A}$, $B \in \mathcal{B}$, the equality

$$\|A - B\| = \max_{\lambda \in \sigma(A), \mu \in \sigma(B)} |\lambda - \mu|$$

Received July 8, 2008; Accepted March 17, 2009
Supported by the National Natural Science Foundation of China (Grant No. 10371049) and the Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20050183002).
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holds. This is easily known false. However, an equivalent condition for the equality is that $\mathcal{A}$ and $\mathcal{B}$ are quantum C^* independent.

Recall that $\mathcal{A}$ and $\mathcal{B}$ are quantum independent if for any state ϕ_1 on $\mathcal{A}$ and ϕ_2 on $\mathcal{B}$, there is a state ϕ on $C^*(\mathcal{A},\mathcal{B})$, such that $\phi|_{\mathcal{A}} = \phi_1$ and $\phi|_{\mathcal{B}} = \phi_2$ as shown in [6].

Theorem 1.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be mutually commutative non-commutative C^*-subalgebras of $\mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

(1) $\|A-B\| = \max_{\lambda\in\sigma(A),\mu\in\sigma(B)} |\lambda-\mu|$, *for all normal operators $A \in \mathcal{A}$, $B \in \mathcal{B}$.*

(2) $\|A+B\| = \|A\| + \|B\|$, *for all positive operators $A \in \mathcal{A}$, $B \in \mathcal{B}$.*

(3) $\mathcal{A}$ and $\mathcal{B}$ are quantum C^* independent.

2. Proof of the theorem

In quantum theory, observables are represented by self-adjoint operators and preparations by states on the $*$-algebra generated by the observables. If $\mathcal{A}$ and $\mathcal{B}$ represent the algebras generated by the observables associated with two quantum subsystems, the quantum independence of $\mathcal{A}$ and $\mathcal{B}$ can be constructed as follows: no choice of a state prepared on one system can prevent the preparation of any state on the other subsystem.

Ross in [10] showed another characterization of quantum independence as follows:

Lemma 2.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be mutually commutative C^*-subalgebras of $\mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

(1) $\mathcal{A}$ and $\mathcal{B}$ are quantum C^* independent.

(2) $0 \neq A \in \mathcal{A}$, $0 \neq B \in \mathcal{B}$ *imply* $AB \neq 0$.

Goldstein, Luczak, and Wilde in [5] showed that the condition 2 in Lemma 2.1 is equivalent to that $A > 0$, $B > 0$ imply $AB > 0$.

To prove the theorem, we need introduce the definition of joint spectrum of operators.

Definition 2.2 *Let $\mathcal{A}$ be a unital commutative C^* algebras. Then the joint spectrum of commuting operator tuple (A,B) is defined as*

$$Sp(A,B) = \{(\phi(A),\phi(B)) : \text{ where } \phi \in M_{\mathcal{A}}\},$$

where $M_{\mathcal{A}}$ is the set of all multiplicative linear functionals on $\mathcal{A}$.

Dash [4] proved the joint spectrum of a commuting operator tuple is a compact set in C^2.

Now we are ready to prove our theorem:

Proof (1) $\Rightarrow$ (2). This is obvious by the fact $\|A\| = \max\{\lambda : \lambda \in \sigma(A)\}$ for any positive operator A in a unital C^* algebra.

(2) $\Rightarrow$ (3). For positive operators $A \in \mathcal{A}$, $B \in \mathcal{B}$,

$$\|A+B\|^2 = (\|A\| + \|B\|)^2,$$

that is,

$$\|AB\| = \|A\| \cdot \|B\|.$$

Thus, we know that if $A > 0, B > 0$, then $AB > 0$. By Lemma 2.1, $\mathcal{A}$ and $\mathcal{B}$ are quantum C^* independent.

(3) $\Rightarrow$ (1). For any normal operator $A \in \mathcal{A}$ and $B \in \mathcal{B}$,

$$\sigma(A - B) \subseteq \sigma(A) - \sigma(B).$$

Suppose that the statement 1 is not valid, then there exists normal operators $A_0 \in \mathcal{A}$, $B_0 \in \mathcal{B}$, such that

$$\|A_0 - B_0\| < \max_{\lambda \in \sigma(A_0), \mu \in \sigma(B_0)} |\lambda - \mu|,$$

then $\sigma(A_0 - B_0) \neq \sigma(A_0) - \sigma(B_0)$. Thus there is $\lambda_0 \in \sigma(A_0)$, $\mu_0 \in \sigma(B_0)$, such that $(\lambda_0, \mu_0) \notin Sp(A_0, B_0)$.

By the Gelafand transformation, we know that the C^* algebra $\mathcal{C}$ generated by A_0, A_0^*, B_0, B_0^* and I is isometry isomorphism to $C(M_{\mathcal{C}})$, which is the set of all complex-valued continuous functions on $M_{\mathcal{C}}$. While $Sp(A_0, B_0)$ is homeomorphism to $M_{\mathcal{C}}$ as shown in [14], thus $\mathcal{C}$ is isomorphism to $C(Sp(A_0, B_0))$.

Since $Sp(A_0, B_0)$ and $\{(\lambda_0, \mu_0)\}$ are compact sets in C^2, there is a $\delta > 0$, such that

$$U((\lambda_0, \mu_0), \delta) \cap Sp(A_0, B_0) = \emptyset.$$

Then if we take $\delta' = \frac{\delta}{2}$, we have

$$U((\lambda_0, \mu_0), \delta) \supseteq U_1 \times U_2 \quad \text{and} \quad (U_1 \times U_2) \cap Sp(A_0, B_0) = \emptyset,$$

where $U_1 = \{z : |z - \lambda_0| < \delta'\}$, $U_2 = \{w : |w - \mu_0| < \delta'\}$.

Note that $\sigma(A_0) \setminus U_1$ and $\sigma(B_0) \setminus U_2$ may be empty. Choose points z_1, z_2, w_1, w_2 in $\mathbb{C}$ such that $z_2 \in (\sigma(A_0) \setminus U_1) \cup \{z_1\}$ and $w_2 \in (\sigma(B_0) \setminus U_2) \cup \{w_1\}$. Put $E_1 = (\sigma(A_0) \setminus U_1) \cup \{z_1\}$, $E_2 = (\sigma(B_0) \setminus U_2) \cup \{w_1\}$, and

$$f(z) = \frac{d(z, E_1)}{d(z, E_1) + d(z, z_2)}, \quad g(w) = \frac{d(w, E_2)}{d(w, E_2) + d(w, w_2)},$$

where $d(x, A) = \inf\{|x - y| : y \in A\}$. Then f and g are continuous everywhere, not zero and $f(E_1) = g(E_2) = \{0\}$. Thus, $f(z) \neq 0$, $g(w) \neq 0$ and $f(z)g(w) = 0$, where $(z, w) \in Sp(A_0, B_0)$. By the continuous functional calculus:

$$\|f(A_0)g(B_0)\| = 0 \neq \|f(A_0)\| \cdot \|g(B_0)\|,$$

which makes a contradiction and thus completes the proof. □

Remark 2.4 By the theorem, we notice that the maximal spectral distance is a very important quantity for a C^* algebra, and the theorem gives us a way to test the independence of two quantum systems.

Acknowledgements We are most indebted to our advisor, Professor JI Youqing, not only for his direct helpful suggestions but primarily for his continual instruction, encouragement and support over all these years.

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