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The First Two Moments of Aggregate Claims in a Markovian Environment

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Abstract We consider the discounted aggregate claims when the insurance risks and financial risks are governed by a discrete-time Markovian environment. We assume that the claim sizes and the financial risks fluctuate over time according to the states of economy, which are interpreted as the states of Markovian environment. We will then determine a system of differential equations for the Laplace-Stieltjes transform of the distribution of discounted aggregate claims under mild assumption. Moreover, using the differentio-integral equation, we will also investigate the first two order moments of discounted aggregate claims in a Markovian environment.

Keywords discrete-time insurance risk model; discounted aggregate claims; circumstance process; moment.

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1. Introduction

In classical risk model, one assumes that the rate of net interest is zero, and aggregate claims are not discounted, though the discounted rate of aggregate claims is the difference of the claim inflation and the interest earned on investment. As pointed by Jang [3], in real phenomena these two components of the net interest rate do not have the same level. In contrast with classical model, recent studies explored discounted aggregate claims under the non-zero rate of net interest, for example, Delbaen and Haezendonck [2], Willmot [10], Léveille and Garrido [4], Jang [3] and Bara Kim and Hwa-Sung Kim [1]. Delbaen & Haezendonck [2] and Willmot [10] analyzed the net premium of compound Poisson discounted aggregate claims. Léveille and Garrido [4] derived an analytical expression for the first two order moments of discounted aggregate claims. Jang [3] obtained the Laplace transform of the distribution of the discounted aggregate claims using the shot noise process. These authors considered discounted aggregate claims under the following common assumptions. First, the claims occur according to a Poisson Process. Secondly, the claims form a sequence of i.i.d. random variables. Thirdly, the claim sizes and the epochs of claim

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occurrences are independent. Bara Kim and Hwa-Sung Kim [1] obtained the explicit expressions for the first two order moments of discounted aggregate claims under weaker conditions. First, it was assumed that the claim arrivals may be correlated. Secondly, the claim sizes were also correlated. Finally, the claim sizes and the epochs of claim occurrences were allowed to be dependent. Most of the literature concerns the discounted aggregate claims under continuous time insurance risk models with zero or a non-zero constant rate of net interest. However, there is not much under a discrete-time insurance risk model and with stochastic interest. The advantage of a discrete-time framework with stochastic interest is its flexibility to insurer features in the model.

The interest rate that will apply in future time is of course not known. Thus it seems reasonable to ask why future interest rates have not been modeled as a stochastic process. Two reasons have led one to refrain from such a model: 1) Insurance is particularly concerned with the long term development of interest rates and no commonly accepted stochastic model exists for making long term predictions. 2) A reasonable assumption is that the remaining time of the insured risks are, essentially, independent random variables. With a fixed interest assumption, the insurer's losses from different policies become independent random variables. The distribution of the aggregate loss can then simply be obtained by convolution. In particular, the variance of the aggregate loss is the sum of the individual variances, which facilitates the use of the normal approximation. Stochastic independency between policies would be lost with the introduction of a stochastic interest rate, since all policies are affected by the same interest development. But the practical evaluation of an insurance cover should analyse different interest scenarios, interest varies over time and can be presented by a sequence of random variables.

In this paper, we consider the discounted aggregate claims when the sizes of claims and discounted factors dynamics of the underlying risk assets are governed by a discrete-time Markovian environment. The discrete-time framework provides a natural and intuitive way to incorporate the effect in the underlying Markovian chain. We assume that the sizes of claims are correlated, the financial risks are correlated and the claim sizes and the financial risks are allowed to be dependent. In doing this, we use a Markovian environment, which affects both claim sizes and the financial risks. The objective of this paper is to investigate the first two order moments of discounted aggregate claims in a Markovian environment. We will derive a system of differential equations for the Laplace-Stieltjes transform of the distribution of discounted aggregate claims. Moreover, using a differentio-integral equation, we will also derive explicit expressions of the first two order moments of discounted aggregate claims.

The rest of the paper is organized as follows. In Section 2, we give a circumstance for the model and some assumptions. In Section 3, we present a system of difference equations for the Laplace-stieltjes transform of the distribution of discounted aggregate claims. In Section 4, the explicit formulae of the first two order moments of discounted aggregate claims are derived. Finally, concluding remarks are given in Section 5.

2. The model

In this section, we present a discrete-time insurance risk model among financial risks and a Markovian environment. Following Nyrhinen [5, 6], Tang and Tsitsiashvili [7, 8] and Tang and Vernic [9], we consider the discrete-time insurance risk model

$$W(n) = \sum_{i=1}^{n} X_i \prod_{k=1}^{i} Y_k, \quad n = 1, 2, \ldots, \tag{1}$$

$$W(0) = 0, \tag{2}$$

where X_n is the aggregate claims within period n and Y_n is the discounted factor from time n to $n-1, n = 1, 2, \ldots$. We call X_n the insurance risks and Y_n the financial risks for $n = 1, 2, \ldots$. It is clear that $W(n)$ introduced above denotes the total discounted amount of claims by the end of the period n. Here we assume X_n are correlated, Y_n are correlated and both X_n and Y_n are allowed to be dependent, for $n = 1, 2, \ldots$.

Next, we describe the laws for X_n and Y_n, $n = 1, 2, \ldots$. First, we introduce a discrete time process $\{J(n) : n = 0, 1, 2, \ldots\}$ that describes the environment for the risk business. Categorize the circumstances for the risk business into m states, say, $1, 2, \ldots, m$. Suppose that $\{J(n) : n = 0, 1, 2, \ldots\}$ is homogeneous Markovian chain, whose state set is $\{1, 2, \ldots, m\}$ and transition matrix is $Q = (q_{ij})_{m\times m}$. Let $J(n)$ denote the state of the business at time n. We call $\{J(n) : n = 0, 1, 2, \ldots\}$ the Markovian environment process. In the following, we give some assumptions.

Assumption A Given $J(n), J(0)$, $(W(n+k) - W(n), J(n+k))$ is independent of $W(n)$.

Assumption B Given $J(n)$, $(X_{n+k}, Y_{n+k})(k \geq 1)$ are independent of $J(0)$, $J(1), \ldots, J(n-1)$. Given $J(1)$, $W(1)$ and $J(0)$ are independent and Y_1 is dependent on X_1.

Assumption C Given $J(0), J(1), \ldots, J(n)$,$(Y_1, Y_2, \ldots, Y_n)$ is independent of $(X_{n+k}, Y_{n+k}, J_{n+k})$, $k \geq 1$.

Assumption D $\{(X_n, Y_n, J_n), n \geq 1\}$ is a stationary process.

We also suppose that $\prod_{h=1}^{n} Y_h$ is independent of $J(n)$, $J(0)$. Finally, throughout the paper, we assume that the moments involved in the paper exist and the integral and differential orders may be interchanged.

3. The Laplace-Stieltjes transform

Let $X^{(i)}$ denote the generic random variable for the claim size X_n given $J(n) = i$. Denote the Laplace-stieltjes transform, the first and the second moments of $X^{(i)}$ by $\hat{g}_i(s)$, g_i and $g_i^{(2)}$, respectively, that is,

$$\begin{aligned}
\hat{g}_i(s) =& E\big[e^{-sX^{(i)}}\big] = E\big[e^{-sX_n}|J(n) = i\big], g_i = E\big[X^{(i)}\big] = E\big[X_n|J(n) = i\big],\\
g_i^{(2)} =& E\big[(X^i)^2\big] = E\Big[\big(X_n\big)^2|J(n) = i\Big], \hat{G}(s) = \mathrm{diag}\big(\hat{g}_1(s), \ldots, \hat{g}_m(s)\big),\\
G =& \mathrm{diag}\big(g_1, \ldots, g_m\big), G^{(2)} = \mathrm{diag}\big(g_1^{(2)}, \ldots, g_m^{(2)}\big).
\end{aligned}$$

Let

$$f_{ij}(s, n) = E\big[e^{-sW(n)}\mathbf{1}_{J(n)=j}|J(0) = i\big]. \tag{3}$$

Denote by $F(s,n)$ the $m\times m$ matrix whose (i,j) entry is $f_{ij}(s,n)$. Now we give a series of propositions for $f_{ij}(s,n)$.

Proposition 3.1 *For any $s\geq 0$, $n=1,2,\ldots,k=1,2,\ldots,$ we have*

$$F(s,n+k)=F(s,n)E\big[F\big(s\prod_{h=1}^{n}Y_h,k\big)\big]. \tag{4}$$

Proof For any $s\geq 0$, $i,j=1,2,\ldots,m$,

$$\begin{aligned}f_{ij}(s,n+k)&=E\big[e^{-sW(n+k)}\mathbf{1}_{J(n+k)=j}|J(0)=i\big]\\&=E\Big[e^{-sW(n)}E\big[e^{-s(W(n+k)-W(n))}\mathbf{1}_{J(n+k)=j}\big|W(n),J(n),J(0)=i\big]\big|J(0)=i\Big].\end{aligned}$$

Since $(W(n+k)-W(n),J(n+k))$ is independent of $W(n)$ when $J(n),J(0)$ are given , the above equation is

$$f_{ij}(s,n+k)=E\Big[e^{-sW(n)}E\big[e^{-s\sum_{h=n+1}^{n+k}X_l\prod_{j=1}^{l}Y_h}\mathbf{1}_{J(n+k)=j}\big|J(n),J(0)=i\big]|J(0)=i\Big].$$

Given $J(n)$, $(X_{n+1},Y_{n+1},\ldots)$ is independent of $J(n-1),J(n-2),\ldots,J(0)$, so

$$\begin{aligned}&E\big[e^{-s\sum_{l=n+1}^{n+k}X_l\prod_{h=n+1}^{l}Y_h}\mathbf{1}_{J(n+k)=j}\big|J(n),J(0)=i\big]\\&=E\big[e^{-s\sum_{l=n+1}^{n+k}X_l\prod_{h=1}^{l}Y_h}\mathbf{1}_{J(n+k)=j}\big|J(n)\big].\end{aligned}$$

For $\{(X_n,Y_n,J(n)),n\geq 1\}$ is a stationary process, the above equation is equal to

$$\begin{aligned}&=E\big[e^{-s\sum_{l=1}^{k}X_l\prod_{h=1}^{l}Y_h}\mathbf{1}_{J(k)=j}\big|J(0)\big]\\&=f_{J(0)j}(s,k).\end{aligned}$$

$(Y_1,Y_2,\ldots,Y_n)$ is independent of $(X_{n+1},Y_{n+1},\ J(n+1),\ X_{n+2},Y_{n+2},J(n+2),\ldots)$ given $J(n)$, $\ldots$, $J(0)$. Then

$$\begin{aligned}&E\big[e^{-s\prod_{h=1}^{n}Y_h\sum_{l=n+1}^{n+k}X_l\prod_{h=n+1}^{l}Y_h}\mathbf{1}_{J(n+k)=j}\big|J(n),J(0)=i\big]\\&=E\Big[E\big[e^{-s\prod_{h=1}^{n}Y_h(\sum_{l=n+1}^{n+k}X_l\prod_{h=n+1}^{l}Y_h)}\mathbf{1}_{J(n+k)=j}\big|J(n),J(0)=i,Y_1,Y_2,\ldots,Y_n\big]\mid J(n),J(0)=i\Big]\\&=E\Big[E\big[e^{-\tilde{s}(\sum_{l=n+1}^{n+k}X_l\prod_{h=n+1}^{l}Y_h)}\mathbf{1}_{J(n+k)=j}\big|J(n),J(0)=i\ |_{\tilde{s}=s\prod_{h=1}^{n}Y_h}\ \big]\mid J(n),J(0)=i\Big]\\&=E\big[f_{J(n)j}(s\prod_{h=1}^{n}Y_h,k)\mid J(n),J(0)=i\big].\end{aligned}$$

Moreover, $\prod_{j=1}^{n}Y_j$ is independent of $J(n)$ and $J(0)$, then

$$\begin{aligned}&E\big[f_{J(n)j}(s\prod_{h=1}^{n}Y_h,k)\mid J(n),J(0)=i\big]=E\big[f_{J(n)j}(s\prod_{h=1}^{n}Y_h,k)\mid J(n)\big]\\&=\sum_{l=1}^{m}\mathbf{1}_{J(n)=l}Ef_{l,j}(s\prod_{h=1}^{n}Y_h,k).\end{aligned}$$

Thus

$$f_{ij}(s,n+k)=E\Big[e^{-sW(n)}Ef_{J_{(n)}j}(s\prod_{h=1}^{n}Y_h,k)|J(0)=i\Big]$$

$$= \sum_{l=1}^{m} E\Big[e^{-sW(n)}\mathbf{1}_{J(n)=l} E f_{l,j}(s\prod_{h=1}^{n} Y_h, k) | J(0)=i\Big]$$
$$= \sum_{l=1}^{m} E\Big[e^{-sW(n)}\mathbf{1}_{J(n)=l} | J(0)=i\Big] E f_{l,j}(s\prod_{h=1}^{n} Y_h, k)$$
$$= \sum_{l=1}^{m} f_{i,l}(s,n) E f_{l,j}(s\prod_{h=1}^{n} Y_h, k),$$

which completes the proof. □

Proposition 3.2 *For any $s \geq 0$, we have*

$$F(s,1) = QE\hat{G}(sY_1), \tag{5}$$
$$F(0,1) = Q. \tag{6}$$

Proof For any $s \geq 0$, we have

$$f_{ij}(s,1) = E\big[e^{-sW(1)}\mathbf{1}_{J(1)=j} | J(0)=i\big]$$
$$= P(J(1)=j|J(0)=i) E\big[e^{-sW(1)}\big|J(1)=j, J(0)=i\big]$$
$$= q_{ij} E\big[e^{-sW(1)}\big|J(1)=j, J(0)=i\big].$$

Since $J(1)$, $W(1)$ and $J(0)$ are mutually independent and $Y(1)$ is dependent of $X(1)$, we have

$$q_{ij} E\big[e^{-sW(1)}\big|J(1)=j, J(0)=i\big] = q_{ij} E\big[e^{-(sY_1)X_1}\big|J(1)=j\big]$$
$$= q_{ij} E\Big[E\big[e^{-(sY_1)X_1}\big|J(1)=j, Y_1\big]\Big] = q_{ij} E\hat{g}_j(sY_1),$$

which yields (5). The equation (6) is trivial.

Proposition 3.3 *We assume that $\tilde{Y}_1$ has the same distribution as Y_1 and is independent of $(Y_1, Y_2, \ldots, Y_n)$. For any $s \geq 0$, $n = 2, 3, \ldots$, we have*

$$F(s,n+1) - F(s,n) = F(s,n)\big[Q(\hat{G}(s\tilde{Y}_1\prod_{h=1}^{n} Y_h) - I)\big] \tag{7}$$
$$F(0,n) = Q^n. \tag{8}$$

Proof For any $s \geq 0$, by Proposition 1, we have

$$F(s,n+1) - F(s,n) = F(s,n) E\big[F(s\prod_{h=1}^{n} Y_h, 1)\big] - F(s,n)$$
$$= F(s,n)\Big[E\big[F(s\prod_{h=1}^{n} Y_h, 1) - I\big]\Big].$$

By Proposition 2, the above equation leads to

$$E\big[f_{ij}(s\prod_{h=1}^{n} Y_h, 1) = E\big[q_{ij} E[\hat{g}_j(s\prod_{h=1}^{n} Y_h \tilde{Y}_1]\big] = q_{ij} E[\hat{g}_j(s\tilde{Y}_1\prod_{h=1}^{n} Y_h],$$
$$F(s,n+1) - F(s,n) = F(s,n)\Big[QE\big[\hat{G}(s\tilde{Y}_1\prod_{h=1}^{n} Y_h)\big] - I\Big].$$

By $F(0,1)=Q$ and the above difference equation, we may obtain

$$\begin{aligned}&F(0,n+1)-F(0,n)=F(0,n)[Q-I]=Q^nF(0,1),\\&F(0,n)=Q^n.\end{aligned}$$

4. Moments

Let $E(n)$ be the $m\times m$ matrix whose (i,j) entry is

$$E_{ij}(n)=E\big[W(n)\mathbf{1}_{J(n)=j}|J(0)=i\big] \tag{9}$$

and $\mu(n)$ be the m-dimensional column vector whose ith component is

$$\mu_i(n)=E\big[W(n)\big|J(0)=i\big]. \tag{10}$$

Then $\mu(n)=E(n)\mathbf{e}$, where $\mathbf{e}=(1,1,\ldots,1)^{\mathrm{T}}$ is the m-dimensional column vector.

Theorem 4.1 *For $n=0,1,2,\ldots$, we have*

$$E(n+1)=\sum_{i=0}^{n}c_{1i}Q^{i+1}GQ^{n-i}, \tag{11}$$

where $c_{1n}=E[\tilde{Y}_1\prod_{j=1}^{n}Y_j], c_{10}=E[\tilde{Y}_1]$.

Proof For $W(0)=0$, $E[W(0)|J(0)]=0$. Since

$$\begin{aligned}E_{ij}(1)=&E[W(1)\mathbf{1}_{J(1)=j}|J(0)=i]=E[X_1Y_1\mathbf{1}_{J(1)=j}|J(0)=i]\\=&q_{ij}E[X_1Y_1|J(1)=j,J(0)=i]=q_{ij}E[X_1Y_1|J(1)=j]=q_{ij}E\Big[E\big[X_1Y_1|J(1)=j,Y_1\big]\Big]\\=&q_{ij}E\Big[Y_1E\big[X_1|J(1)=j,Y_1\big]\Big]=q_{ij}E\Big[Y_1E\big[X_1|J(1)=j\big]\Big]\\=&q_{ij}E\big[Y_1g_j\big]=q_{ij}c_{10}g_j,\end{aligned}$$

we have $E(1)=c_{10}QG$. Let

$$M_k(n)=\frac{\partial^kF(s,n)}{\partial s^k}\Big|_{s=0},\quad k=1,2, \tag{12}$$

and denote the probability density functions of $\tilde{Y}_1,(Y_1,Y_2,\ldots,Yn)$, $X^{(i)}$ by $p_1(y)$, $p_n(y_1,\ldots,y_n)$, $p(x|i)$, respectively,

$$\begin{aligned}&\frac{\partial E[\hat{G}(s\tilde{Y}_1\prod_{h=1}^{n}Y_h)]}{\partial s}=\frac{\partial}{\partial s}\int_{R^{n+1}}\hat{G}(sy\prod_{h=1}^{n}y_hp_1(y)p_n(y_1,\ldots,y_n))\mathrm{d}y\mathrm{d}y_1\cdots\mathrm{d}y_n\\&=\int_{R^{n+1}}y\prod_{h=1}^{n}y_h\frac{\partial\hat{G}(\tilde{s})}{\partial\tilde{s}}|_{\tilde{s}=sy\prod_{h=1}^{n}y_h}p_1(y)p_n(y_1,\ldots,y_n)\mathrm{d}y\mathrm{d}y_1\cdots\mathrm{d}y_n.\end{aligned}$$

Note that $\hat{g}_i(s)$ is the ith diagonal element of $m\times m$ matrix $\hat{G}(s)$, we have

$$\frac{\partial\hat{g}_i(\tilde{s})}{\partial\tilde{s}}=\frac{\partial E[e^{-\tilde{s}X^{(i)}}]}{\partial\tilde{s}}=\frac{\partial}{\partial\tilde{s}}\int_Re^{-\tilde{s}x}p(x|i)\mathrm{d}x=\int_R(-x)e^{-\tilde{s}x}p(x|i)\mathrm{d}x.$$

From the above equations,

$$\frac{\partial E[\hat{G}(s\tilde{Y}_1\prod_{h=1}^{n}Y_h)]}{\partial s}|_{s=0}$$

$$= \int_{R^{n+1}} y\prod_{h=1}^{n} y_h[\int_R(-x)e^{-\tilde{s}x}\mathrm{diag}(p(x|1),p(x|2),\dots,p(x|m))\mathrm{d}x]\cdot$$

$$p_1(y)p_n(y_1,\dots,y_n)\mathrm{d}y\mathrm{d}y_1\cdots\mathrm{d}y_n|_{\tilde{s}=0}$$

$$= \int_{R^{n+1}} y\prod_{h=1}^{n} y_h(-G)p_1(y)p_n(y_1,\dots,y_n)\mathrm{d}y\mathrm{d}y_1\cdots\mathrm{d}y_n$$

$$= -E[\tilde{Y}_1\prod_{h=1}^{n}Y_h]G = -c_{1n}G.$$

By (7) and (12),

$$M_1(n+1)-M_1(n) = [\frac{\partial F(s,n+1)}{\partial s}-\frac{\partial F(s,n)}{\partial s}]|_{s=0}$$

$$= \frac{\partial\big(F(s,n+1)-F(s,n)\big)}{\partial s}|_{s=0}$$

$$= \frac{\partial}{\partial s}F(s,n)\Big[QE\big[\hat{G}(s\tilde{Y}_1\prod_{h=1}^{n}Y_h)\big]-I\Big]|_{s=0}$$

$$= \frac{\partial F(s,n)}{\partial s}|_{s=0}(Q-I)-F(s,n)Q\frac{\partial E[\hat{G}(s\tilde{Y}_1\prod_{h=1}^{n}Y_h)]}{\partial s}|_{s=0}$$

$$= M_1(n)(Q-I)-c_{1n}Q^{n+1}G.$$

Since $M_1(0)=\mathbf{0}, F(0,n)=Q^n$ and $E(n)=-M_1(n)$, the above equations lead to

$$E(n+1)-E(n) = E(n)(Q-I)+c_{1n}Q^{n+1}G. \tag{13}$$

Then,

$$E(n+1) = E(n)Q+c_{1n}Q^{n+1}G = \sum_{i=0}^{n} c_{1i}Q^{i+1}GQ^{n-i}.$$

Next we consider the second order moment of the discounted value of aggregate claims. Let $\mu_i^{(2)}(n)$ be the second moment of $W(n)$ given $J(0)=i$, that is,

$$\mu_i^{(2)}(n) = E\big[(W(n))^2|J(0)=i\big]. \tag{14}$$

Denote by $\mu^{(2)}(n)$ the m-dimensional column vector whose ith component is $\mu_i^{(2)}(n)$, then given $J(0)=i$ variance $\sigma_i^2(n)$ of $W(n)$ is expressed as

$$\sigma_i^2(n) = \mu_i^{(2)}(n)-\big(\mu_i(n)\big)^2, \quad i=1,\dots,m. \tag{15}$$

The following theorem gives an explicit expression for $\mu_i^{(2)}(n)$.

Theorem 4.2 *For $n=1,2,\dots,$ we have*

$$\mu_i^{(2)}(n+1) = \sum_{i=1}^{n}\Big[c_{2i}Q^{i+1}G^{(2)}+2c_{1i}\big[\sum_{j=0}^{i-1}c_{1j}Q^{j+1}GQ^{j-i-1}\big]QG\Big]Q^{n-i}\boldsymbol{e}+c_{20}QG^{(2)}Q^n\boldsymbol{e}, \tag{16}$$

where $c_{1n}=E[\tilde{Y}_1\prod_{h=1}^{n}Y_h]$, $c_{10}=E[\tilde{Y}_1]$, $c_{2n}=E[(\tilde{Y}_1\prod_{h=1}^{n})^2Y_h]$, $c_{20}=E[(\tilde{Y}_1)^2]$.

Proof The proof is similar to that of Theorem 1, except for a more careful differentio-integral operation. Note that

$$\frac{\partial^2 E[\hat{G}(s\tilde{Y}_1\prod_{h=1}^n Y_h)]}{\partial s^2}=\frac{\partial^2}{\partial s^2}\int_{R^{n+1}}\hat{G}(sy\prod_{h=1}^n y_h p_1(y)p_n(y_1,\ldots,y_n))\mathrm{d}y\mathrm{d}y_1\cdots\mathrm{d}y_n$$

$$=\int_{R^{n+1}}(y\prod_{h=1}^n y_h)^2\frac{\partial^2\hat{G}(\tilde{s}^2)}{\partial\tilde{s}}|_{\tilde{s}=sy\prod_{h=1}^n y_h}p_1(y)p_n(y_1,\ldots,y_n)\mathrm{d}y\mathrm{d}y_1\cdots\mathrm{d}y_n$$

and

$$\frac{\partial^2\hat{g}_i(\tilde{s})}{\partial\tilde{s}^2}=\frac{\partial^2 E[e^{-\tilde{s}X^{(i)}}]}{\partial\tilde{s}^2}=\frac{\partial^2}{\partial\tilde{s}^2}\int_R e^{-\tilde{s}x}p(x|i)\mathrm{d}x=\int_R x^2e^{-\tilde{s}x}p(x|i)\mathrm{d}x,$$

we have

$$\begin{aligned}&\frac{\partial^2 E[\hat{G}(s\tilde{Y}_1\prod_{h=1}^n Y_h)]}{\partial s^2}|_{s=0}\\&=\int_{R^{n+1}}(y\prod_{h=1}^n y_h)^2[\int_R x^2e^{-\tilde{s}x}\mathrm{diag}(p(x|1),p(x|2),\ldots,p(x|m))\mathrm{d}x]\cdot\\&\quad p_1(y)p_n(y_1,\ldots,y_n)\mathrm{d}y\mathrm{d}y_1\cdots\mathrm{d}y_n|_{\tilde{s}=0}\\&=\int_{R^{n+1}}(y\prod_{h=1}^n y_h)^2G^{(2)}p_1(y)p_n(y_1,\ldots,y_n)\mathrm{d}y\mathrm{d}y_1\cdots\mathrm{d}y_n\\&=E[(\tilde{Y}_1\prod_{h=1}^n Y_h)^{(2)}]G^{(2)}=c_{2n}G^{(2)}.\end{aligned}$$

By the equation (5),

$$\begin{aligned}&M_2(1)\\&=\frac{\partial^2F(s,1)}{\partial s^2}|_{s=0}=Q\int_R y^2\int_R x^2e^{-sx}\mathrm{diag}(p(x|1),p(x|2),\ldots,p(x|m))\mathrm{d}x\mathrm{d}y|_{s=0}\\&=Q\int_R y^2\int_R x^2\mathrm{diag}(p(x|1),p(x|2),\ldots,p(x|m))\mathrm{d}x\mathrm{d}y|_{s=0}\\&=Q\int_R y^2G^{(2)}\mathrm{d}y=c_{20}QG^{(2)}.\end{aligned}$$

Furthermore, by (12)

$$\begin{aligned}&M_2(n+1)-M_2(n)\\&=[\frac{\partial^2F(s,n+1)}{\partial s^2}-\frac{\partial^2F(s,n)}{\partial s^2}]|_{s=0}=\frac{\partial^2\big(F(s,n+1)-F(s,n)\big)}{\partial s^2}|_{s=0}\\&=\frac{\partial^2}{\partial s^2}\{F(s,n)\Big[QE\big[\hat{G}(s\tilde{Y}_1\prod_{h=1}^n Y_h\big]-I\Big]\}|_{s=0}\\&=\frac{\partial^2F(s,n)}{\partial s^2}\Big[QE\big[\hat{G}(s\tilde{Y}_1\prod_{h=1}^n Y_h\big]-I\Big]\}|_{s=0}+\frac{\partial F(s,n)}{\partial s}Q\frac{\partial E\big[\hat{G}(s\tilde{Y}_1\prod_{h=1}^n Y_h}{\partial s}|_{s=0}+\\&\quad F(s,n)Q\frac{\partial E\big[\hat{G}(s\tilde{Y}_1\prod_{h=1}^n Y_h\big]}{\partial s}|_{s=0}\\&=M_2(n)(Q-I)-2c_{1n}M_1(n)QG+c_{2n}Q^{n+1}G^{(2)},\end{aligned}$$

where $M_2(0) = \mathbf{0}$ and $M_1(n) = -E(n)$. So

$$\begin{aligned}
M_2(n+1) &= M_2(n)Q + 2c_{1n}E(n)QG + c_{2n}Q^{n+1}G^{(2)} \\
&= \sum_{i=0}^{n} \Big(c_{2i}Q^{i+1}G^{(2)} + 2c_{1i}E(i)QG\Big)Q^{n-i} \\
&= \sum_{i=1}^{n} \Big(c_{2i}Q^{i+1}G^{(2)} + 2c_{1i}E(i)QG\Big)Q^{n-i} + c_{20}QG^{(2)}Q^{n} \\
&= \sum_{i=1}^{n} \Big[c_{2i}Q^{i+1}G^{(2)} + 2c_{1i}\big[\sum_{j=0}^{i-1} c_{1j}Q^{j+1}GQ^{j-i-1}\big]QG\Big]Q^{n-i} + c_{20}QG^{(2)}Q^{n}.
\end{aligned}$$

Substituting $u^{(2)}(n) = M_2(n)\mathbf{e}$ into the above equation leads to

$$\mu_i^{(2)}(n+1) = \sum_{i=1}^{n} \Big[c_{2i}Q^{i+1}G^{(2)} + 2c_{1i}\big[\sum_{j=0}^{i-1} c_{1j}Q^{j+1}GQ^{j-i-1}\big]QG\Big]Q^{n-i}\mathbf{e} + c_{20}QG^{(2)}Q^{n}\mathbf{e}.$$

The proof of the theorem is completed. □

5. Conclusions

In this paper, we derived a system of differential equations for the Laplace-Stieltjes transform of the distribution of discounted aggregate claims. Moreover, using the differentio-integral equation, we investigated the first two order moments of discounted aggregate claims in a Markovian environment. The explicit expressions of the first two order moments have been obtained. These results may have some practical implications on the pricing and the investment of insurance.

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