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The Weighted Estimates of the Schrödinger Operators on the Nilpotent Lie Group

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Abstract In this paper we consider the Schrödinger operator $-\Delta_G+W$ on the nilpotent Lie group G where the nonnegative potential W belongs to the reverse Hölder class B_{q_1} for some $q_1\geq\frac{D}{2}$ and D is the dimension at infinity of G. The weighted L^p-L^q estimates for the operators $W^\alpha(-\Delta_G+W)^{-\beta}$ and $W^\alpha\nabla_G(-\Delta_G+W)^{-\beta}$ are obtained.

Keywords nilpotent Lie group; Schrödinger operators; reverse Hölder class.

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1. Introduction

As we know, Schrödinger operators on the Euclidean space $\mathbb{R}^n$ with non-negative potentials which belong to the reverse Hölder class have been investigated by a number of scholars [1, 2]. Now the investigation of Schrödinger operators has been generalized to two direction. On the one hand, Kurata and Sugano generalized Shen's results to uniformly elliptic operators in [3]. on the other hand, Lu [4] and Li [5] investigated the Schrödinger operators in a more general setting.

The main purpose of this paper is to investigate the weighted L^p-L^q boundedness of the operators

$$T_1=W^\alpha(-\Delta_G+W)^{-\beta},\quad 0\leq\alpha\leq\beta\leq 1,$$

and

$$T_2=W^\alpha\nabla_G(-\Delta_G+W)^{-\beta},\quad 0\leq\alpha\leq\frac{1}{2}\leq\beta\leq 1,\ \beta-\alpha\geq\frac{1}{2},$$

on the nilpotent Lie group G. Note that Sugano [6] has studied the weighted estimates of the above two operators on the Euclidean space and Liu [7] has obtained the same estimates on the stratified Lie group.

Assume G is a simple connected nilpotent Lie group and $\mathfrak{g}$ is its Lie algebra which is identified with the space of left invariant vector fields. Given $X=\{X_1,\ldots,X_k\}\subseteq\mathfrak{g}$ a Hörmander system

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of left invariant vector fields on G. This means that there exists an integer s such that the vector fields $X_1, \ldots, X_k$ together with their commutators of order at most s span the tangent space of G at every point x. Let $\Delta_G = \sum_{i=1}^k X_i^2$ be the sub-Laplacian on G associated to X. The gradient operator ∇_G is denoted by $\nabla_G = (X_1, \ldots, X_k)$. Following [8], one can define a left invariant metric d associated to X which is called the Carnot-Caratheodory metric: let $x, y \in G$, and

$$d(x,y) = \inf\{\delta \mid \gamma : [0,\delta] \to G \,|\, \gamma(0) = x, \gamma(\delta) = y\},$$

where γ is a piecewise smooth curve satisfying $\gamma'(s) = \sum_{i=1}^k a_i(s) X_i(\gamma(s))$ with $\sum_{i=1}^k |a_i(s)|^2 \le 1$, for all $s \in [0,t]$.

If $x \in G$ and $r > 0$, we will denote by $B(x,r) = \{h \in G | d(x,y) < r\}$ the metric balls. Assume $\mathrm{d}x$ is the Haar measure on G. Then for every measurable set $E \subseteq G$, $|E|$ denotes the measure of E. Suppose e is the unit element of G. Note that $V(t) = |B(e,t)| = |B(x,t)|$ for any $x \in G$ and $t > 0$. Let d and D be the local dimension and the dimension at infinity of G. Note that $D \ge d$ and we always assume $d \ge 2$ throughout the paper. It follows from (1.1) in [5] that there exists a constant $C_1 > 0$ such that

$$C_1^{-1} t^d \le V(t) \le C_1 t^d, \quad \forall 0 \le t \le 1,$$

$$C_1^{-1} t^D \le V(t) \le C_1 t^D, \quad \forall 1 \le t < \infty.$$

Also, there exists a constant $C_2 > 1$ such that for any $r > 0$,

$$V(2r) \le C_2 V(r). \tag{1}$$

Definition 1 *A nonnegative locally L^q integrable function W on G is said to belong to the reverse Hölder class B_q $(1 < q < \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality*

$$\Big(\frac{1}{|B|}\int_B W(x)^q \mathrm{d}x\Big)^{\frac{1}{q}} \le C\Big(\frac{1}{|B|}\int_B W(x)\mathrm{d}x\Big) \tag{2}$$

holds for every ball B in G.

It is important that the B_q class has a property of "self improvement"; that is, if $W \in B_q$, then $W \in B_{q+\varepsilon}$ for some $\varepsilon > 0$ ([5]).

Now we recall the definitions of fractional maximal operator M_γ and $A_{p,q}$-weight on G.

Definition 2 *Let $f \in L^1_{\mathrm{loc}}(G)$. For $\gamma > 0$, the fractional maximal operator is defined by*

$$M_\gamma f(x) = \sup_{x\in B} \frac{1}{|B|^{1-\gamma}} \int_B |f(y)|\mathrm{d}y, \quad x \in G,$$

where the supremum on the right side is taken over all balls B such that $x \in B$.

Definition 3 *Let $1 < p < \infty$ and $1 < q < \infty$. For a non-negative function $w(x)$, we say $w \in A_{p,q}$ if*

$$\Big(\frac{1}{|B|}\int_B w(x)^q \mathrm{d}x\Big)^{\frac{1}{q}} \Big(\frac{1}{|B|}\int_B w(x)^{-p/(p-1)} \mathrm{d}x\Big)^{\frac{p-1}{p}} \le C$$

holds for every ball B in G, where C is a positive constant independent of B.

We obtain the estimates for the adjoint operators T_1^* and T_2^* with the potential $W \in B_{q_1}$ for some $q_1 > \frac{D}{2}$.

Theorem 1 *Suppose $W \in B_{q_1}$ for some $q_1 > \frac{D}{2}$, $0 < \alpha \leq \beta \leq 1$ and let $\frac{1}{q_2} = 1 - \frac{\alpha}{q_1}$. Then there exists a constant $C > 0$ such that*

$$|T_1^* f(x)| \leq C\{M_{\varepsilon q_2}(|f|^{q_2})(x)\}^{\frac{1}{q_2}}, \quad f \in C_0^\infty(G),$$

where $\varepsilon = \frac{2(\beta-\alpha)}{\theta}$, $\theta \in [d, D]$.

Theorem 2 *Suppose $V \in B_{q_1}$ for some $q_1 > \frac{D}{2}$, $0 < \alpha \leq \frac{1}{2} < \beta \leq 1$ and $\beta - \alpha \geq \frac{1}{2}$. And let*

$$\frac{1}{q_2} = \begin{cases} 1 - \frac{\alpha}{q_1}, & \text{if } q_1 \geq D, \\ 1 - \frac{(\alpha+1)}{q_1} + \frac{1}{D}, & \text{if } \frac{D}{2} < q_1 < D. \end{cases}$$

Then there exists a constant $C > 0$ such that

$$|T_2^* f(x)| \leq C\{M_{\varepsilon q_2}(|f|^{q_2})(x)\}^{\frac{1}{q_2}}, \quad f \in C_0^\infty(G),$$

where $\varepsilon = \frac{2(\beta-\alpha)-1}{\theta}$, $\theta \in [d, D]$.

The above theorems will yield the weighted L^p estimates for T_1 and T_2 which generalize the main results in [6] and [7] to the nilpotent Lie group.

Corollary 1 *Assume that $W \in B_{q_1}$ for $q_1 > \frac{D}{2}$, and $0 \leq \alpha \leq \beta \leq 1$. Let $1 < p < \frac{1}{\frac{\alpha}{q_1} + \frac{\gamma}{\theta}}$, $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{\theta}$ and $\frac{1}{q_2} = 1 - \frac{\alpha}{q_1}$, where $\gamma = 2(\beta - \alpha)$ and $\theta \in [d, D]$. We suppose w satisfies*

(A) $\alpha > 0$, $w^{-q_2} \in A_{\frac{q'}{q_2}, \frac{p'}{q_2}}$ and $w^{-\frac{q_2 q'}{q_2 - q'}} \in A_\infty$;

(B) $\alpha = 0$, $w^{-q_2} \in A_{\frac{q'}{q_2}, \frac{p'}{q_2}}$, $w^{-p'}$ and $w^{-\frac{q_2 q'}{q_2 - q'}} \in A_\infty$,

where $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$. Then there exists a positive constant C such that for any $f \in C_0^\infty(G)$,

$$\| (T_1 f)w \|_{L^q(G)} \leq C \| fw \|_{L^p(G)} .$$

Corollary 2 *Assume that $W \in B_{q_1}$ for $q_1 > \frac{D}{2}$, and*

$$\begin{cases} 0 \leq \alpha \leq \frac{1}{2} \leq \beta \leq 1, & \text{if } q_1 \geq D, \\ 0 \leq \alpha \leq \frac{1}{2} < \beta \leq 1, & \text{if } \frac{D}{2} < q_1 < D. \end{cases}$$

Let $\gamma = 2(\beta - \alpha) - 1$ and $\beta - \alpha \geq \frac{1}{2}$, and let $1 < p < \frac{1}{\frac{1}{p_1} + \frac{\gamma}{\theta}}$, $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{\theta}$, $\frac{1}{q_2} = 1 - \frac{1}{p_1}$, where $\theta \in [d, D]$ and

$$\frac{1}{p_1} = \begin{cases} \frac{\alpha}{q_1}, & \text{if } q_1 > D, \\ \frac{(\alpha+1)}{q_1} - \frac{1}{D}, & \text{if } \frac{D}{2} < q_1 < D. \end{cases}$$

We suppose w satisfies

(A) $\alpha > 0$, $w^{-q_2} \in A_{\frac{q'}{q_2}, \frac{p'}{q_2}}$ and $w^{-\frac{q_2 q'}{q_2 - q'}} \in A_\infty$;

(B) $\alpha = 0$, $w^{-q_2} \in A_{\frac{q'}{q_2}, \frac{p'}{q_2}}$, $w^{-p'}$ and $w^{-\frac{q_2 q'}{q_2 - q'}} \in A_\infty$,

where $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$. Then there exists a positive constant C such that for any

$f \in C_0^\infty(G)$,

$$\| (T_2 f)w \|_{L^q(G)} \leq C \| fw \|_{L^p(G)} .$$

Throughout this paper, unless otherwise indicated, we will use C to denote constants, which are not necessarily the same at each occurrence. By $A \sim B$, we mean that there exist $C > 0$ and $c > 0$ such that $c \leq \frac{A}{B} \leq C$.

2. Preliminaries

First we briefly recall the definition of the auxiliary function $m(x, V)$ and its basic properties on the nilpotent Lie group in [5].

Let $W \in B_{q_1}$ for some $q_1 > \frac{D}{2}$, where D is the dimension at infinity of G. Then the auxiliary function $\rho(x, W) = \rho(x)$ is defined by

$$\rho(x) = \frac{1}{m(x, W)} \doteq \sup_{r>0} \Big\{ r : \frac{r^2}{W(r)} \int_{B(x,r)} W(y)\mathrm{d}y \leq 1 \Big\}, \quad x \in G.$$

Lemma 1 *The measure $W(x)\mathrm{d}x$ satisfies the doubling condition, that is, there exists $C > 0$ such that*

$$\int_{B(x,2r)} W(y)\mathrm{d}y \leq C \int_{B(x,r)} W(y)\mathrm{d}y$$

for all balls $B(x, r)$ in G.

Lemma 2 *There exists $C > 0$ such that, for $0 < r < R < \infty$,*

$$\frac{r^2}{V(r)} \int_{B(x,r)} W(y)\mathrm{d}y \leq C(\frac{r}{R})^{2-\frac{D}{q_1}} \frac{R^2}{V(R)} \int_{B(x,R)} W(y)\mathrm{d}y.$$

Lemma 3 *If $r = \rho(x)$, then*

$$\frac{r^2}{V(r)} \int_{B(x,r)} W(y)\mathrm{d}y = 1.$$

Moreover,

$$\frac{r^2}{V(r)} \int_{B(x,r)} W(y)\mathrm{d}y \sim 1 \quad \text{if and only if} \quad r \sim \rho(x).$$

Lemma 4 *There exist $C > 0$ and $l_0 > 0$ such that, for any x and y in G,*

$$\frac{1}{C}\Big(1 + \frac{\mathrm{d}(x, y)}{\rho(x)}\Big)^{-l_0} \leq \frac{\rho(y)}{\rho(x)} \leq C\Big(1 + \frac{\mathrm{d}(x, y)}{\rho(x)}\Big)^{\frac{l_0}{l_0+1}}.$$

In particular, $\rho(x) \sim \rho(y)$ if $\mathrm{d}(x, y) < C\ \rho(x)$.

Lemma 5 *There exist $C > 0$ and $l_1 > 0$ such that*

$$\int_{B(x,R)} \frac{\mathrm{d}(x, y)^2 W(y)}{V(\mathrm{d}(x, y))}\mathrm{d}y \leq \frac{CR^2}{V(R)} \int_{B(x,R)} W(y)\mathrm{d}y \leq C\Big(1 + \frac{R}{\rho(x)}\Big)^{l_1}.$$

See [5] for the proofs of Lemmas 1–5.

Let $\Gamma(x, y, \lambda)$ denote the fundamental solution for the operator $-\Delta_G + W + \lambda$, where $\lambda \geq 0$. The following estimates of the fundamental solution for the Schrödinger operator on the nilpotent Lie group have been proved in [5].

Lemma 6 *Let $l > 0$ be an integer. Suppose $W \in B_{\frac{D}{2}}$. Then there exists $C_l > 0$ such that for $x \neq y$,*

$$\big|\Gamma(x,y,\lambda)\big| \leq \frac{C_l}{\big(1+\mathrm{d}(x,y)\lambda^{\frac{1}{2}}\big)^l\big(1+\mathrm{d}(x,y)\rho(x)^{-1}\big)^l}\frac{\mathrm{d}(x,y)^2}{V(\mathrm{d}(x,y))}.$$

Lemma 7 *Let $l > 0$ be an integer. Suppose $W \in B_{\frac{D}{2}}$. Then there exists $C_l > 0$ such that for $x \neq y$,*

$$\begin{aligned}\big|\nabla_{G,\,y}\Gamma(y,x,\lambda)\big| \leq & \frac{C_l}{\big(1+\mathrm{d}(x,y)\,\lambda^{\frac{1}{2}}\big)^l\big(1+\mathrm{d}(x,y)\rho(x)^{-1}\big)^l}\frac{\mathrm{d}(x,y)^2}{V(\mathrm{d}(x,y))}\times \\ & \Big\{\int_{B(y,\frac{1}{4}\mathrm{d}(x,y))}\frac{\mathrm{d}(y,h)}{V(\mathrm{d}(y,h))}W(h)\mathrm{d}h+\frac{1}{\mathrm{d}(x,y)}\Big\}.\end{aligned}$$

In particular, when $W \in B_{\frac{D}{2}}$, there exists $C_l > 0$ such that for $x \neq y$,

$$\big|\nabla_{G,y}\Gamma(y,x,\lambda)\big| \leq \frac{C_l}{\big(1+\mathrm{d}(x,y)\lambda^{\frac{1}{2}}\big)^l\big(1+\mathrm{d}(x,y)\rho(x)^{-1}\big)^l}\frac{\mathrm{d}(x,y)}{V(\mathrm{d}(x,y))}.$$

In order to prove Corollarys 1–4, we need to introduce the theory of the weighted norm inequalities for fractional maximal operators and fractional integral operators on spaces of homogeneous type in [9].

Let (X,d,μ) be a space of homogeneous type, where d is a quasi-distance and μ is a positive measure defined on a σ-algebra of subsets of X and satisfies the doubling condition. It follows from [9] that the nilpotent Lie group G endowed with the Carnot-Carathedory metric d is also a space of homogeneous type. let M_δ be the fractional maximal operator on the space of homogeneous type X which is defined, for each $\delta \in [0,1)$, by

$$M_\delta f(x) = \sup_{x\in B}\frac{1}{\mu(B)^{1-\delta}}\int_B |f(y)|\mathrm{d}\mu(y), \quad f \in L^1_{\mathrm{loc}}(X,\mathrm{d}\mu).$$

Let I_δ be the fractional integral operator on the space of homogeneous type X which is defined, for each $\delta \in (0,1)$, by

$$I_\delta f(x) = \int_X \frac{f(y)}{\mu(B(y,\mathrm{d}(x,y)))^{1-\delta}}\mathrm{d}\mu(y), \quad f \in L^1(X,\mathrm{d}\mu).$$

A weight ω is a nonnegative function in $L^1_{\mathrm{loc}}(X,\mathrm{d}\mu)$ and we shall use $\omega(A)$ to denote $\int_A \omega\mathrm{d}\mu$. We say that a weight ω belongs to A_∞ if there exist positive constants $C > 0$ and $\delta > 0$ such that

$$\frac{\mu(E)}{\mu(B)} \leq C\Big(\frac{\omega(E)}{\omega(B)}\Big)^\delta$$

holds for every ball B and every measurable set $E \subseteq B$.

Proposition 1 *(1) Suppose $0 \leq \delta < 1$ and $1 < p \leq q < \infty$. Let (w,v) be a pair of weight with $v^{-\frac{1}{p-1}} \in A_\infty$. Then*

$$\| M_\delta f \|_{L^q(X,w\mathrm{d}\mu)} \leq C \| f \|_{L^p(X,v\mathrm{d}\mu)},$$

if and only if

$$\frac{1}{\mu(B)^{(1-\delta)p}}\Big(\int_B w\mathrm{d}\mu\Big)^{\frac{p}{q}}\Big(\int_B v^{-\frac{1}{p-1}}\mathrm{d}\mu\Big)^{p-1} \leq C < \infty, \quad \textit{for every ball } B \subseteq X.$$

(2) Suppose $0<\delta<1$, (w,v) *be a pair of weight with* $w\in A_\infty$ *and* $v^{-\frac{1}{p-1}}\in A_\infty$. *Then*

$$\| I_\delta f \|_{L^q(X,w\mathrm{d}\mu)}\le C \| f \|_{L^p(X,v\mathrm{d}\mu)},$$

if and only if

$$\frac{1}{\mu(B)^{(1-\delta)p}}\Big(\int_B w\mathrm{d}\mu\Big)^{\frac{p}{q}}\Big(\int_B v^{-\frac{1}{p-1}}\mathrm{d}\mu\Big)^{p-1}\le C<\infty,\quad \textit{for every ball } B\subseteq X.$$

3. The proof of the main results

Proof of Theorem 1 By the functional calculus, we may write, for all $0<\beta<1$,

$$(-\Delta_G+W)^{-\beta}=\frac{1}{\pi}\int_0^\infty \lambda^{-\beta}(-\Delta_G+W+\lambda)^{-1}\mathrm{d}\lambda. \tag{3}$$

Let $f\in C_0^\infty(G)$. From $(-\Delta_G+W+\lambda)^{-1}f(x)=\int_G\Gamma(x,y,\lambda)f(y)\mathrm{d}y$, it follows that

$$T_1 f(x)=\int_G K_1(x,y)W(x)^\alpha f(y)\mathrm{d}y, \tag{4}$$

where

$$K_1(x,y)=\begin{cases}\frac{1}{\pi}\int_0^\infty \lambda^{-\beta}\Gamma(x,y,\lambda)\mathrm{d}\lambda, & \text{for } 0<\beta<1,\\ \Gamma(x,y,0), & \text{for } \beta=1.\end{cases} \tag{5}$$

Let $f\in C_0^\infty(G)$. The adjoint of T_1 is given by

$$T_1^*f(x)=\int_G \overline{K_1(y,x)}W(y)^\alpha f(y)\mathrm{d}y.$$

By Lemma 6, for all $0<\beta\le 1$ and all integer $l\ge 2$, there exists a constant $C_l>0$ such that

$$\big|\overline{K_1(y,x)}\big|\le \frac{C_l}{\big(1+\mathrm{d}(x,y)\rho(x)^{-1}\big)^l}\frac{\mathrm{d}(x,y)^{2\beta}}{V(\mathrm{d}(x,y))}. \tag{6}$$

Let $r=\rho(x)$. It follows from Hölder's inequality that

$$\begin{aligned}\big|T_1^*f(x)\big|\le&\int_G\frac{C_l}{\big(1+\mathrm{d}(x,y)\rho(x)^{-1}\big)^l}\frac{\mathrm{d}(x,y)^{2\beta}}{V(\mathrm{d}(x,y))}W(y)^\alpha|f(y)|\mathrm{d}y\\ \le& C_l\sum_{j=-\infty}^{\infty}\int_{2^{j-1}r<\mathrm{d}(x,y)\le 2^jr}\frac{1}{\big(1+2^{j-1}\big)^l}\frac{(2^{j-1}r)^{2\beta}}{V(2^{j-1}r)}W(y)^\alpha|f(y)|\mathrm{d}y\\ \le& CC_l\sum_{j=-\infty}^{\infty}\frac{(2^jr)^{2\beta}V(2^{j-1}r)^{-\varepsilon}}{(1+2^{j-1})^l}\Big\{\frac{1}{V(2^{j-1}r)}\int_{B(x,2^jr)}W(y)^{q_1}\mathrm{d}y\Big\}^{\frac{\alpha}{q_1}}\\ &\Big\{\frac{1}{V(2^{j-1}r)^{1-\varepsilon q_2}}\int_{B(x,2^jr)}|f(y)|^{q_2}\mathrm{d}y\Big\}^{\frac{1}{q_2}}.\end{aligned}$$

Letting $\varepsilon=\frac{2(\beta-\alpha)}{\theta}$, where $\theta\in[d,D]$ and using (2) we know that

$$\begin{aligned}\big|T_1^*f(x)\big|\le& CC_l\{M_{\varepsilon q_2}(|f|^{q_2})(x)\}^{q_2}\sum_{j=-\infty}^{\infty}\frac{(2^jr)^{2\beta-2\alpha}V(2^{j-1}r)^{-\varepsilon}}{(1+2^{j-1})^l}\Big\{\frac{(2^jr)^2}{|B(x,2^jr)|}\int_{B(x,2^jr)}W(y)\mathrm{d}y\Big\}^\alpha\\ \le& CC_l\{M_{\varepsilon q_2}(|f|^{q_2})(x)\}^{q_2}\Big\{\sum_{j\le 1+\log_2\frac{1}{r}}\frac{(2^jr)^{2\beta-2\alpha}V(2^{j-1}r)^{-\varepsilon}}{(1+2^{j-1})^l}\Big\{\frac{(2^jr)^2}{|B(x,2^jr)|}\int_{B(x,2^jr)}W(y)\mathrm{d}y\Big\}^\alpha+\end{aligned}$$

$$\sum_{j>1+\log_2\frac{1}{r}}\frac{(2^jr)^{2\beta-2\alpha}V(2^{j-1}r)^{-\varepsilon}}{(1+2^{j-1})^l}\Big\{\frac{(2^jr)^2}{|B(x,2^jr)|}\int_{B(x,2^jr)}W(y)\mathrm{d}y\Big\}^\alpha\Big\}$$

$$\leq CC_l\{M_{\varepsilon q_2}(|f|^{q_2})(x)\}^{q_2}\Big\{\sum_{j\leq 1+\log_2\frac{1}{r}}\frac{(2^jr)^{(2\beta-2\alpha)(1-\frac{d}{\theta})}}{(1+2^{j-1})^l}\Big\{\frac{(2^jr)^2}{|B(x,2^jr)|}\int_{B(x,2^jr)}W(y)\mathrm{d}y\Big\}^\alpha+$$

$$\sum_{j>1+\log_2\frac{1}{r}}\frac{(2^jr)^{(2\beta-2\alpha)(1-\frac{D}{\theta})}}{(1+2^{j-1})^l}\Big\{\frac{(2^jr)^2}{|B(x,2^jr)|}\int_{B(x,2^jr)}W(y)\mathrm{d}y\Big\}^\alpha\Big\}$$

$$\leq CC_l\{M_{\varepsilon q_2}(|f|^{q_2})(x)\}^{q_2}\sum_{j=-\infty}^{\infty}\frac{1}{(1+2^{j-1})^l}\Big\{\frac{(2^jr)^2}{|B(x,2^jr)|}\int_{B(x,2^jr)}W(y)\mathrm{d}y\Big\}^\alpha.$$

By Lemma 5 we conclude that for the case $j\geq 1$ there exists a constant $C>0$ such that

$$\frac{(2^jr)^2}{|B(x,2^jr)|}\int_{B(x,2^jr)}W(y)\mathrm{d}y\leq C(2^j)^{l_1}. \tag{7}$$

For the case $j\leq 0$, by using Lemma 2 we see that

$$\frac{(2^jr)^2}{|B(x,2^jr)|}\int_{B(x,2^jr)}V(y)\mathrm{d}y\leq C\big(\frac{r}{2^jr}\big)^{\frac{D}{q_1}-2}\frac{r^2}{|B(x,r)|}\int_{B(x,r)}V(y)\mathrm{d}y=C(2^j)^{2-\frac{D}{q_1}}. \tag{8}$$

Thus,

$$\begin{aligned}\big|T_1^*f(x)\big|&\leq CC_l\{M_{\varepsilon q_2}(|f|^{q_2})(x)\}^{q_2}\Big\{\sum_{j=1}^{\infty}\frac{(2^j)^{l_1}}{(1+2^{j-1})^l}+\sum_{j=-\infty}^{0}(2^j)^{2-\frac{D}{q_1}}\Big\}\\&\leq C\{M_{\varepsilon q_2}(|f|^{q_2})(x)\}^{\frac{1}{q_2}},\end{aligned}$$

where we take l sufficiently large. □

Proof of Theorem 2 Let $f\in C_0^\infty(G)$. Similar to (4) and (5), the adjoint of T_2 is also given by

$$T_2^*f(x)=\int_G\overline{K_2(y,x)}W(y)^\alpha f(y)\mathrm{d}y.$$

Case $q_1\geq D$: By Lemma 7, for all $0<\beta\leq 1$ and all integer $l\geq 2$, there exists a positive constant C_l such that

$$|\overline{K_2(y,x)}|\leq\frac{C_l}{\big(1+\mathrm{d}(x,y)\rho(x)^{-1}\big)^l}\frac{\mathrm{d}(x,y)^{2\beta-1}}{V(\mathrm{d}(x,y))}.$$

Let $r=\rho(x)$. Then similar to the proof of Theorem 1 we have

$$\big|T_2^*f(x)\big|\leq CC_l\sum_{j=-\infty}^{\infty}\frac{(2^jr)^{2\beta-1}}{(1+2^{j-1})^l}\big\{\frac{1}{V(2^jr)}\int_{B(x,2^jr)}W(y)^{q_1}\mathrm{d}y\big\}^{\frac{\alpha}{q_1}}\big\{\frac{1}{V(2^jr)}\int_{B(x,2^jr)}|f(y)|^{q_2}\mathrm{d}y\big\}^{\frac{1}{q_2}}.$$

Letting $\varepsilon=\frac{2(\beta-\alpha)-1}{\theta}$, where $\theta\in[d,D]$. Similar to the estimates of $\big|T_1^*f(x)\big|$ we conclude that

$$\big|T_2^*f(x)\big|\leq CC_l\{M_{\varepsilon q_2}(|f|^{q_2})(x)\}^{\frac{1}{q_2}}.$$

Case $\frac{D}{2}<q_1<D$: Fix $x_0,y_0\in G$. Let $R=\frac{\mathrm{d}(x_0,y_0)}{4}$. By Lemma 7 we get, for all positive

integer l, there exists a positive constant C_l such that

$$\left|\nabla_{G,y}\Gamma(y_0,x_0,\lambda)\right| \le \frac{C_l}{(1+R\lambda^{\frac{1}{2}})^l\big(1+R\,\rho(x_0)^{-1}\big)^l}\Big(\frac{R^2}{V(R)}\int_{B(y_0,\frac{1}{4}R)}\frac{\mathrm{d}(y_0,y)W(y)\mathrm{d}y}{V(\mathrm{d}(y_0,y))}+\frac{R}{V(R)}\Big).$$

Then we see that there exists a positive constant C_l such that for all integer $l \ge 2$,

$$\left|\overline{K_2(y_0,x_0)}\right| \le \frac{C_l}{\big(1+R\,\rho(x_0)^{-1}\big)^l}\Big(\frac{R^{2\beta}}{V(R)}\int_{B(y_0,\frac{1}{4}R)}\frac{\mathrm{d}(y_0,y)W(y)\mathrm{d}y}{V(\mathrm{d}(y_0,y))}+\frac{R^{2\beta-1}}{V(R)}\Big).$$

Let $r=\rho(x)$ and choose p_1 such that $\frac{1}{p_1}=\frac{1}{q_1}-\frac{1}{D}$. Note that $\frac{1}{p_1}+\frac{\alpha}{q_1}+\frac{1}{q_2}=1$. By Hölder inequality, we obtain

$$\begin{aligned}\left|T_2^*f(x)\right| &\le \sum_{j=-\infty}^{\infty}\int_{2^{j-1}r<\mathrm{d}(x,y)\le 2^jr}\left|\overline{K_2(y,x)}\right|W(y)^\alpha|f(y)|\mathrm{d}y\\ &\le \sum_{j=-\infty}^{\infty}V(2^jr)\Big\{\frac{1}{V(2^jr)}\int_{2^{j-1}r<\mathrm{d}(x,y)\le 2^jr}\left|\overline{K_2(y,x)}\right|^{p_1}\mathrm{d}y\Big\}^{\frac{1}{p_1}}\\ &\quad\Big\{\frac{1}{V(2^jr)}\int_{B(x,2^jr)}W(y)^{q_1}\mathrm{d}y\Big\}^{\frac{\alpha}{q_1}}\Big\{\frac{1}{V(2^jr)}\int_{B(x,2^jr)}|f(y)|^{q_2}\mathrm{d}y\Big\}^{\frac{1}{q_2}}.\end{aligned}$$

Using Minkowski's inequality and the well known theorem on fractional integrals on the nilpotent Lie group (see (1.7) in [5]), we obtain

$$\begin{aligned}&V(2^jr)\Big\{\frac{1}{V(2^jr)}\int_{2^{j-1}r<\mathrm{d}(x,y)\le 2^jr}\left|\overline{K_2(y,x)}\right|^{p_1}\mathrm{d}y\Big\}^{\frac{1}{p_1}}\\ &\le \frac{CC_lV(2^jr)}{(1+2^{j-3})^l}\Big\{\frac{(2^jr)^{2\beta+1}}{V(2^jr)}\Big[\frac{1}{V(2^jr)}\int_{B(x,2^{j-2}r)}W(y)^{q_1}\mathrm{d}y\Big]^{\frac{1}{q_1}}+\frac{(2^jr)^{2\beta-1}}{V(2^jr)}\Big\}\\ &\le \frac{C'C_l(2^jr)^{2\beta-1}}{(1+2^{j-3})^l}\Big[\frac{(2^{j-2}r)^2}{V(2^{j-2}r)}\int_{B(x,2^{j-2}r)}W(y)\mathrm{d}y+1\Big].\end{aligned}$$

For the case $j \ge 1$, using (7) we have

$$V(2^jr)\Big\{\frac{1}{V(2^jr)}\int_{2^{j-1}r<\mathrm{d}(x,y)\le 2^jr}\left|\overline{K_2(y,x)}\right|^{p_1}\mathrm{d}y\Big\}^{\frac{1}{p_1}} \le C'C_l\frac{2^{jl_1}(2^jr)^{2\beta-1}}{(1+2^{j-3})^l}.$$

For the case $j \le 0$, using (8) we obtain

$$V(2^jr)\Big\{\frac{1}{V(2^jr)}\int_{2^{j-1}r<\mathrm{d}(x,y)\le 2^jr}\left|\overline{K_2(y,x)}\right|^{p_1}\mathrm{d}y\Big\}^{\frac{1}{p_1}} \le C'C_l\frac{(2^jr)^{2\beta-1}}{(1+2^{j-3})^l}.$$

Then it follows that

$$\begin{aligned}\left|T_2^*f(x)\right| \le& C'C_l\{M_{\varepsilon q_2}(|f|^{q_2})(x)\}^{\frac{1}{q_2}}\Big\{\sum_{j=1}^{\infty}\frac{2^{jk_0}}{(1+2^{j-3})^l}+\sum_{j=-\infty}^{0}\frac{1}{(1+2^{j-3})^l}\Big\}\\ &\Big[\frac{(2^jr)^2}{V(2^jr)}\int_{B(x,2^{j-2}r)}W(y)\mathrm{d}y\Big]^\alpha\frac{(2^jr)^{2\beta-2\alpha-1}}{V(2^jr)^\varepsilon},\end{aligned}$$

where $\varepsilon=\frac{2(\beta-\alpha)-1}{\theta}, \theta\in[d,D]$. Combining (7) and (8) again and similar to the estimates of $\left|T_1^*f(x)\right|$, we get

$$|T_2^*f(x)| \le C\{M_{\varepsilon q_2}(|f|^{q_2})(x)\}^{\frac{1}{q_2}}. \quad \square$$

Proof of Corollary 1 Case $\alpha > 0$: Note that $\theta \in [d, D]$. Let $\gamma = 2(\beta - \alpha)$ and $\frac{1}{q_2} = 1 - \frac{\alpha}{q_1}$. For q' such that $q_2 < q' < \frac{\theta}{\gamma}$ and $\frac{1}{p'} = \frac{1}{q'} - \frac{\gamma}{\theta}$, then it follows from the assumptions that

$$0 < \gamma q_2 < \theta,\ 1 < \frac{q'}{q_2} < \frac{\theta}{\gamma q_2},\ \frac{1}{p'/q_2} = \frac{1}{q'/q_2} - \frac{\gamma q_2}{\theta}.$$

By Theorem 1 and Proposition 1(1), there exists a positive constant C such that for any $f \in C_0^\infty(G)$,

$$\| (T_1^* f) w^{-1} \|_{L^{p'}(G)} \leq C \| f w^{-1} \|_{L^{q'}(G)} .$$

Since $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$, the desired estimate follows by duality.

Case $\alpha = 0$: Since $q_2 = 1$, so the condition for w is $w^{-1} \in A_{p',q'}$, which is equivalent to $w \in A_{p,q}$. Then following the same idea of the proof of Corollary 1 in [7], we get the desired estimate. □

Proof of Corollary 2 Case $\alpha > 0$: Let $\gamma = 2(\beta - \alpha) - 1$ and $\frac{1}{q_2} = 1 - \frac{1}{p_1}$,

$$\frac{1}{p_1} = \begin{cases} \frac{\alpha}{q_1}, & \text{if } q_1 > D, \\ \frac{(\alpha+1)}{q_1} - \frac{1}{D}, & \text{if } \frac{D}{2} < q_1 < D. \end{cases}$$

Note that $\theta \in [d, D]$. For q' such that $q_2 < q' < \frac{\theta}{\gamma}$ and $\frac{1}{p'} = \frac{1}{q'} - \frac{\gamma}{\theta}$, then it follows from the assumptions that

$$0 < \gamma q_2 < \theta,\ 1 < \frac{q'}{q_2} < \frac{\theta}{\gamma q_2},\ \frac{1}{p'/q_2} = \frac{1}{q'/q_2} - \frac{\gamma q_2}{\theta}.$$

By Theorem 2 and Proposition 1(1), there exists a positive constant C such that for any $f \in C_0^\infty(G)$,

$$\| (T_2^* f) w^{-1} \|_{L^{p'}(G)} \leq C \| f w^{-1} \|_{L^{q'}(G)} .$$

Since $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$, so the desired estimate follows by duality.

Case $\alpha = 0$ and $\frac{1}{2} < \beta \leq 1$: Using the estimates of the kernel $K_2(x, y)$ and following the same idea of the proof of Corollary 3 in [7], we get the desired estimate.

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