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On Skew Armendariz Matrix Rings

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Abstract Let R be a ring. We show in the paper that the subring $U_n(R)$ of the upper triangular matrix ring $T_n(R)$ is $\overline{\alpha}$-skew Armendariz if and only if R is α-rigid, also it is maximal in some non $\overline{\alpha}$-skew Armendariz rings, where α is a ring endomorphism of R with $\alpha(1)=1$.

Keywords Armendariz ring; skew Armendariz ring; reduced ring; matrix ring.

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1. Introduction and definitions

Throughout this paper, all the rings are associative with an identity 1 ($\neq 0$). Rege and Chhawchharia [7] introduced the notion of an Armendariz ring and called a ring R Armendariz if whenever polynomials $f(x)=a_0+a_1x+\cdots+a_mx^m, g(x)=b_0+b_1x+\cdots+b_nx^n\in R[x]$ satisfy $f(x)g(x)=0$, then $a_ib_j=0$ for each i,j. The name "Armendariz" was chosen because Armendariz [5, Lemma 1] had noted that a reduced ring (has no nonzero nilpotent elements) satisfies this condition. The interest of this notion lies in its natural and useful role in understanding the relation between the annihilators of the ring R and the annihilators of polynomial ring $R[x]$. For example, for an Armendariz ring R, R is Baer if and only if $R[x]$ is Baer [5, 8], and R is right Goldie if and only if $R[x]$ is right Goldie [11]. Some other properties of Armendariz rings have been studied by Huh et al. [1], Kim and Lee [8] and Anderson and Camillo [4]. Hong et al. [2] and Lee and Wong [9] have studied generalizations of Armendariz rings, respectively, skew Armendariz and weak Armendariz. Lee and Zhou [10] have considered some "relative maximal" Armendariz subrings of matrix rings over reduced rings.

Recall that an endomorphism α of a ring R is called rigid if $a\alpha(a)=0$ implies $a=0$ for $a\in R$ (see [6]). We call a ring R α-rigid if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is a monomorphism and α-rigid rings are reduced rings by Hong et al. [3]. Recall that for a ring R with a ring endomorphism $\alpha: R\to R$, a skew

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polynomial ring (also called an Ore extension of endomorphism type) $R[x;\alpha]$ of R is the ring obtained by giving the polynomial ring over R with the new multiplication $xr = \alpha(r)x$ for all $r \in R$. According to Hong et al. [2], a ring R is called α-skew Armendariz, if whenever elements $f = a_0 + a_1x^1 + \cdots + a_nx^n$ and $g = b_0 + b_1x^1 + \cdots + b_mx^m \in R[\alpha;x]$ satisfy $fg = 0$, then $a_i\alpha^i(b_j) = 0$ for each i, j. Clearly, if $\alpha = 1$, then the notion of α-skew Armendariz coincides with the Armendariz.

In this paper, we are motivated by results in Lee and Zhou [10] and Hong et al. [2] to continue the study of skew Armendariz rings and try to find some "relatively maximal" skew Armendariz subrings of matrix rings over a ring R.

We first fix some notations. Let R be a ring. We write $M_n(R)$ and $T_n(R)$ for the $n\times n$ matrix ring and $n \times n$ upper triangular matrix ring over R, respectively. The $n \times n$ identity matrix is denoted by I_n. For any $A \in M_n(R)$, let $RA = \{rA : r \in R\}$. For $n \geq 2$, let $\{E_{i,j} : 1 \leq i, j \leq n\}$ be the set of the matrix units. Let $\alpha : R \to R$ be a ring endomorphism with $\alpha(1) = 1$. For any $A = (a_{i,j}) \in M_n(R)$, we define $\overline{\alpha} : M_n(R) \to M_n(R)$ by $\overline{\alpha}((a_{i,j})_{n\times n}) = (\alpha(a_{i,j}))_{n\times n}$, and so $\overline{\alpha}$ is a ring endomorphism of the ring $M_n(R)$.

Define a subring R_n of the $n \times n$ matrix ring $M_n(R)$ over R as follows:

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} : a, a_{ij} \in R \right\}.$$

Note that R_3 is $\overline{\alpha}$-skew Armendariz when R is α-rigid, but R_n is not for $n \geq 4$ (see [2, Proposition 17, Example 18]). In addition, Hong et al. [2] gave an example to show that the condition of α-rigid cannot be weakened to be reduced. In the paper we try to find some "relatively maximal" $\overline{\alpha}$-skew Armendariz subrings of $T_n(R)$ for $n \geq 2$ when R is a α-rigid ring. For this purpose, we introduce the following notation.

For a positive integer $n \geq 2$, let

$$U_n(R) = \sum_{i=1}^{k} \sum_{j=k+1}^{n} RE_{i,j} + \sum_{j=k+2}^{n} RE_{k+1,j} + RI_n,$$

where $k = [n/2]$, i.e., k satisfies $n = 2k$ when n is an even integer, and $n = 2k + 1$ when n is an odd integer.

Note that if $n = 3$, then the ring $U_3(R) = R_3$ is $\overline{\alpha}$-skew Armendariz when R is an α-rigid ring.

Lemma 1.1 ([2, Proposition 3]) *Let $\alpha : R \to R$ be a ring endomorphism. Then $R[x;\alpha]$ is reduced if and only if R is α-rigid.*

Lemma 1.2 ([3, Lemma 4]) *Let R be an α-rigid ring and $a, b \in R$. If $ab = 0$ then $a\alpha^n(b) = \alpha^n(a)b = 0$ for any positive integer n.*

Lemma 1.3 ([3, Proposition 6]) *Suppose that R is an α-rigid ring. Let $f(x)=\sum_{i=0}^{m}a_ix^i$ and $g(x)=\sum_{j=0}^{n}b_jx^j$ in $R[\alpha;x]$. Then $f(x)g(x)=0$ if and only if $a_ib_j=0$ for all $0\le i\le m$, $0\le j\le n$.*

Corollary 1.4 *Let $\alpha:R\to R$ be a ring endomorphism. If R is α-rigid, then the following statements hold:*

1) *R is reduced;*
2) *R is α-skew Armendariz.*

Proof 1) By Lemma 1.1, it is trivial.

2) It follows easily from Lemmas 1.2 and 1.3. □

2. α-skew Armendariz properties of matrix rings

Lemma 2.1 *Let $\alpha:R\to R$ be a ring endomorphism. Then the ring $U_n(R)+RE_{l,k}$ is not $\overline{\alpha}$-skew Armendariz for every $n\ge 4$ and any $l\in\{1,2,\ldots,k-1\}$, where $k=[n/2]$.*

Proof It is easy to check that $U_n(R)+RE_{l,k}$ is a ring. Suppose that $f=E_{l,k}+(E_{l,k}-E_{l,k+1})x$ and $g=E_{k+1,k+2}+(E_{k,k+2}+E_{k+1,k+2})x$ in $(U_n(R)+RE_{l,k})[x;\alpha]$. Then we have $fg=0$, but $(E_{l,k}-E_{l,k+1})\overline{\alpha}(E_{k+1,k+2})\neq 0$. This proves that $U_n(R)+RE_{l,k}$ is not $\overline{\alpha}$-skew Armendariz for every $n\ge 4$ and any $l\in\{1,2,\ldots,k-1\}$, where $k=[n/2]$. □

Lemma 2.2 *Let $\alpha:R\to R$ be a ring endomorphism. If R is α-rigid, then the ring $U_n(R)$ is $\overline{\alpha}$-skew Armendariz for every $n=2k+1\ge 3$.*

Proof Let $f=\sum_{i=0}^{s}A_ix^i$, $g=\sum_{j=0}^{t}B_jx^j\in U_n(R)[x;\overline{\alpha}]$ be such that $fg=0$, where $A_i,B_j\in U_n(R)$. We claim that $A_i\overline{\alpha}^{\,i}(B_j)=0$ for each i,j. Suppose that $A_i=(a_{uv}^{(i)})\in U_n(R)$, $i=0,\ldots,s$, and $B_j=(b_{uv}^{(j)})\in U_n(R)$, $j=0,\ldots,t$. We use $a^{(i)}$ and $b^{(j)}$, respectively, to denote $a_{uv}^{(i)}$ and $b_{uv}^{(j)}$ for $u=v$.

Write $f=(f_{uv})$. Then $f_{uu}=\sum_{i=0}^{s}a^{(i)}x^i$, $f_{uv}=\sum_{i=0}^{s}a_{u,v}^{(i)}x^i$, $g_{uu}=\sum_{j=0}^{t}b^{(j)}x^j$, $g_{uv}=\sum_{j=0}^{t}b_{u,v}^{(j)}x^j$, where $1\le u\le k+1\le v\le 2k+1$ with $u\neq v$, and other f_{uv}'s and g_{uv}'s are all zero. For convenience, let $f_0=f_{uu}$ and $g_0=g_{uu}$. Thus we have

$$f_0g_0=0 \tag{1}$$

and for any $p,q\in\{1,2,\ldots,k\}$

$$f_0g_{p,k+1}+f_{p,k+1}g_0=0, \tag{2}$$

$$f_0g_{p,k+1+q}+f_{p,k+1}g_{k+1,k+1+q}+f_{p,k+1+q}g_0=0, \tag{3}$$

$$f_0g_{k+1,k+1+q}+f_{k+1,k+1+q}g_0=0. \tag{4}$$

From equation (1), we have that $a^{(i)}\alpha^i(b^{(j)})=0$ for all i and j since R is α-skew Armendariz by Corollary 1.4. We also have that $g_0f_0=0$ since $R[x;\alpha]$ is reduced by Lemma 1.1. If we multiply equation (2) by g_0 on the left side, then $g_0f_0g_{p,k+1}+g_0f_{p,k+1}g_0=0$ for $p=1,2,\ldots,k$. Thus we get that $g_0f_{p,k+1}g_0=0$. Since $R[x;\alpha]$ is reduced, it follows that $f_{p,k+1}g_0=0$ (hence

$g_0 f_{p,k+1} = 0$) and so $f_0 g_{p,k+1} = 0$ (hence $g_{p,k+1} f_0 = 0$) for $p = 1, 2, \ldots, k$. Thus we get that $a^{(i)}_{p,k+1}\alpha^i(b^{(j)}) = 0$ and $a^{(i)}\alpha^i(b^{(j)}_{p,k+1}) = 0$ for $p = 1, 2, \ldots, k$ and all i and j since R is α-skew Armendariz. Similarly, for equation (4), continue using the same manner, we can show that $f_0 g_{k+1,k+1+q} = g_{k+1,k+1+q} f_0 = 0$ and $f_{k+1,k+1+q} g_0 = g_0 f_{k+1,k+1+q} = 0$ for $q = 1, 2, \ldots, k$. Hence we have that $a^{(i)}\alpha^i(b^{(j)}_{k+1,k+1+q}) = a^{(i)}_{k+1,k+1+q}\alpha^i(b^{(j)}) = 0$ for $q = 1, 2, \ldots, k$ and all i, j. If we multiply equation (3) on the left side by g_0, then we obtain that

$$0 = g_0 f_0 g_{p,k+1+q} + g_0 f_{p,k+1} g_{k+1,k+1+q} + g_0 f_{p,k+1+q} g_0 = g_0 f_{p,k+1+q} g_0$$

for any $p, q \in \{1, 2, \ldots, k\}$, we have $f_{p,k+1+q} g_0 = 0$. Thus $a^{(i)}_{p,k+1+q}\alpha^i(b^{(j)}) = 0$ for all i, j and $p, q \in \{1, 2, \ldots, k\}$ since R is α-skew Armendariz. Hence

$$f_0 g_{p,k+1+q} + f_{p,k+1} g_{k+1,k+1+q} = 0 \text{ for any } p, q \in \{1, 2, \ldots, k\}. \tag{*}$$

Multiplying equation (*) on the right side by f_0, we obtain

$$0 = f_0 g_{p,k+1+q} f_0 + f_{p,k+1} g_{k+1,k+1+q} f_0 = f_0 g_{p,k+1+q} f_0$$

for any $p, q \in \{1, 2, \ldots, k\}$. Since $R[x; \alpha]$ is reduced, we have $f_0 g_{p,k+1+q} = 0$. Thus

$$a^{(i)}\alpha^i(b^{(j)}_{p,k+1+q}) = 0$$

for all i, j and $p, q \in \{1, 2, \ldots, k\}$, since R is α-skew Armendariz. It also follows from (*) that $f_{p,k+1} g_{k+1,k+1+q} = 0$ for any $p, q \in \{1, 2, \ldots, k\}$. Since R is α-skew Armendariz, we have that $a^{(i)}_{p,k+1}\alpha^i(b^{(j)}_{k+1,k+1+q}) = 0$ for all i, j and $p, q \in \{1, 2, \ldots, k\}$.

Now it is easy to see that $A_i \overline{\alpha}^{\,i}(B_j) = 0$ for all $i = 0, \ldots, s$ and $j = 0, \ldots, t$. Thus $U_n(R)$ is an $\overline{\alpha}$-skew Armendariz ring for every $n = 2k + 1 \geq 3$. □

Lemma 2.3 *If R is an α-rigid ring, then the ring $U_n(R)$ is $\overline{\alpha}$-skew Armendariz for every $n = 2k \geq 2$.*

Proof It is similar to the proof of Lemma 2.2. □

Remark For a positive even integer $n = 2k \geq 2$, if we let

$$V_n(R) = \sum_{i=1}^{k}\sum_{j=k+1}^{n} RE_{i,j} + \sum_{j=1}^{k-1} RE_{j,k} + RI_n,$$

then we have that the ring $V_n(R)$ is $\overline{\alpha}$-skew Armendariz for every $n = 2k \geq 2$ under the conditions of the above lemma.

The following theorem gives our main result that generalizes the result of Hong et al. [2, Proposition 17].

Theorem 2.4 *Let $\alpha : R \to R$ be a ring endomorphism. Then the following statements are equivalent:*

1) *R is α-rigid;*
2) *$U_n(R)$ is $\overline{\alpha}$-skew Armendariz for every $n \geq 2$.*

Proof 1) $\Rightarrow$ 2). It follows from Lemmas 2.2 and 2.3.

2) $\Rightarrow$ 1). Suppose that R is not α-rigid. Then there exists an element $0 \neq a \in R$ such that $a\alpha(a) = 0$. Let $f = E_{1,n} + aI_n x, g = E_{1,n} - aI_n x \in U_n(R)[x;\overline{\alpha}]$, we have $fg = 0$, but $E_{1,n}aI_n \neq 0$, which implies that $U_n(R)$ is not $\overline{\alpha}$-skew Armendariz, which is a contradiction. So the proof is completed. □

The following obvious result gives some Armendariz subrings of $T_n(R)$ for $n \geq 2$ and generalizes the result of Kim and Lee [8, Proposition 2] since reduced rings are Armendariz (hence 1-skew Armendariz).

Corollary 2.5 *If the ring R is reduced, then $U_n(R)$ is Armendariz for every $n \geq 2$.*

Theorem 2.6 *Let $\alpha : R \to R$ be a ring endomorphism. If R is α-rigid, then $U_n(R)$ is a maximal $\overline{\alpha}$-skew Armendariz subring of $U_n(R) + RE_{l,k}$ for every $n \geq 4$ and any $l \in \{1, 2, \dots, k-1\}$, where $k = [n/2]$.*

Proof Suppose that $U_n(R)$ is not a maximal $\overline{\alpha}$-skew Armendariz subring for some $n \geq 4$ and $l \in \{1, 2, \dots, k-1\}$, where $k = [n/2]$. Then there is some $\overline{\alpha}$-skew Armendariz subring W such that W properly contains $U_n(R)$ since $U_n(R)$ is $\overline{\alpha}$-skew Armendariz (by Theorem 2.4). So there is an element $0 \neq a \in R$ such that $0 \neq aE_{l,k} \in RE_{l,k}$. Now let $f = aE_{l,k} + a(E_{l,k} - E_{l,k+1})x$ and $g = E_{k+1,k+2} + (E_{k,k+2} + E_{k+1,k+2})x$ in $W[x;\overline{\alpha}]$. Then we have $fg = 0$, but $a(E_{l,k} - E_{l,k+1})\overline{\alpha}(E_{k+1,k+2}) = aE_{l,k+2} \neq 0$, which is a contradiction. So the proof is completed. □

Corollary 2.7 ([2, Proposition 17]) *Let $\alpha : R \to R$ be a ring endomorphism. If R is α-rigid, then the ring*

$$R_3 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in R \right\}$$

is $\overline{\alpha}$-skew Armendariz.

Given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M is the ring $R \propto M = R \bigoplus M$ with the usual addition and the following multiplication:

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).$$

This is isomorphic to the ring of all matrices

$$\begin{pmatrix} r & m \\ 0 & r \end{pmatrix},$$

where $r \in R$ and $m \in M$ and the usual matrix operations are used.

Corollary 2.8 ([2, Proposition 15]) *Let $\alpha : R \to R$ be a ring endomorphism. If R is α-rigid, then the trivial extension of R by R, $R \propto R$ is $\overline{\alpha}$-skew Armendariz.*

Remark Let $\alpha : R \to R$ be a ring endomorphism. Then the same way can be used to show the following:

1) $V_n(R)$ is a maximal $\overline{\alpha}$-skew Armendariz subring of $V_n(R)+RE_{k+1,l}$ for every $n=2k\geq 4$ and any $l\in\{k+2,k+3,\ldots,n\}$.

2) $U_n(R)$ is a maximal $\overline{\alpha}$-skew Armendariz subring of $U_n(R)+RE_{k+2,l}$ for every $n=2k+1\geq 5$ and any $l\in\{k+3,k+4,\ldots,n\}$.

We end this paper by providing a characterization of an abelian ring to be α-skew Armendariz in terms of its idempotents. Recall that a ring R is abelian if all its idempotents are central.

Proposition 2.9 *Let R be an abelian ring with $\alpha(e)=e$ for any $e=e^2\in R$. Then the following statements are equivalent:*

1) R is α-skew Armendariz;

2) eR and $(1-e)R$ are α-skew Armendariz for some $e=e^2\in R$;

3) eR and $(1-e)R$ are α-skew Armendariz for any $e=e^2\in R$.

Proof It suffices to show 2)$\Rightarrow$1). Let $f=a_0+a_1x+\cdots+a_nx^n$ and $g=b_0+b_1x+\cdots+b_mx^m\in R[x;\alpha]$ satisfy $fg=0$. Then $(ef)(eg)=0$ and $((1-e)f)((1-e)g)=0$ for some $e=e^2\in R$ by hypothesis. Since eR and $(1-e)R$ are α-skew Armendariz, we have $ea_i\alpha^i(b_j)=0$ and $(1-e)a_i\alpha^i(b_j)=0$. Hence $a_i\alpha^i(b_j)=0$ for each i,j, and so R is α-skew Amendariz. $\square$

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