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Strong Convergence by the Shrinking Projection Method for a Generalized Equilibrium Problems and Hemi-Relatively Nonexpansive Mappings

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Abstract Motivated by the recent result obtained by Takahashi and Zembayashi in 2008, we prove a strong convergence theorem for finding a common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a hemi-relatively nonexpansive mapping in a Banach space by using the shrinking projection method. The main results obtained in this paper extend some recent results.

Keywords hemi-relatively nonexpansive mapping; generalized equilibrium problem; α-inverse-strongly monotone mapping.

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1. Introduction

Let C be a nonempty closed convex subset of a real Banach space E, and T a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. Let f be an equilibrium bifunction from $C\times C$ into R, and $A: C\to E^*$ a nonlinear mapping. Now we consider the following generalized equilibrium problem: find $z\in C$ such that

$$f(z,y)+\langle Az, y-z\rangle \geqslant 0,\quad \forall y\in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by EP, i.e.,

$$EP=\{z\in C: f(z,y)+\langle Az, y-z\rangle \geqslant 0,\quad \forall y\in C\}.$$

In the case of $f\equiv 0$, EP is denoted by $VI(C,A)$. In the case of $A\equiv 0$, EP is denoted by $EP(f)$, Takahashi-Zembayashi [1] in 2008 proved a strong convergence theorem for finding a common element of $EP(f)$ and the set of fixed points of a relatively nonexpansive mapping in the framework of uniformly smooth and uniformly convex Banach spaces by using the shrinking projection method. Now, in this paper, we imitatively prove a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem (1.1) and the set

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of fixed points of a hemi-relatively nonexpansive mapping in the same framework by using the similar shrinking projection method.

2. Preliminaries

Let E be a real Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing. It is well known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E. In this case, J is singe valued and also one to one.

Now in the framework of smooth Banach spaces, we consider the function defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \text{ for } x, y \in E.$$

Following Alber [2], the generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) = \arg\min_{y\in C} \phi(y, x), \quad \forall\, x \in E.$$

The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the function $\phi(y, x)$, that is, $\Pi_C x = \widetilde{x}$, where $\widetilde{x}$ is the solution to the minimization problem

$$\phi(\widetilde{x}, x) = \min_{y\in C} \phi(y, x).$$

Existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping J (see [2, 6, 10]). The generalized projection Π_C from E onto C is well defined, single valued and satisfies

$$(\|x\| - \|y\|)^2 \leqslant \phi(y, x) \leqslant (\|x\| + \|y\|)^2, \quad \forall\, x, y \in E. \tag{2.1}$$

If E is a Hilbert space, then $\phi(y, x) = \|y - x\|^2$ and Π_C is the metric projection of H onto C.

T is called hemi-relatively nonexpansive if $\phi(p, Tx) \leqslant \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. A point $p \in C$ is said to be an asymptotic fixed point of T if there exists $\{x_n\}$ in C which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$. Following Matsushita-Takahashi [3], a mapping T is said to be relatively nonexpansive if the following conditions are satisfied:

(1) $F(T)$ is nonempty;

(2) $\phi(p, Tx) \leqslant \phi(p, x)$, for all $p \in F(T)$, $x \in C$;

(3) $\hat{F}(T) = F(T)$.

It is obvious that the class of hemi-relatively nonexpansive mappings contains the class of relatively nonexpansive mappings.

For solving the equilibrium problem for bifunction $f : C \times C \to R$, let us assume that f satisfies the following conditions:

(A_1) $f(x, x) = 0$ for all $x \in C$;

(A_2) f is monotone, i.e., $f(x,y)+f(y,x)\leqslant 0$ for all $x,y\in C$;

(A_3) for each $x,y,z\in C$,

$$\lim_{t\to 0+} f(tz+(1-t)x,y)\leqslant f(x,y);$$

(A_4) for each $x\in C$, $y\to f(x,y)$ is a convex and lower semicontinuous.

Lemma 2.1 *Let E be a strictly convex and smooth real Banach space, C a closed convex subset of E. Let T be a hemi-relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex..*

Proof We firstly prove that $F(T)$ is closed.

Indeed, if $\{x_n\}\subset F(T)$ with $x_n\to x$, then we have $\phi(x_n,Tx)\leqslant\phi(x_n,x)$. Hence,

$$\phi(x,Tx)=\lim_{n\to\infty}\phi(x_n,Tx)\leqslant\lim_{n\to\infty}\phi(x_n,x)=\phi(x,x)=0.$$

This implies $\phi(x,Tx)=0$, and hence $x\in F(T)$.

Finally, we show that $F(T)$ is convex.

Indeed, for any $x,y\in F(T)$, taking $z=tx+(1-t)y$ for $t\in[0,1]$, we have

$$\begin{aligned}\phi(z,Tz)&=\|z\|^2-2\langle z,J(Tz)\rangle+\|Tz\|^2\\&=\|z\|^2-2\langle tx+(1-t)y,J(Tz)\rangle+\|Tz\|^2\\&=\|z\|^2-2t\langle x,J(Tz)\rangle-2(1-t)\langle y,J(Tz)\rangle+\|Tz\|^2\\&=\|z\|^2+t\phi(x,Tz)+(1-t)\phi(y,Tz)-t\|x\|^2-(1-t)\|y\|^2\\&\leqslant\|z\|^2+t\phi(x,z)+(1-t)\phi(y,z)-t\|x\|^2-(1-t)\|y\|^2\\&=\|z\|^2-2\langle tx+(1-t)y,Jz\rangle+\|z\|^2=\phi(z,z)=0.\end{aligned}$$

This implies $z\in F(T)$.

Lemma 2.2 ([4]) *Let C be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E, and let f be a bifunction from $C\times C$ to R satisfying (A_1)–(A_4). Let $r>0$ and $x\in E$. Then there exists $z\in C$ such that*

$$f(z,y)+\frac{1}{r}\langle y-z,Jz-Jx\rangle\geqslant 0,\quad \forall\, y\in C.$$

Lemma 2.3 ([5]) *Let C be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E, and let f be a bifunction from $C\times C$ to R satisfying (A_1)–(A_4). For $r>0$ and $x\in E$, define a mapping $T_r:E\to 2^C$ as follows:*

$$T_r(x)=\{z\in C: f(z,y)+\frac{1}{r}\langle y-z,Jz-Jx\rangle\geqslant 0,\quad \forall\, y\in C\}$$

for all $x\in E$. Then the following holds:

(1) T_r is single-valued;

(2) T_r is a firmly nonexpansive-type mapping, that is, for all $x,y\in E$,

$$\langle T_rx-T_ry,JT_rx-JT_ry\rangle\leqslant\langle T_rx-T_ry,Jx-Jy\rangle;$$

(3) $F(T_r)=\hat{F}(T_r)=EP(f)$;

(4) $EP(f)$ *is closed and convex.*

Lemma 2.4 ([5]) *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E and let f be a bifunction from $C \times C$ to R satisfying (A_1)–(A_4). Then for $r > 0$, $x \in E$, and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leqslant \phi(q, x).$$

Lemma 2.5 ([2, 6]) *Let C be nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leqslant \phi(x, y), \quad \forall\, x \in C, y \in E.$$

Lemma 2.6 ([6]) *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.7 ([7–9]) *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $h : [0, 2r] \to R$ such that $h(0) = 0$ and*

$$h(\|x - y\|) \leqslant \phi(x, y)$$

for all $x, y \in B_r$, where $B_r = \{x \in E : \|x\| \leqslant r\}$.

Recall that an operator S in a Banach space is called closed. If $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

3. The main results

Theorem 3.1 *Let E be a uniformly smooth and uniformly convex Banach space, and C a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4), and S a closed hemi-relatively nonexpansive mapping from C into itself such that $F(S) \cap EP \neq \emptyset$. Assume, $A : C \to E^*$ is α-inverse-strongly monotone mapping. $\{x_n\}$ is a sequence generated by $x_0 = x \in C$, $C_0 = C$ and*

$$\begin{cases} y_n = J^{-1}(a_n J x_n + (1 - a_n) J S x_n), \\ u_n \in C \ \text{ such that } f(u_n, y) + \langle A u_n, y - u_n \rangle + \dfrac{1}{r_n} \langle y - u_n, J u_n - J y_n \rangle \geqslant 0, \ \forall\, y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leqslant \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x, \end{cases} \tag{3.1}$$

for every $n \in \{0\} \cup \mathbb{N}$, where J is the duality mapping on E, $\{a_n\} \subset [0, 1]$ satisfies $\liminf_{n\to\infty} a_n(1 - a_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap EP} x$, where $\Pi_{F(S)\cap EP}$ is the generalized projection of E onto $F(S) \cap EP$.

Proof Firstly, we may define a bifunction $g : C \times C \to R$ by

$$g(x, y) = f(x, y) + \langle Ax, y - x \rangle, \quad \forall\, x, y \in C.$$

We claim that the bifunction g satisfies conditions (A_1)–(A_4).

Indeed, we can see easily that $g(x,x)=0$ for all $x\in C$, i.e., (A_1) holds. Next, we can prove easily that $g(z,y)+g(y,z)\leqslant 0$ for all $y,z\in C$ by way of the assumption that A is α-inverse-strongly monotone. By virtue of the continuity of $x\to\langle Ax,y-x\rangle$, we can conclude g satisfies (A_3). Below, we may prove $y\mapsto g(x,y)$ is convex for any $x\in C$. Indeed,

$$\begin{aligned} g(x,ty+(1-t)z) &= f(x,ty+(1-t)z)+\langle Ax,ty+(1-t)z-x\rangle \\ &\leqslant tf(x,y)+(1-t)f(x,z)+t\langle Ax,y-x\rangle+(1-t)\langle Ax,z-x\rangle \\ &= tg(x,y)+(1-t)g(x,z). \end{aligned}$$

Next, we prove that $y\mapsto g(x,y)$ is lower semi-continuous.

Indeed, if $\{y_n\}\subset C$ with $y_n\to y\in C$, then

$$g(x,y)=f(x,y)+\langle Ax,y-x\rangle\leqslant\liminf_{n\to\infty}f(x,y_n)+\lim_{n\to\infty}\langle Ax,y_n-x\rangle=\liminf_{n\to\infty}g(x,y_n).$$

Thus, (A_4) also holds for $g(x,y)$.

From all the proof above, (3.1) can actually be equivalent to

$$\begin{cases} y_n=J^{-1}(a_nJx_n+(1-a_n)JSx_n), \\ u_n\in C \ \text{ such that } g(u_n,y)+\dfrac{1}{r_n}\langle y-u_n,Ju_n-Jy_n\rangle\geqslant 0,\ \forall\, y\in C, \\ C_{n+1}=\{z\in C_n:\phi(z,u_n)\leqslant\phi(z,x_n)\}, \\ x_{n+1}=\Pi_{C_{n+1}}x, \end{cases}\tag{3.2}$$

where $S:C\to C$ is a nonexpansive mapping defined by (3.2), and $g(x,y)$ is a bifunction satisfying the conditions (A_1)–(A_4). Now we have $EP=EP(g)$, for

$$EP(g)=\{z\in C:g(z,y)\geqslant 0,\ \forall y\in C\}=\{z\in C:f(z,y)+\langle Az,y-z\rangle\geqslant 0,\ \forall y\in C\}=EP.$$

Below, we shall prove $\{x_n\}$ generated by (3.2) converges strongly to $\Pi_{F(S)\cap EP(g)}x$.

Since the bifunction g satisfies conditions (A_1)–(A_4), we know by Lemma 2.3(4) that $EP(g)$ is closed and convex. In addition, Lemma 2.1 tells us that $F(S)$ is also closed and convex so that $\Pi_{F(S)\cap EP(g)}$ is well defined.

Secondly, since the bifunction g satisfies conditions (A_1)–(A_4), we may still denote $u_n=T_{r_n}y_n$ for all $n\in\mathbb{N}$. Then Lemmas 2.3 and 2.4 yield that each T_{r_n} is relatively nonexpansive. We claim that each C_n is closed and convex.

Indeed, since

$$\phi(z,u_n)\leqslant\phi(z,x_n)\Leftrightarrow\|u_n\|^2-\|x_n\|^2-2\langle z,Ju_n-Jx_n\rangle\geqslant 0,$$

C_n is closed and convex for all $n\in\{0\}\cup\mathbb{N}$. This implies each $\Pi_{C_{n+1}}$ is well defined.

Next, we show by induction that $EP(g)\cap F(S)\subset C_n$ for all $n\in\{0\}\cup\mathbb{N}$.

Indeed, from $C_0=C$, we have $F(S)\cap EP(g)\subset C_0$.

Suppose that $F(S)\cap EP(g)\subset C_k$ for some $k\in\{0\}\cup\mathbb{N}$. Let $u\in F(S)\cap EP(g)\subset C_k$. Since T_{r_k} is relatively nonexpansive, and S is hemi-relatively nonexpansive, we get by Lemmas 2.3 and

2.4

$$\begin{aligned}\phi(u,u_k) =&\phi(u,T_{r_k}y_k)\leqslant \phi(u,y_k)\\ =&\phi(u,J^{-1}(a_kJx_k+(1-a_k)JSx_k))\\ =&\|u\|^2-2\langle u,a_kJx_k+(1-a_k)JSx_k\rangle+\|a_kJx_k+(1-a_k)JSx_k\|^2\\ \leqslant&\|u\|^2-2a_k\langle u,Jx_k\rangle-2(1-a_k)\langle u,JSx_k\rangle+a_k\|x_k\|^2+(1-a_k)\|Sx_k\|^2\\ =&a_k\phi(u,x_k)+(1-a_k)\phi(u,Sx_k)\leqslant\phi(u,x_k).\end{aligned}$$

Hence, we have $u\in C_{k+1}$. This implies

$$EP(g)\cap F(S)\subset C_n,\quad \forall n\in\{0\}\cup\mathbb{N}.$$

So, $\{x_n\}$ is well defined.

From the definition of x_n, we get by Lemma 2.5

$$\phi(x_n,x)=\phi(\Pi_{C_n}x,x)\leqslant\phi(u,x)-\phi(u,\Pi_{C_n}x)\leqslant\phi(u,x)$$

for all $u\in F(S)\cap EP(g)\subset C_n$. Then $\phi(x_n,x)$ is bounded. Thereby, both $\{x_n\}$ and $\{Sx_n\}$ are bounded.

From $x_{n+1}\in C_{n+1}\subset C_n$ and $x_n=\Pi_{C_n}x$, we have

$$\phi(x_n,x)\leqslant\phi(x_{n+1},x),\quad \forall\, n\in\{0\}\cup\mathbb{N}.$$

Thus, the limit of $\{\phi(x_n,x)\}$ exists owing to the boundedness of the monotone real sequence $\{\phi(x_n,x)\}$. Denote

$$\lim_{n\to\infty}\phi(x_n,x)=d. \tag{3.3}$$

From Lemma 2.5, we know that for any positive integer m,

$$\phi(x_{n+m},x_n)=\phi(x_{n+m},\Pi_{C_n}x)\leqslant\phi(x_{n+m},x_0)-\phi(x_n,x_0),\quad\forall\, n\in\mathbb{N}, \tag{3.4}$$

and hence

$$\lim_{n\to\infty}\phi(x_{n+m},x_n)=0.$$

Next, we claim that $\{x_n\}$ is a Cauchy sequence. If not, there exists a constant $\varepsilon_0>0$ and subsequences $\{n_k\},\{m_k\}\subset\{n\}$ such that

$$\|x_{n_k+m_k}-x_{n_k}\|\geqslant\varepsilon_0,$$

for all $k\geqslant 1$.

In addition, we get by (3.3) and (3.4)

$$\begin{aligned}\phi(x_{n_k+m_k},x_{n_k})&\leqslant\phi(x_{n_k+m_k},x)-\phi(x_{n_k},x)\\ &\leqslant|\phi(x_{n_k+m_k},x)-d|+|\phi(x_{n_k},x)-d|\to 0,\quad\text{as } k\to\infty.\end{aligned}$$

The boundedness of $\{x_n\}$ can be obtained by (2.1) and (3.3). Hence, we get by Lemma 2.6 that

$$\|x_{n_k+m_k}-x_{n_k}\|\to 0,\quad\text{as } k\to\infty.$$

The contradiction implies that $\{x_n\}$ is a Cauchy sequence.

Since

$$\phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{C_n} x) \leqslant \phi(x_{n+1}, x) - \phi(\Pi_{C_n} x, x) = \phi(x_{n+1}, x) - \phi(x_n, x)$$

for all $n \in \{0\} \cup \mathbb{N}$, we have $\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0$. From $x_{n+1} = \Pi_{C_{n+1}} x \in C_{n+1}$, we get by (3.2)

$$\phi(x_{n+1}, u_n) \leqslant \phi(x_{n+1}, x_n), \quad \forall\, n \in \{0\} \cup \mathbb{N}.$$

Thereby,

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0.$$

Thus, $\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0$ and Lemma 2.6 yield

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0 = \lim_{n\to\infty} \|x_{n+1} - u_n\|,$$

and hence

$$\lim_{n\to\infty} \|x_n - u_n\| = 0.$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|J(x_n) - J(u_n)\| = 0.$$

Let $r = \sup_{n\in\mathbb{N}} \{\|x_n\|, \|Sx_n\|\}$. Since E is a uniformly smooth Banach space, we know that E^* is a uniformly convex Banach space. Therefore, from Lemma 2.7, there exists a continuous, strictly increasing, and convex function h with $h(0) = 0$ such that

$$\|\alpha x^* + (1-\alpha) y^*\|^2 \leqslant \alpha \|x^*\|^2 + (1-\alpha)\|y^*\|^2 - \alpha(1-\alpha) h(\|x^* - y^*\|)$$

for all $x^*, y^* \in B_r^*$ and $\alpha \in [0,1]$, where $B_r^* = \{x^* \in E^* : x^* = Jx,\ x \in B_r\}$. Thanks to the assumptions on the Banach space E, the normalized duality mapping is really a single-valued and one-to-one surjection of E onto E^*, which deduces $B_r^* = \{x^* \in E^* : \|x^*\| \leqslant r\}$. So, for $u \in F(S) \cap EP(g)$, we have

$$\begin{aligned}
\phi(u, u_n) =& \phi(u, T_{r_n} y_n) \leqslant \phi(u, y_n) = \phi(u, J^{-1}(a_n J x_n + (1-a_n) JSx_n)) \\
=& \|u\|^2 - 2\langle u, a_n J x_n + (1-a_n) JSx_n\rangle + \|a_n J x_n + (1-a_n) JSx_n\|^2 \\
\leqslant& \|u\|^2 - 2a_n \langle u, Jx_n\rangle - 2(1-a_n)\langle u, JSx_n\rangle + a_n \|x_n\|^2 + (1-a_n)\|Sx_n\|^2 - \\
& a_n(1-a_n) h(\|Jx_n - JSx_n\|) \\
=& a_n \phi(u, x_n) + (1-a_n)\phi(u, Sx_n) - a_n(1-a_n) h(\|Jx_n - JSx_n\|) \\
\leqslant& \phi(u, x_n) - a_n(1-a_n) h(\|Jx_n - JSx_n\|).
\end{aligned}$$

Therefore, we have

$$a_n(1-a_n) h(\|Jx_n - JSx_n\|) \leqslant \phi(u, x_n) - \phi(u, u_n). \tag{3.5}$$

Since

$$\begin{aligned}
\phi(u, x_n) - \phi(u, u_n) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle u, Jx_n - Ju_n\rangle \\
&\leqslant |\|x_n\| - \|u_n\||(\|x_n\| + \|u_n\|) + 2\|u\| \cdot \|Jx_n - Ju_n\| \\
&\leqslant \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|u\| \cdot \|Jx_n - Ju_n\|,
\end{aligned}$$

we have

$$\lim_{n\to\infty}(\phi(u,x_n)-\phi(u,u_n))=0. \tag{3.6}$$

From $\liminf_{n\to\infty} a_n(1-a_n)>0$, we get by (3.5)

$$\lim_{n\to\infty} h(\|Jx_n-JSx_n\|)=0.$$

The property of h yields

$$\lim_{n\to\infty}\|Jx_n-JSx_n\|=0.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}\|x_n-Sx_n\|=0. \tag{3.7}$$

Since $\{x_n\}$ is a Cauchy sequence, there exists a point $p\in C$ such that $\{x_n\}$ converges strongly to p, i.e.,

$$\lim_{n\to\infty}\|x_n-p\|=0. \tag{3.8}$$

Since S is a closed operator, we know by (3.7) and (3.8) that

$$p\in F(S).$$

Next, we shall show $p\in EP(g)$ so that

$$p\in F(S)\cap EP(g). \tag{3.9}$$

Indeed, since $u_n=T_{r_n}y_n$ and $\phi(u,y_n)\leqslant\phi(u,x_n)$, we get by Lemma 2.4

$$\phi(u_n,y_n)\leqslant\phi(u,y_n)-\phi(u,T_{r_n}y_n)\leqslant\phi(u,x_n)-\phi(u,T_{r_n}y_n)=\phi(u,x_n)-\phi(u,u_n).$$

Then we get by (3.6)

$$\lim_{n\to\infty}\phi(u_n,y_n)=0.$$

So we get by the boundedness of $\{u_n\}$ and Lemma 2.6

$$\lim_{n\to\infty}\|u_n-y_n\|=0. \tag{3.10}$$

Thus, all the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ converge strongly to the same element $p\in F(S)$.

Since J is uniformly norm-to-norm continuous on bounded sets, we get by (3.10) and $r_n\geqslant a$

$$\lim_{n\to\infty}\frac{\|Ju_n-Jy_n\|}{r_n}=0. \tag{3.11}$$

From $u_n=T_{r_n}y_n$, we have

$$g(u_n,y)+\frac{1}{r_n}\langle y-u_n,Ju_n-Jy_n\rangle\geqslant 0,\quad \forall\, y\in C. \tag{3.12}$$

Since g satisfies the conditions (A_1)–(A_4), we can get by (3.11), (3.12) and by letting $n\to\infty$ that

$$g(y,p)\leqslant 0,\quad \forall\, y\in C. \tag{3.13}$$

For t with $0<t\leqslant 1$ and $y\in C$, let $y_t=ty+(1-t)p$. Since $y\in C$ and $p\in C$, we have $y_t\in C$, and hence $g(y_t,p)\leqslant 0$. So, we get by (A_1) and (A_4)

$$0=g(y_t,y_t)\leqslant tg(y_t,y)+(1-t)g(y_t,p)\leqslant tg(y_t,y).$$

Thus,

$$g(y_t, y) \geqslant 0, \quad \forall\, y \in C.$$

Letting $t \to 0^+$, we get by (A_3)

$$g(p, y) \geqslant 0, \quad \forall\, y \in C.$$

Therefore, $p \in EP(g)$, and hence (3.9) holds.

Finally, we show that $p = \Pi_{F(S)\cap EP(g)}x$.

Indeed, we can get by Lemma 2.5

$$\phi(p, \Pi_{F(S)\cap EP(g)}x) + \phi(\Pi_{F(S)\cap EP(g)}x, x) \leqslant \phi(p, x). \tag{3.14}$$

On the other hand, since $x_{n+1} = \Pi_{C_{n+1}}x$ and $F(S) \cap EP(g) \subset C_n$ for all n, we get by Lemma 2.5

$$\phi(\Pi_{F(S)\cap EP(g)}x, x_{n+1}) + \phi(x_{n+1}, x) \leqslant \phi(\Pi_{F(S)\cap EP(g)}x, x). \tag{3.15}$$

Then we can get by (3.14) and (3.15) that both $\phi(p, x) \leqslant \phi(\Pi_{F(S)\cap EP(g)}x, x)$ and $\phi(p, x) \geqslant \phi(\Pi_{F(S)\cap EP(g)}x, x)$ hold, and hence $\phi(p, x) = \phi(\Pi_{F(S)\cap EP(g)}x, x)$. It follows by the uniqueness of $\Pi_{F(S)\cap EP(g)}x$ that $p = \Pi_{F(S)\cap EP(g)}x$. This completes the proof. □

Remark Letting $A \equiv 0$ in Theorem 3.1, and replacing the closed hemi-relatively nonexpansive mapping with relatively nonexpansive mapping, we see, Theorem 3.1 is reduced to Takahashi-Zembayashi [1, Theorem 3.1].

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