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Almost Periodic Solution and Global Stability for Cooperative L-V Diffusion System

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Abstract In this paper a nonautonomous two-species n-patches system is studied. Within each patch, there are two cooperative species and their dynamics are described by the Lotka-Volterra model. Each species can diffuse independently and discretely between its interpatch and intrapatch. By constructing a suitable Liapunov function, some sufficient conditions are obtained for the existence of a unique globally asymptotically stable positive almost periodic solution.

Keywords almost periodic solution; global stability; cooperative; diffusion.

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1. Introduction

Lotka-Volterra cooperative system is one of the celebrated population dynamics models on species interaction, and it has been studied by many authors [1, 2, 6, 7]. Most of the authors investigated the periodic solutions and almost periodic solutions or permanence of system. Liu and Chen [1] considered a two-species, two-patches with periodic coefficients Lotka-volterra cooperative diffusion system

$$\begin{cases} \dot{x}_1 = x_1[r_1(t) - a_{11}(t)x_1 + a_{12}(t)y_1] + D_1(t)(x_2 - x_1), \\ \dot{y}_1 = y_1[r_2(t) + a_{21}(t)x_1 - a_{22}(t)y_1] + D_2(t)(y_2 - y_1), \\ \dot{x}_2 = x_2[s_1(t) - b_{11}(t)x_2 + b_{12}(t)y_2] + D_1(t)(x_1 - x_2), \\ \dot{y}_2 = y_2[s_2(t) + b_{21}(t)x_2 - b_{22}(t)y_2] + D_2(t)(y_1 - y_2). \end{cases} \tag{1.1}$$

They derived some appropriate conditions to guarantee the system (1.1) had a unique positive periodic solution which was globally and asymptotically stable. Wei [2] extended the system (1.1) from periodic system to asymptotical periodic system and studied its asymptotical periodic solutions. However because of the difference of living environment in different patches and influence of mankind, the same species in different patches perhaps has different diffusion rates,

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and species probably causes loss in the course of diffusion. So the model (1.1) may be too ideal, it omits the different diffusion rates caused by diffusion loss. Lu [3] introduced a single species diffusion model with diffusion loss

$$\dot{x}_i = x_i[b_i(t) - a_i(t)] + \sum_{j=1}^{n}(1 - \lambda_{ij}(t))D_{ij}(t)x_j - \sum_{j=1}^{n} D_{ji}(t)x_i. \tag{1.2}$$

Motivated by the work of [1–3], in this paper, we investigate a two-species, n-patches Lotka-Volterra cooperative diffusion system with diffusion loss

$$\begin{aligned} \dot{x}_i &= x_i[r_1^{(i)}(t) - a_{11}^{(i)}(t)x_i + a_{12}^{(i)}(t)y_i] + \sum_{j=1}^{n}(1 - \lambda_{ij}(t))C_{ij}(t)x_j - \sum_{j=1}^{n} C_{ji}(t)x_i, \\ \dot{y}_i &= y_i[r_2^{(i)}(t) + a_{21}^{(i)}(t)x_i - a_{22}^{(i)}(t)y_i] + \sum_{j=1}^{n}(1 - u_{ij}(t))D_{ij}(t)y_j - \sum_{j=1}^{n} D_{ji}(t)y_i, \end{aligned} \tag{1.3}$$

where $i = 1, 2, \ldots, n$. In this system, i stands for the ith patch; x_i and y_i represent the densities of species x and y in the ith patch; $r_1^{(i)}(t)$ and $r_2^{(i)}(t)$ denote the intrinsic growth rates of x and y in patch i; $a_{11}^{(i)}(t)$ and $a_{22}^{(i)}(t)$ show the intraspecies competition rates of x and y in patch i; $a_{12}^{(i)}(t)$ and $a_{21}^{(i)}(t)$ are the cooperative coefficients of y to x and x to y in patch i; $C_{ij}(t)$ and $D_{ij}(t)$ are the diffusion rates of x and y from patch j to patch i without diffusion loss; $\lambda_{ij}(t)$ and $u_{ij}(t)$ are the loss rates from patch j to patch i of x and y.

In this paper, for the system (1.3), we always assume that

(H1) $r_1^{(i)}(t), r_2^{(i)}(t), a_{11}^{(i)}(t), a_{22}^{(i)}(t), a_{12}^{(i)}(t), a_{21}^{(i)}(t), C_{ij}(t), D_{ij}(t), \lambda_{ij}(t), u_{ij}(t)$ $(i, j = 1, 2, \ldots, n,$ $t \in [0, +\infty))$ are nonnegative continuous bounded almost periodic functions.

(H2) $a_{11}^{(i)}(t)$, $a_{22}^{(i)}(t)$, $a_{12}^{(i)}(t)$, $a_{21}^{(i)}(t)$ $(i = 1, 2, \ldots, n,\ t \in [0, +\infty))$ have the strictly positive lower bounds; $0 \le \lambda_{ij}(t) < 1$, $0 \le u_{ij}(t) < 1$ $(i, j = 1, 2, \ldots, n)$; $C_{ij}(t) = D_{ij}(t) = 0$ $(i = j)$; $u_{ij}(t) = \lambda_{ij}(t) = 0$ $(i = j)$.

2. Preliminaries

Let

$$R_+^{2n} = \{(x_1, y_1, \ldots, x_n, y_n) \in R^{2n} | x_i > 0, y_i > 0\ (i = 1, 2, \ldots, n)\} \tag{2.1}$$

and

$$R_{+0}^{2n} = \{(x_1, y_1, \ldots, x_n, y_n) \in R^{2n} | x_i \ge 0, y_i \ge 0\ (i = 1, 2, \ldots, n)\}. \tag{2.2}$$

We define the usual Euclidean norm by $\|\cdot\|$,

$$\|z\| = \sqrt{x_1^2 + y_1^2 + \cdots + x_n^2 + y_n^2}\ ,\ \text{as}\ \ z = (x_1, y_1, \ldots, x_n, y_n)^{\mathrm{T}}. \tag{2.3}$$

If $f(t)$ is a continuous function with respect to $t \in [0, +\infty)$, we denote

$$f^L = \inf_{t \in [0, +\infty)} f(t), \quad f^M = \sup_{t \in [0, +\infty)} f(t). \tag{2.4}$$

Definition 2.1 *If there exists a compact region Ω $(\Omega \subset R_{+0}^{2n})$ and a finite time $T = T(t_0, z_0)$ such that for any $(t_0, z_0) \in R \times R_{+0}^{2n}$, it follows that $z(t, t_0, z_0) \in \Omega$ for all $t \ge T(t_0, z_0)$, then the solutions $z(t, t_0, z_0)$ of the system (1.3) are called to be ultimately bounded with respect to the* *region R_{+0}^{2n}.*

Definition 2.2 *If all the solutions of the system (1.3) are ultimately bounded in a compact region which has strictly positive distance from coordinate hyper-planes, then the system (1.3) is called permanent.*

3. Some results

Let

$$A_i(t) = \begin{pmatrix} a_{11}^{(i)}(t) & -a_{12}^{(i)}(t) \\ -a_{21}^{(i)}(t) & a_{22}^{(i)}(t) \end{pmatrix}, \quad \hat{A}_i = \begin{pmatrix} a_{11}^{(i)^L} & -a_{12}^{(i)^M} \\ -a_{21}^{(i)^M} & a_{22}^{(i)^L} \end{pmatrix} \tag{3.1}$$

and

$$B_i(t) = \frac{1}{2}(W_i A_i(t) + A_i^{\mathrm{T}}(t) W_i), \quad \hat{B}_i = \frac{1}{2}(W_i \hat{A}_i + \hat{A}_i{}^{\mathrm{T}} W_i), \tag{3.2}$$

where W_i is real constant positive diagonal matrix, W_i=diag$(W_{i1},\ W_{i2})$, and $\hat{B}_i$ is positive definitive matrix, for $i = 1, 2, \ldots, n$.

Lemma 3.1 *Under the above two assumptions, one can obtain $\sum_{i=1}^n z_i^{\mathrm{T}} B_i(t) z_i \geq \lambda \|z\|^2$, $t \in [0, +\infty)$, where $z_i = (x_i, y_i)^{\mathrm{T}}$ is any two-dimensional nonnegative column vector and λ is the minimum value of all real positive eigenvalues of all $\hat{B}_i$ $(i = 1, 2, \ldots, n)$.*

Proof For any $z_i = (x_i, y_i)^{\mathrm{T}} \in R_{+0}^2$, according to the assumptions (1)–(2), one can yield

$$z_i^{\mathrm{T}} B_i(t) z_i = W_{i1} a_{11}^{(i)}(t) x_i^2 - (W_{i1} a_{12}^{(i)}(t) + W_{i2} a_{21}^{(i)}(t)) x_i y_i + W_{i2} a_{22}^{(i)}(t) y_i^2, \tag{3.3}$$

$$z_i^{\mathrm{T}} \hat{B}_i z_i = W_{i1} a_{11}^{(i)^L} x_i^2 - (W_{i1} a_{12}^{(i)^M} + W_{i2} a_{21}^{(i)^M}) x_i y_i + W_{i2} a_{22}^{(i)^L} y_i^2. \tag{3.4}$$

We set $\lambda(\hat{B}_i)$ to represent the set of all real positive eigenvalues of $\hat{B}_i$, $\lambda_i = \min \lambda(\hat{B}_i)$, $\lambda = \min\{\lambda_i\}$ $(i = 1, 2, \ldots, n)$.

Combining (3.3) and (3.4), one can easily derive that

$$z_i^{\mathrm{T}} B_i(t) z_i \geq z_i^{\mathrm{T}} \hat{B}_i z_i \geq \lambda_i \|z_i\|^2 \geq \lambda \|z_i\|^2, \quad t \in [0, +\infty),\ i = 1, 2, \ldots, n. \tag{3.5}$$

Therefore one has

$$\sum_{i=1}^n z_i^{\mathrm{T}} B_i(t) z_i \geq \sum_{i=1}^n \lambda \|z_i\|^2 = \lambda \|z\|^2. \tag{3.6}$$

The proof of Lemma 3.1 is completed. □

Lemma 3.2 *R_{+0}^{2n} is a positive invariant set with respect to the system (1.3).*

Proof For all $t \in [0, +\infty)$ and $z \in R_{+0}^{2n}$, one can obtain

$$\begin{cases} \dot{x}_i|_{x_i=0} = \displaystyle\sum_{j=1, j\neq i}^n (1 - \lambda_{ij}(t)) C_{ij}(t) x_j \geq 0, \\ \dot{y}_i|_{y_i=0} = \displaystyle\sum_{j=1, j\neq i}^n (1 - u_{ij}(t)) D_{ij}(t) y_j \geq 0, \end{cases} \tag{3.7}$$

where $i = 1, 2, \ldots, n$.

Obviously R_{+0}^{2n} is a positive invariant set. The proof of Lemma 3.2 is completed. □

Theorem 3.1 *Under the above assumptions (1)–(2), the solutions of the system (1.3) are ultimately bounded with respect to the region* R_{+0}^{2n}.

Proof We define the function $S:\ R_{+0}^{2n} \to R_{+0}$

$$S = \sum_{i=1}^{n}(W_{i1}x_i + W_{i2}y_i). \tag{3.8}$$

One can easily derive that

$$\dot{S}|_{(1.3)} = \sum_{i=1}^{n}(W_{i1}\dot{x}_i + W_{i2}\dot{y}_i). \tag{3.9}$$

Now we set $W = \mathrm{diag}(W_1, W_2, \ldots, W_n)$, where $W_i = \mathrm{diag}(W_{i1}, W_{i2})$. Furthermore we set

$$z = (z_1, z_2, \ldots, z_n)^{\mathrm{T}}, \quad z_i = (x_i, y_i)^{\mathrm{T}}, \quad e(t) = (r_1^{(1)}, r_2^{(1)}, \ldots, r_1^{(n)}, r_2^{(n)})^{\mathrm{T}}, \tag{3.10}$$

and denote

$$A(t) = We(t),$$

$$B(t) = (-\sum_{j=1}^{n} C_{j1}(t)W_{11}, \ldots, -\sum_{j=1}^{n} C_{jn}(t)W_{n1}\ ,\ \ -\sum_{j=1}^{n} D_{j1}(t)W_{12}, \ldots, -\sum_{j=1}^{n} D_{jn}(t)W_{n2})^{\mathrm{T}},$$

$$C(t) = (\sum_{j=1}^{n} W_{j1}(1-\lambda_{j1}(t))C_{j1}(t), \ldots, \sum_{j=1}^{n} W_{j1}(1-\lambda_{jn}(t))C_{jn}(t), \sum_{j=1}^{n} W_{j2}(1-u_{j1}(t))D_{j1}(t) \ldots\ ,$$

$$\sum_{j=1}^{n} W_{j2}(1-u_{jn}(t))D_{jn}(t))^{\mathrm{T}}. \tag{3.11}$$

By complicated calculations, one can prove

$$\begin{aligned}\dot{S}|_{(1.3)} &= \sum_{i=1}^{n}(W_{i1}\dot{x}_i + W_{i2}\dot{y}_i) = -\sum_{i=1}^{n} z_i^{\mathrm{T}} W_i A_i(t) z_i + z^{\mathrm{T}}A(t) + z^{\mathrm{T}}B(t) + z^{\mathrm{T}}C(t) \\ &= -\sum_{i=1}^{n} z_i^{\mathrm{T}} B_i(t) z_i + z^{\mathrm{T}}A(t) + z^{\mathrm{T}}B(t) + z^{\mathrm{T}}C(t). \end{aligned} \tag{3.12}$$

Taking into account the assumptions (H1)–(H2), one can easily know there must exist adequate positive constants M_1, M_2, M_3 such that the following inequalities hold true

$$\|A(t)\| < M_1, \quad \|B(t)\| < M_2, \quad \|C(t)\| < M_3. \tag{3.13}$$

Let $M = M_1 + M_2 + M_3$. By (3.12), (3.13) and (3.6), one can know that

$$\dot{S}|_{(1.3)} \le -\lambda\|z\|^2 + M\|z\|. \tag{3.14}$$

We set

$$\alpha = [M + (M^2 + 4\lambda\varepsilon)^{\frac{1}{2}}](2\lambda)^{-1}, \tag{3.15}$$

then the following implication holds true

$$\|z\| > \alpha \Rightarrow \dot{S}|_{(1.3)} < -\varepsilon. \tag{3.16}$$

We define the following compact set

$$\Phi(\beta) = \{z:\ z \in R^{2n}_{+0},\ S(z) \le \beta\}, \tag{3.17}$$

where $\beta = \eta\alpha$, $\eta = \sum_{i=1}^{n}(W_{i1} + W_{i2})$. Furthermore note the fact that

$$S(z) \le \eta\|z\|. \tag{3.18}$$

By means of (3.15)–(3.17) one has

$$\dot{S}|_{(1.3)} \ge -\varepsilon \Rightarrow \|z\| \le \alpha \Rightarrow S(z) \le \eta\alpha = \beta \Rightarrow z \in \Phi(\beta). \tag{3.19}$$

Integrating from t_0 to t, in view of the inequality $\dot{S}|_{(1.3)} \ge -\varepsilon$ and $S(z) \le \beta$, one can derive $t \ge t_0 + \varepsilon^{-1}(S(z_0) - \beta)$.

This means that $\Phi(\beta)$ is an ultimately bounded region according to the Definition 2.1 and we can take

$$T = T(t_0, z_0) = \max\{t_0, t_0 + \varepsilon^{-1}(S(z_0) - \beta)\}. \tag{3.20}$$

The proof of Theorem 3.1 is completed. □

Theorem 3.2 *If* $\inf_{t>0}(r_1^{(i)}(t) - \sum_{j=1,j\ne i}^{n} C_{ji}(t)) > 0$ *and* $\inf_{t>0}(r_2^{(i)}(t) - \sum_{j=1,j\ne i}^{n} D_{ji}(t)) > 0$ *for* $i = 1, 2, \ldots, n$, *then there must exist a strictly positive compact region which attracts all the solutions of the system (1.3) with positive initial values.*

Proof Fron the system (1.3), when $x_i = h_i$ and $y_i = g_i$, one can find

$$\begin{aligned}\dot{x}_i|_{x_i=h_i>0} &= h_i[r_1^{(i)}(t) - a_{11}^{(i)}(t)h_i + a_{12}^{(i)}(t)y_i] + \sum_{j=1}^{n}(1-\lambda_{ij}(t))C_{ij}(t)x_j - \sum_{j=1}^{n} C_{ji}(t)h_i \\ &\ge h_i(r_1^{(i)}(t) - \sum_{j=1,j\ne i}^{n} C_{ji}(t) - a_{11}^{(i)}(t)h_i) + a_{12}^{(i)}(t)h_i y_i,\end{aligned} \tag{3.21}$$

$$\begin{aligned}\dot{y}_i|_{y_i=g_i>0} &= g_i[r_2^{(i)}(t) + a_{21}^{(i)}(t)x_i - a_{22}^{(i)}(t)g_i] + \sum_{j=1}^{n}(1-u_{ij}(t))D_{ij}(t)y_j - \sum_{j=1}^{n} D_{ji}(t)g_i \\ &\ge g_i(r_2^{(i)}(t) - \sum_{j=1,j\ne i}^{n} D_{ji}(t) - a_{22}^{(i)}(t)g_i) + a_{21}^{(i)}(t)g_i x_i.\end{aligned} \tag{3.22}$$

We define

$$x_i^L = \inf_{t>0}\frac{r_1^{(i)}(t) - \sum_{j=1,j\ne i}^{n} C_{ji}(t)}{a_{11}^{(i)}(t)},\quad y_i^L = \inf_{t>0}\frac{r_2^{(i)}(t) - \sum_{j=1,j\ne i}^{n} D_{ji}(t)}{a_{22}^{(i)}(t)}. \tag{3.23}$$

Obviously one can judge

$$\dot{x}_i|_{x_i=x_i^L>0} \ge 0,\quad \dot{y}_i|_{y_i=y_i^L>0} \ge 0,\ i = 1, 2, \ldots, n. \tag{3.24}$$

Set

$$\Omega = \{z, z \in R^{2n}_{+0}, S(z) \le \beta, x_i \ge x_i^L, y_i \ge y_i^L\ (i = 1, 2, \ldots, n)\}. \tag{3.25}$$

Apparently Ω is also a positive invariant set. Further Ω is still a bounded closed convex set and it attracts all the solutions of the system (1.3) with positive initial values. Moreover, one can easily obtain the system (1.3) is permanent by definition 2.2. The proof of Theorem 3.2 is completed. □

4. Existence and asymptotical stability of almost periodic solution

We discuss the system (1.3) with almost periodic coefficients in Ω. The adjoint system of the system (1.3) is as follows

$$\begin{aligned} \dot{x}_i &= x_i[r_1^{(i)}(t) - a_{11}^{(i)}(t)x_i + a_{12}^{(i)}(t)y_i] + \sum_{j=1}^{n}(1-\lambda_{ij}(t))C_{ij}(t)x_j - \sum_{j=1}^{n} C_{ji}(t)x_i, \\ \dot{y}_i &= y_i[r_2^{(i)}(t) + a_{21}^{(i)}(t)x_i - a_{22}^{(i)}(t)y_i] + \sum_{j=1}^{n}(1-u_{ij}(t))D_{ij}(t)y_j - \sum_{j=1}^{n} D_{ji}(t)y_i, \\ \dot{\tilde{x}}_i &= \tilde{x}_i[r_1^{(i)}(t) - a_{11}^{(i)}(t)\tilde{x}_i + a_{12}^{(i)}(t)\tilde{y}_i] + \sum_{j=1}^{n}(1-\lambda_{ij}(t))C_{ij}(t)\tilde{x}_j - \sum_{j=1}^{n} C_{ji}(t)\tilde{x}_i, \\ \dot{\tilde{y}}_i &= \tilde{y}_i[r_2^{(i)}(t) + a_{21}^{(i)}(t)\tilde{x}_i - a_{22}^{(i)}(t)\tilde{y}_i] + \sum_{j=1}^{n}(1-u_{ij}(t))D_{ij}(t)\tilde{y}_j - \sum_{j=1}^{n} D_{ji}(t)\tilde{y}_i, \end{aligned} \tag{4.1}$$

where $i = 1, 2, \dots, n$.

Lemma 4.1 ([4]) *Let D be an open set of R_{+0}^{2n}. Function $V(t,x,y)$ is defined on the region $R_+ \times D \times D$ or $R_+ \times R_+^{2n} \times R_+^{2n}$, satisfying:*

(i) $a(|x-y|) \le V(t,x,y) \le b(|x-y|)$, where $a(r)$, $b(r)$ are continuous increasing positive definite functions;

(ii) $|V(t,x_1,y_1) - V(t,x_2,y_2)| \le K(|x_1-x_2| + |y_1-y_2|)$, $K > 0$ is a constant;

(iii) $V'(t,x,y) \le -cV(t,x,y)$, where $c > 0$ is a constant.

Let the solutions of the system (1.3) lie in compact set Ω for all $t \ge t_0 > 0$, $\Omega \subseteq D$. Then the system (1.3) has a unique almost periodic solution $z(t)$ in D, $z(t)$ lies in Ω, and it is asymptotically stable.

Theorem 4.1 *If $\sup_{t>0}(-a_{11}^{(i)}(t)+a_{21}^{(i)}(t)+\sum_{j=1,j\ne i}^{n} C_{ji}(t)\frac{1}{x_j^L}) < 0$ and $\sup_{t>0}(-a_{22}^{(i)}(t)+a_{12}^{(i)}(t)+\sum_{j=1,j\ne i}^{n} D_{ji}(t)\frac{1}{y_j^L}) < 0$ for $i = 1, 2, \dots, n$, then the system (1.3) has a unique positive almost periodic solution which is asymptotically stable.*

Proof Let $z(t) = (x_1(t), y_1(t), \dots, x_n(t), y_n(t))^{\mathrm{T}}$ be any solution of the system (1.3). We consider adjoint system (4.1) of the system (1.3) and let $X_i(t) = \ln x_i(t)$, $Y_i(t) = \ln y_i(t)$, $\tilde{X}_i(t) = \ln \tilde{x}_i(t)$, $\tilde{Y}_i(t) = \ln \tilde{y}_i(t)$ $(i = 1, 2, \dots, n)$, where $x_i(t)$, $y_i(t)$, $\tilde{x}_i(t)$, $\tilde{y}_i(t)$ $(i = 1, 2, \dots, n)$ are the solutions of the system (4.1). We denote $Z(t) = (X_1(t), Y_1(t), \dots, X_n(t), Y_n(t))^{\mathrm{T}}$.

Construct Liapunov function

$$V(t) = V(t, Z(t), \tilde{Z}(t)) = \sum_{i=1}^{n} |X_i(t) - \tilde{X}_i(t)| + \sum_{i=1}^{n} |Y_i(t) - \tilde{Y}_i(t)|. \tag{4.2}$$

Let

$$a(r) = b(r) = \sum_{i=1}^{n} |X_i(t) - \tilde{X}_i(t)| + \sum_{i=1}^{n} |Y_i(t) - \tilde{Y}_i(t)|. \tag{4.3}$$

Clearly, $a(r)$, $b(r)$, $V(t)$ are continuous increasing positive definite functions, $V(t)$ satisfies condition (1) of Lemma 4.1. Further,

$$\begin{aligned}
&\Big|\sum_{i=1}^{n}|X_{i1}(t)-\tilde{X}_{i1}(t)|+\sum_{i=1}^{n}|Y_{i1}(t)-\tilde{Y}_{i1}(t)|-(\sum_{i=1}^{n}|X_{i2}(t)-\tilde{X}_{i2}(t)|+\sum_{i=1}^{n}|Y_{i2}(t)-\tilde{Y}_{i2}(t)|)\Big|\\
&\quad\le\sum_{i=1}^{n}|X_{i1}(t)-X_{i2}(t)|+\sum_{i=1}^{n}|Y_{i1}(t)-Y_{i2}(t)|+\\
&\qquad\sum_{i=1}^{n}|\tilde{X}_{i1}(t)-\tilde{X}_{i2}(t)|+\sum_{i=1}^{n}|\tilde{Y}_{i1}(t)-\tilde{Y}_{i2}(t)|. \qquad (4.4)
\end{aligned}$$

$V(t)$ satisfies condition (2) of Lemma 4.1. To verify condition (3) of Lemma 4.1, we compute the upper right derivation of function $V(t)$.

$$\begin{aligned}
&D^{+}V(t)|_{(1.3)}\\
&=\sum_{i=1}^{n}\frac{X_i(t)-\tilde{X}_i(t)}{|X_i(t)-\tilde{X}_i(t)|}[-a_{11}^{(i)}(t)(x_i-\tilde{x}_i)+a_{12}^{(i)}(t)(y_i-\tilde{y}_i)+\sum_{j=1,j\neq i}^{n}(1-\lambda_{ij}(t))C_{ij}(t)(\frac{x_j}{x_i}-\frac{\tilde{x}_j}{\tilde{x}_i})]+\\
&\quad\sum_{i=1}^{n}\frac{Y_i(t)-\tilde{Y}_i(t)}{|Y_i(t)-\tilde{Y}_i(t)|}[a_{21}^{(i)}(t)(x_i-\tilde{x}_i)-a_{22}^{(i)}(t)(y_i-\tilde{y}_i)+\sum_{j=1,j\neq i}^{n}(1-u_{ij}(t))D_{ij}(t)(\frac{y_j}{y_i}-\frac{\tilde{y}_j}{\tilde{y}_i})]\\
&\le\sum_{i=1}^{n}(-a_{11}^{(i)}(t)+a_{21}^{(i)}(t))|x_i-\tilde{x}_i|+\sum_{i=1}^{n}(-a_{22}^{(i)}(t)+a_{12}^{(i)}(t))|y_i-\tilde{x}_i|+\sum_{i=1}^{n}H_i(t)+\sum_{i=1}^{n}G_i(t), \quad (4.5)
\end{aligned}$$

where for $i=1,2,\dots,n$,

$$H_i(t)=\begin{cases}\displaystyle\sum_{j=1,j\neq i}^{n}(1-\lambda_{ij}(t))C_{ij}(t)(\frac{x_j}{x_i}-\frac{\tilde{x}_j}{\tilde{x}_i}), & X_i\ge\tilde{X}_i,\\ \displaystyle\sum_{j=1,j\neq i}^{n}(1-\lambda_{ij}(t))C_{ij}(t)(\frac{\tilde{x}_j}{\tilde{x}_i}-\frac{x_j}{x_i}), & X_i<\tilde{X}_i,\end{cases} \qquad (4.6)$$

$$G_i(t)=\begin{cases}\displaystyle\sum_{j=1,j\neq i}^{n}(1-u_{ij}(t))D_{ij}(t)(\frac{y_j}{y_i}-\frac{\tilde{y}_j}{\tilde{y}_i}), & Y_i\ge\tilde{Y}_i,\\ \displaystyle\sum_{j=1,j\neq i}^{n}(1-u_{ij}(t))D_{ij}(t)(\frac{\tilde{y}_j}{\tilde{y}_i}-\frac{y_j}{y_i}), & Y_i<\tilde{Y}_i.\end{cases} \qquad (4.7)$$

By complicated calculations one can obtain

$$\sum_{i=1}^{n}H_i(t)+\sum_{i=1}^{n}G_i(t)\le\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}C_{ji}(t)\frac{1}{x_j^L}|x_i-\tilde{x}_i|+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}D_{ji}(t)\frac{1}{y_j^L}|y_i-\tilde{y}_i|. \qquad (4.8)$$

So one can get the following inequality by (4.5)–(4.8),

$$\begin{aligned}
D^{+}V(t)|_{(1.3)}\le&\sum_{i=1}^{n}(-a_{11}^{(i)}(t)+a_{21}^{(i)}(t)+\sum_{j=1,j\neq i}^{n}C_{ji}(t)\frac{1}{x_j^L})|x_i-\tilde{x}_i|+\\
&\sum_{i=1}^{n}(-a_{22}^{(i)}(t)+a_{12}^{(i)}(t)+\sum_{j=1,j\neq i}^{n}D_{ji}(t)\frac{1}{y_j^L})|y_i-\tilde{y}_i|. \qquad (4.9)
\end{aligned}$$

From conditions of Theorem 4.1, one can obtain there must exist two positive constants A_1 and A_2 such that for each i $(i = 1, 2, \ldots, n)$,

$$\sup_{t>0}(-a_{11}^{(i)}(t) + a_{21}^{(i)}(t) + \sum_{j=1,j\neq i}^{n} C_{ji}(t)\frac{1}{x_j^L}) < -A_1;$$

$$\sup_{t>0}(-a_{22}^{(i)}(t) + a_{12}^{(i)}(t) + \sum_{j=1,j\neq i}^{n} D_{ji}(t)\frac{1}{y_j^L}) < -A_2. \tag{4.10}$$

We take $A = \min(A_1, A_2)$, hence one can obtain for each i $(i = 1, 2, \ldots, n)$,

$$\sup_{t>0}(-a_{11}^{(i)}(t) + a_{21}^{(i)}(t) + \sum_{j=1,j\neq i}^{n} C_{ji}(t)\frac{1}{x_j^L}) < -A \ ;$$

$$\sup_{t>0}(-a_{22}^{(i)}(t) + a_{12}^{(i)}(t) + \sum_{j=1,j\neq i}^{n} D_{ji}(t)\frac{1}{y_j^L}) < -A. \tag{4.11}$$

Therefore

$$D^+V(t)|_{(1.3)} \leq -A(\sum_{i=1}^{n}|x_i - \tilde{x}_i| + \sum_{i=1}^{n}|y_i - \tilde{y}_i|). \tag{4.12}$$

Note the fact that

$$|x_i - \tilde{x}_i| = |\mathrm{e}^{X_i} - \mathrm{e}^{\tilde{X}_i}| = \xi_i(t)|X_i - \tilde{X}_i| \geq x_i^L|X_i - \tilde{X}_i|, \tag{4.13}$$

$$|y_i - \tilde{y}_i| = |\mathrm{e}^{Y_i} - \mathrm{e}^{\tilde{Y}_i}| = \zeta_i(t)|Y_i - \tilde{Y}_i| \geq y_i^L|Y_i - \tilde{Y}_i|, \tag{4.14}$$

where $i = 1, 2, \ldots, n$, $\xi_i(t)$ lies in between $x_i(t)$ and $\tilde{x}_i(t)$, $\zeta_i(t)$ lies in between $y_i(t)$ and $\tilde{y}_i(t)$. Taking $c = \min(Ax_i^L, Ay_i^L\ (i = 1, 2, \ldots, n)) > 0$ yields

$$D^+V(t)|_{(1.3)} \leq -c(\sum_{i=1}^{n}|X_i - \tilde{X}_i| + \sum_{i=1}^{n}|Y_i - \tilde{Y}_i|) = -cV(t), \tag{4.15}$$

which manifests that $V(t)$ satisfies condition (3) of Lemma 4.1. By lemma 4.1, the system (1.3) has a unique asymptotically stable almost periodic solution $z(t)$, and $z(t)$ lies in compact region Ω. The proof of theorem 4.1 is completed. $\square$

5. Global attractivity

Lemma 5.1 ([5]) *Suppose function $f(x)$ is uniformly continuous on $[a, +\infty)$, and $\int_a^{+\infty} f(x)\mathrm{d}x$ is convergent, then we have $\lim_{x\to+\infty} f(x) = 0$.*

Theorem 5.1 *If the system (1.3) satisfies the conditions of Theorem 4.1, then the unique almost periodic solution of the system (1.3) is globally attractive.*

Proof Let $z(t) = (x_1(t), y_1(t), \ldots, x_n(t), y_n(t))^{\mathrm{T}}$ be a definitive almost periodic solution of the system (1.3), $\tilde{z}(t) = (\tilde{x}_1(t), \tilde{y}_1(t), \ldots, \tilde{x}_n(t), \tilde{y}_n(t))^{\mathrm{T}}$ be any solution of the system (1.3).

Construct the same Liapunov function as Theorem 4.1,

$$V(t) = V(t, Z(t), \tilde{Z}(t)) = \sum_{i=1}^{n}|X_i(t) - \tilde{X}_i(t)| + \sum_{i=1}^{n}|Y_i(t) - \tilde{Y}_i(t)|. \tag{5.1}$$

Integrating both sides of (4.12) from 0 to t, one can derive

$$V(t)+A\int_0^t(\sum_{i=1}^{n}|x_i(s)-\tilde{x}_i(s)|+\sum_{i=1}^{n}|y_i(s)-\tilde{y}_i(s)|)\mathrm{d}s\le V(0). \tag{5.2}$$

The expression (5.2) shows that

$$0\le V(t)\le V(0)=\sum_{i=1}^{n}|X_i(0)-\tilde{X}_i(0)|+\sum_{i=1}^{n}|Y_i(0)-\tilde{Y}_i(0)|<+\infty,\quad t\ge 0, \tag{5.3}$$

and

$$\int_0^t(\sum_{i=1}^{n}|x_i(s)-\tilde{x}_i(s)|+\sum_{i=1}^{n}|y_i(s)-\tilde{y}_i(s)|)\mathrm{d}s\le\frac{V(0)}{A}<+\infty,\quad t\ge 0. \tag{5.4}$$

The expression (5.4) implies that

$$\sum_{i=1}^{n}|x_i(s)-\tilde{x}_i(s)|+\sum_{i=1}^{n}|y_i(s)-\tilde{y}_i(s)|\in L^1[0,+\infty). \tag{5.5}$$

Obviously $x_i(t)$ and $y_i(t)$ $(i=1,2,\ldots,n)$ are uniformly bounded, so $X_i(t)$ and $Y_i(t)$ $(i=1,2,\ldots,n)$ are also uniformly bounded. In addition, by (5.1)–(5.3), one knows $\tilde{X}_i(t)$ and $\tilde{Y}_i(t)$ are uniformly bounded, so $\tilde{x}_i(t)$ and $\tilde{y}_i(t)$ are also uniformly bounded. Combining this fact with the system (1.3), one has $\dot{x}_i,\ \dot{y}_i,\ \dot{\tilde{x}}_i,\ \dot{\tilde{y}}_i$ are uniformly bounded. Therefore one can easily check $\sum_{i=1}^{n}|x_i(t)-\tilde{x}_i(t)|+\sum_{i=1}^{n}|y_i(t)-\tilde{y}_i(t)|$ is uniformly continuous on $[0,+\infty)$. From the expression (5.5), it follows that $\sum_{i=1}^{n}|x_i(t)-\tilde{x}_i(t)|+\sum_{i=1}^{n}|y_i(t)-\tilde{y}_i(t)|$ is integrable on $[0,+\infty)$. Hence one can conclude

$$\lim_{t\to+\infty}|x_i(t)-\tilde{x}_i(t)|=0,\quad \lim_{t\to+\infty}|y_i(t)-\tilde{y}_i(t)|=0. \tag{5.6}$$

Therefore

$$\|z(t)-\tilde{z}(t)\|\to 0,\quad t\to+\infty. \tag{5.7}$$

The proof of theorem (5.1) is completed. □

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