Journal of Mathematical Research & Exposition
Mar., 2011, Vol. 31, No. 2, pp. 233–241
DOI:10.3770/j.issn:1000-341X.2011.02.006
Http://jmre.dlut.edu.cn

Fractional Type Marcinkiewicz Integral on Hardy Spaces

Zeng Yan SI[1], **Li Na WANG**[2], **Yin Sheng JIANG**[3,*]
1. *School of Mathematical Science, Beijing Normal University, Beijing* 100875*, P. R. China;*
2. *Army Avation Insititute of PLA, Beijing* 101123*, P. R. China;*
3. *College of Mathematics and System Sciences, Xinjiang University, Xinjiang* 830046*, P. R. China*

Abstract The authors in the paper proved that if Ω is homogeneous of degree zero and satisfies some certain logarithmic type Lipschitz condition, then the fractional type Marcinkiewicz Integral $\mu_{\Omega,\alpha}$ is an operator of type $(H\dot{K}_{q_1}^{n(1-1/q_1),p}, \dot{K}_{q_2}^{n(1-1/q_1),p})$ and of type $(H^1(R^n), L^{n/(n-\alpha)})$.

Keywords fractional type Marcinkiewicz Integral; Herz type Hardy space; Hardy space.

Document code A
MR(2010) Subject Classification 42B20; 42B25
Chinese Library Classification O174.2

1. Introduction and results

Suppose that S^{n-1} is the unit sphere of R^n $(n \geq 2)$ equipped with the normalized Lebesgue mesure $\mathrm{d}\sigma = \mathrm{d}\sigma(x')$ and let Ω be homogeneous of degree zero and satisfy

$$\int_{S^{n-1}} \Omega(x')\mathrm{d}\sigma(x') = 0, \tag{1}$$

where $x' = x/|x|$ for any $x \neq 0$.

Then the fractional type Marcinkiewicz integral of higher dimension is defined by

$$\mu_{\Omega,\alpha}(f)(x) = \Big(\int_0^\infty |F_{\Omega,t,\alpha}(x)|^2 \frac{\mathrm{d}t}{t^3}\Big)^{1/2},$$

where

$$F_{\Omega,t,\alpha}(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\alpha-1}} f(y)\mathrm{d}y.$$

Let us now give some definitions and formulate our results.

Definition 1 ([1]) *We call $a(x)$ an H^1 atom, if there is a ball $B \subset R^n$ such that*

(i) $\mathrm{supp}\,(a) \subset B$*; (ii)* $\|a\|_{L^\infty} \leq |B|^{-1}$*; (iii)* $\int_{R^n} a(x)\mathrm{d}x = 0$.

A measurable $f \in L^1(R^n)$ is said to belong to the homogeneous Hardy spaces $H^1(R^n)$, if $f = \sum_{j=-\infty}^{+\infty} \lambda_j a_j$ in the sense of distributions, where a_j is an H^1 atom, $\lambda_j \in C$ and $\sum_{j=-\infty}^{+\infty} |\lambda_j| < \infty$.

Received May 21, 2009; Accepted July 13, 2009
Supported by the National Natural Science Foundation of China (Grant Nos. 10861010; 10871024).
* Corresponding author
E-mail address: yinshengjiang@msn.com (Y. S. JIANG)

Furthermore, the seminorm of H^1 is defined by

$$\|f\|_{H^1} \doteq \inf \sum_{j=-\infty}^{+\infty} |\lambda_j| < \infty,$$

where inf *is taken over all above decomposition of f.*

For $k \in Z$, let $B_k = \{x \in R^n : |x| \le 2^k\}$, $C_k = B_k/B_{k-1}$, X_k denote the characteristic function of the set C_k.

Definition 2 ([2]) *Let $0 < p \le \infty$, $0 < q < \infty$ and $\alpha \in R$. Then the homogeneous Herz space $\dot{K}_q^{\alpha,p}(R^n)$ is defined by*

$$\dot{K}_q^{\alpha,p}(R^n) \doteq \{f \in L^q_{\mathrm{loc}}(R^n\backslash\{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}} \doteq \{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|fX_k\|^p_{L^q}\}^{1/p}$$

with usual modification made with $p = \infty$.

Definition 3 ([2]) *Let $0 < p \le \infty$, $0 < q < \infty$ and $\alpha \in R$, $G(f)$ denote the Grand maximal function of f. Then the homogeneous Herz type Hardy space $H\dot{K}_q^{\alpha,p}(R^n)$ is defined by*

$$H\dot{K}_q^{\alpha,p}(R^n) \doteq \{f \in S'(R^n) : G(f) \in \dot{K}_q^{\alpha,p}\}$$

and

$$\|f\|_{H\dot{K}_q^{\alpha,p}} \doteq \|G(f)\|_{\dot{K}_q^{\alpha,p}}.$$

Definition 4 ([3]) *Let $1 < q < \infty$, $n(1 - 1/q) \le \alpha < \infty$ and $s \ge [\alpha + n(1/q - 1)]$.*

A function $a(x)$ on R^n is called a central (α, q)-atom, if it satisfies

(a) $\operatorname{supp} a \in B(0, r) \doteq \{x \in R^n : |x| < r\}$;

(b) $\|a\|_{L^q} \le |B(0, r)|^{-\alpha/n}$;

(c) $\int_{R^n} a(x)x^\gamma \mathrm{d}x = 0$, *for any multi-index with $|\gamma| \le s$.*

Theorem A ([3]) *Let $0 < p < \infty$, $1 < q < \infty$ and $n(1 - 1/q) \le \alpha < \infty$. Then we have $f \in H\dot{K}_q^{\alpha,p}(R^n)$ if and only if*

$$f = \sum_{k=-\infty}^{+\infty} \lambda_k a_k; \quad \text{in the sense of } \ S'(R^n),$$

where a_k is a central (α, q)-atom with the $\operatorname{supp} B_k$ *and $\sum_{k=-\infty}^{+\infty} |\lambda_k|^p < \infty$.*

Furthermore, $\|f\|_{H\dot{K}_q^{\alpha,p}} \sim \inf\{(\sum_{k=-\infty}^{+\infty} |\lambda_k|^p)^{1/p}\}$, where the infimum is taken over all above decompositions of f.

The results in the paper are formulated as follows.

Theorem 1 *Let $0 < \alpha < n$, $r \ge n/(n - \alpha)$ and let $\Omega \in L^r(S^{n-1})$ be homogeneous of degree zero on R^n. If Ω satisfies*

(i) $\int_{S^{n-1}} \Omega(x')\mathrm{d}x' = 0$;

(ii) There is a constant $C > 0$ and $\rho > 1$ such that $|\Omega(y_1) - \Omega(y_2)| \le \frac{C}{(\log \frac{1}{|x-y|})^\rho}$, where $y_1, y_2 \in S^{n-1}$. Then there is a $C > 0$ such that

$$\|\mu_{\Omega,\alpha} f\|_{L^{n/(n-\alpha)}} \le C\|f\|_{H^1}.$$

Theorem 2 *Let $0 < \alpha < n$, $0 < p < \infty$, $1 < q_1$, $q_2 < \infty$ and $1/q_2 = 1/q_1 - \alpha/n$. If $r \ge \max\{q_2, n/(n-\alpha)\}$ such that $\Omega \in L^r(S^{n-1})$ is homogeneous of degree zero on R^n and Ω satisfies*

(i) $\int_{S^{n-1}} \Omega(x')\mathrm{d}x' = 0$;

(ii) There is a constant $C > 0$ and $\rho > \max\{1, 1/p\}$ such that $|\Omega(y_1) - \Omega(y_2)| \le \frac{C}{(\log \frac{1}{|x-y|})^\rho}$, where $y_1, y_2 \in S^{n-1}$. Then there is a $C > 0$ such that

$$\|\mu_{\Omega,\alpha} f\|_{\dot{K}_{q_2}^{n(1-1/q_1),p}} \le C\|f\|_{H\dot{K}_{q_1}^{n(1-1/q_1),p}}$$

.

In the following the letter C will denote a constant which may vary at each occurrence.

2. Some basic lemmas

In the proofs of Theorems 1 and 2, we need the following lemmas.

Lemma 1 ([4]) *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/r = 1/p - \alpha/n$ and $q \ge n/(n-\alpha)$. If $\Omega \in L^q(S^{n-1})$, then the fractional integral operator $T_{\Omega,\alpha}$ defined by*

$$T_{\Omega,\alpha} f(x) = \int_{R^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)\mathrm{d}y$$

is bounded from $L^p(R^n)$ to $L^r(R^n)$.

We establish the boundedness of $\mu_{\Omega,\alpha}$ on Lebesgue spaces first, which is the key estimate for the proofs of Theorems 1 and 2.

Lemma 2 *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ and let $\Omega \in L^r(S^{n-1})$ with $r \ge n/(n-\alpha)$ be homogeneous of degree zero on R^n. Then $\mu_{\Omega,\alpha}$ maps $L^p(R^n)$ continuously into $L^q(R^n)$.*

Proof By Minkowski's inequality, we have

$$\begin{aligned}(\mu_{\Omega,\alpha} f)(x) &= \Big(\int_0^\infty \Big|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\alpha-1}} f(y)\mathrm{d}y\Big|^2 \frac{\mathrm{d}t}{t^3}\Big)^{1/2} \\ &\le C\int_{R^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha-1}}|f(y)|\Big(\int_{|x-y|}^\infty \frac{\mathrm{d}t}{t^3}\Big)^{1/2}\mathrm{d}y \\ &\le C\int_{R^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|\mathrm{d}y.\end{aligned}$$

By Lemma 1, we have

$$\|\mu_{\Omega,\alpha}(f)\|_{L^q(R^n)} \le C\|T_{|\Omega|,\alpha}|f|\|_{L^q} \le C\|f\|_{L^p(R^n)}. \ \square$$

We introduce some notations, for any $s \in Z_+$

$$D_s(R^n) = \{f \in D(R^n) = \bigcap_{k=1}^{\infty} C_0^k(R^n) : \int f(x)x^\beta \mathrm{d}x = 0, \text{ for any } |\beta| \le s\};$$

$$\dot{D}_s(R^n) = \{f \in D(R^n) : 0 \notin \operatorname{supp} f\}.$$

Lemma 3 ([3]) *Let $0 < p \le +\infty$, $1 < q < \infty, n(1-1/q) \le \alpha < \infty$, s be a nonnegative integer with $s \ge [\alpha + n(1/q - 1)]$. Suppose that $f \in D_s(R^n)$ and suppose $f \subset B_{k-1}$ for some $k_0 \in N$.*

(i) There exist a sequence of numbers $\{\lambda_k\}_{k\in Z}$ and a sequence of central (α, q)-atoms $\{a_k\}_{k\in Z} \subset B_{k+2} \setminus B_{k-1}$ such that

$$f(x) = \sum_{k\in Z} \lambda_k a_k(x)$$

for all $x \in R^n \setminus \{0\}$ and in the sense of $\phi'(R^n)$ and

$$\sum_{k\in Z} |\lambda_k|^p \le C\|f\|^p_{H\dot{K}_q^{\alpha,p}(R^n)}.$$

Moreover, if $\operatorname{supp} f \subset B_{k_0-1} \setminus B_{k_1+1}$ for some $k_1 \in Z$, then $\lambda_k = 0$ for all $k > k_0$ and $k < k_1$.

(ii) There exist a sequences of numbers $\{\lambda_k\}_{k=0}^{k_0}$ and a sequence of central (α, q)-atoms of restrict type $\{a_k\}_{k\in Z} \subset D_s(R^n)$ with $\operatorname{supp} a_k \subset B_{k+1}$, such that

$$f(x) = \sum_{k=0}^{k_0} \lambda_k a_k(x)$$

for all $x \in R^n$, and

$$\sum_{k} |\lambda_k|^p \le C\|f\|^p_{HK_q^{\alpha,p}(R^n)}.$$

Lemma 4 ([3]) *Let $0 < p < \infty$, $1 < q < \infty$, $n(1-1/q) < \alpha < \infty$ and $s \ge [\alpha + n(1/q - 1)]$. Then*

(i) $\dot{D}_s(R^n)$ is dense in $H\dot{K}_q^{\alpha,p}(R^n)$;

(ii) $D_s(R^n)$ is dense in $HK_q^{\alpha,p}(R^n)$.

3. Proofs of Theorems 1 and 2

We first give the proof of Theorem 1.

Proof of Theorem 1 By the atomic decomposition theory of Hardy spaces, it suffices to prove that there is a constant C such that for any $(1, l, 0)$-atom $a(x)$, the inequality

$$\|(\mu_{\Omega,\alpha} a)(x)\|_{L^q} \le C \tag{2}$$

holds, where $l > 1$ and $q = n/(n - \alpha)$. To do so, we take $1 < l_1 < l_2 < \infty$, such that $1/l_1 - 1/l_2 = \alpha/n$. Without loss of generality, we may assume that $a(x)$ is a $(1, l_1, 0)$-atom, supported in a ball $B = B(0, d)$ with center at zero and radius d, which means

(i) $\operatorname{supp}(a) \in B$; (ii) $\|a\|_{L^{l_1}} \leq |B|^{1/l_1 - 1}$; (iii) $\int_{R^n} a(x)\mathrm{d}x = 0$.

We have

$$\begin{aligned}\|(\mu_{\Omega,\alpha}a)(x)\|_{L^q} &\leq \Big(\int_{2B} |(\mu_{\Omega,\alpha}a)(x)|^q \mathrm{d}x\Big)^{1/q} + \Big(\int_{(2B)^C} |(\mu_{\Omega,\alpha}a)(x)|^q \mathrm{d}x\Big)^{1/q} \\ &= I_1 + I_2.\end{aligned}$$

By Hölder's inequality and Lemma 2, we get

$$I_1 \leq C||(\mu_{\Omega,\alpha}a)(x)||_{L^{l_2}}|B|^{1/q-1/l_2} \leq C||a||_{L^{l_1}}|B|^{1/q-1/l_2} \leq C. \tag{3}$$

For I_2, we have

$$\begin{aligned}I_2 =&\Big\{\int_{(2B)^C}\Big[\int_0^\infty \Big|\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-\alpha-1}}a(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big]^{\frac{q}{2}}\mathrm{d}x\Big\}^{1/q} \\ \leq&\Big\{\int_{(2B)^C}\Big[\int_0^{|x|} \Big|\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-\alpha-1}}a(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big]^{\frac{q}{2}}\mathrm{d}x\Big\}^{1/q} + \\ &\Big\{\int_{(2B)^C}\Big[\int_{|x|}^\infty \Big|\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-\alpha-1}}a(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big]^{\frac{q}{2}}\mathrm{d}x\Big\}^{1/q} \\ =&I_{21} + I_{22}.\end{aligned} \tag{4}$$

By Minkowski's inequality, we get

$$\begin{aligned}I_{21} &\leq \Big\{\int_{(2B)^C}\Big[\int_B\Big|\frac{\Omega(x-y)}{|x-y|^{n-\alpha-1}}\Big||a(y)|\Big(\int_{|x-y|}^{|x|}\frac{\mathrm{d}t}{t^3}\Big)^{\frac{1}{2}}\mathrm{d}y\Big]^q\mathrm{d}x\Big\}^{1/q} \\ &\leq \int_B |a(y)|\Big[\int_{(2B)^C}\Big|\frac{\Omega(x-y)}{|x-y|^{n-\alpha-1}}\Big|^q\Big(\int_{|x-y|}^{|x|}\frac{\mathrm{d}t}{t^3}\Big)^{q/2}\mathrm{d}x\Big]^{1/q}\mathrm{d}y \\ &\leq \int_B |a(y)|\Big[\sum_{j=1}^\infty\int_{2^jd\leq|x|\leq2^{j+1}d}\Big|\frac{\Omega(x-y)}{|x-y|^{n-\alpha-1}}\Big|^q\Big(\int_{|x-y|}^{|x|}\frac{\mathrm{d}t}{t^3}\Big)^{q/2}\mathrm{d}x\Big]^{1/q}\mathrm{d}y \\ &\leq \int_B |a(y)|\sum_{j=1}^\infty\Big[\int_{2^jd\leq|x|\leq2^{j+1}d}\Big|\frac{\Omega(x-y)}{|x-y|^{n-\alpha-1}}\Big|^q\frac{|y|^{q/2}}{|x-y|^{3q/2}}\mathrm{d}x\Big]^{1/q}\mathrm{d}y \\ &\leq \int_B |a(y)|\sum_{j=1}^\infty 2^{-j/2}(2^jd)^{-(n-\alpha)}\Big[\int_{2^jd\leq|x|\leq2^{j+1}d}\Big|\Omega(x-y)\Big|^q\mathrm{d}x\Big]^{1/q}\mathrm{d}y.\end{aligned}$$

Noting that $r \geq n/(n-\alpha) = q$ and $\Omega \in L^r(S^{n-1})$, we get

$$\begin{aligned}&\Big[\int_{2^jd\leq|x|\leq2^{j+1}d}|\Omega(x-y)|^q\mathrm{d}x\Big]^{1/q} \\ &\quad\leq C\Big[\int_{2^jd\leq|x|\leq2^{j+1}d}|\Omega(x-y)|^r\mathrm{d}x\Big]^{1/r}\Big(\int_{|x|\leq2^{j+1}d}\Big)^{\frac{1}{q}-\frac{1}{r}} \\ &\quad\leq C||\Omega||_{L^r(S^{n-1})}(2^{j+1}d)^{n/r}(2^{j+1}d)^{n(1/q-1/r)} \leq C(2^{j+1}d)^{n/q}.\end{aligned}$$

So we can get

$$\begin{aligned}I_{21} &\leq C\int_B|a(y)|\sum_{j=1}^\infty 2^{-j/2}(2^jd)^{-(n-\alpha)}(2^{j+1}d)^{n/q}\mathrm{d}y \leq C\int_B|a(y)|\sum_{j=1}^\infty 2^{-j/2}\mathrm{d}y \\ &\leq C\int_B|a(y)|\mathrm{d}y \leq C.\end{aligned} \tag{5}$$

Similarly to I_{21} and noting that

$$\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x|^{n-1}}\Big|\le\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x-y|^{n-1}}\Big|+\Big|\frac{\Omega(x)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x|^{n-1}}\Big|$$
$$\le\frac{C(1+|\Omega(x)|)}{|x|^{n-1}(\log\frac{|x|}{d})^{\rho}},$$

by the vanishing condition of $a(x)$, we can get

$$\begin{aligned}
I_{22}&\le\Big\{\int_{(2B)^C}\Big[\int_B\Big|\frac{\Omega(x-y)}{|x-y|^{n-\alpha-1}}-\frac{\Omega(x)}{|x|^{n-\alpha-1}}\Big||a(y)|\Big(\int_{|x|}^{\infty}\frac{\mathrm{d}t}{t^3}\Big)^{\frac12}\mathrm{d}y\Big]^q\mathrm{d}x\Big\}^{1/q}\\
&\le C\int_B|a(y)|\sum_{j=1}^{\infty}(2^jd)^{-1}(2^{j+1}d)^{\alpha}\Big[\int_{2^jd\le|x|\le2^{j+1}d}\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x|^{n-1}}\Big|^q\mathrm{d}x\Big]^{1/q}\mathrm{d}y\\
&\le\int_B|a(y)|\sum_{j=1}^{\infty}(2^jd)^{-1}(2^{j+1}d)^{\alpha}\Big[\int_{2^jd\le|x|\le2^{j+1}d}\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x|^{n-1}}\Big|^r\mathrm{d}x\Big]^{1/r}\times\\
&\qquad\Big[\int_{2^jd\le|x|\le2^{j+1}d}\Big]^{1/q-1/r}\mathrm{d}y\\
&\le C\int_B|a(y)|\sum_{j=1}^{\infty}(2^jd)^{-1}(2^{j+1}d)^{\alpha}(2^{j+1}d)^{n(1/q-1/r)}\times\\
&\qquad\Big[\int_{2^jd\le|x|\le2^{j+1}d}\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x|^{n-1}}\Big|^r\mathrm{d}x\Big]^{1/r}\mathrm{d}y\\
&\le C\int_B|a(y)|\sum_{j=1}^{\infty}(2^jd)^{\alpha-1}(2^{j+1}d)^{n(1/q-1/r)}(2^jd)^{-(n-1)}(\log 2^j)^{-\rho}(2^jd)^{n/r}\times\\
&\qquad(\|\Omega\|_{L^r(S^{n-1})}+1)\mathrm{d}y\\
&\le C\int_B|a(y)|\sum_{j=1}^{\infty}j^{-\rho}\mathrm{d}y\le C.
\end{aligned}\tag{6}$$

Togethering with (4)–(6) yields the desired estimate. □

Let us turn our attention to the proof of Theorem 2.

Proof of Theorem 2 Let $\beta=n(1-1/q_1)$ and $f\in\dot{D}_s(R^n)$, $\operatorname{supp}f\subset B_{k_0-1}\backslash B_{k_1+1}$. By Lemma 3, there exist a sequence of numbers $\{\lambda_k\}_{k\in Z}$ and a sequence of central (β,q_1)-atoms $\{a_k\}_{k\in Z}\subset\dot{D}_s(R^n)$, such that

$$f(x)=\sum_{k\in Z}\lambda_ka_k(x),\qquad\sum_{|k|\le k_0}|\lambda_k|^p\le C\|f\|^p_{H\dot{K}_q^{\alpha,p}(R^n)}.$$

If $|i|>k_0$, we let $\lambda_i=0$

$$\begin{aligned}
&\|\mu_{\Omega,\alpha}f\|^p_{\dot{K}_{q_2}^{n(1-1/q_1),p}(R^n)}=\sum_{|k|\le k_0}2^{k\beta p}\|\mu_{\Omega,\alpha}fX_k\|^p_{L^q(R^n)}\\
&\le C\Big[\sum_{k=-\infty}^{\infty}2^{k\beta p}\Big(\sum_{i=k-1}^{\infty}|\lambda_i|\|\mu_{\Omega,\alpha}(a_i)X_k\|_{q_2}\Big)^p+\sum_{k=-\infty}^{\infty}2^{k\beta p}\Big(\sum_{i=-\infty}^{k-2}|\lambda_i|\|\mu_{\Omega,\alpha}(a_i)X_k\|_{q_2}\Big)^p\Big]\\
&\doteq C(I_1+I_2).
\end{aligned}\tag{7}$$

By Lemma 2, we have

$$
\begin{aligned}
I_1 &\le C \sum_{k=-\infty}^{\infty} 2^{k\beta p}\Big(\sum_{i=k-1}^{\infty} |\lambda_i| \|a_i\|_{q_1}\Big)^p \le C \sum_{k=-\infty}^{\infty} \Big(\sum_{i=k-1}^{\infty} |\lambda_i| 2^{(k-i)\beta}\Big)^p \\
&\le C \sum_{i=-\infty}^{\infty} |\lambda_i|^p. \qquad (8)
\end{aligned}
$$

For I_2, we write

$$
\begin{aligned}
D_i^1 &= \Big[\int_0^{|x|+2^{i+1}} \Big| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\alpha-1}} a_i(y)\mathrm{d}y\Big|^2 \frac{\mathrm{d}t}{t^3}\Big]^{1/2}, \\
D_i^2 &= \Big[\int_{|x|+2^{i+1}}^{+\infty} \Big| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\alpha-1}} a_i(y)\mathrm{d}y\Big|^2 \frac{\mathrm{d}t}{t^3}\Big]^{1/2}.
\end{aligned}
$$

Then

$$
\begin{aligned}
I_2 &= \sum_{k=-\infty}^{\infty} 2^{k\beta p}\Big(\sum_{i=-\infty}^{k-2} |\lambda_i| \|\mu_{\Omega,\alpha}(a_i) X_k\|_{q_2}\Big)^p \\
&\le \sum_{k=-\infty}^{\infty} 2^{k\beta p}\Big(\sum_{i=-\infty}^{k-2} |\lambda_i| \|D_i^1 X_k\|_{q_2}\Big)^p + \sum_{k=-\infty}^{\infty} 2^{k\beta p}\Big(\sum_{i=-\infty}^{k-2} |\lambda_i| \|D_i^2 X_k\|_{q_2}\Big)^p \\
&\doteq C(I_{21} + I_{22}). \qquad (9)
\end{aligned}
$$

Noting that $x \in C_k,\ y \in B_i,\ k \ge i+2$, we have $|x-y| \sim |x| \sim |x| + 2^{i+1}$. By Minkowski's inequality we have

$$
\begin{aligned}
\|D_i^1 X_k\|_{q_2} &\le \Big\{\int_{C_k} \Big[\int_{R^n} \Big(\int_{|x-y|}^{|x|+2^{i+1}} \frac{\mathrm{d}t}{t^3}\Big)^{1/2} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha-1}} |a_i(y)|\mathrm{d}y\Big]^{q_2} \mathrm{d}x\Big\}^{1/q_2} \\
&\le C\Big\{\int_{C_k} \Big[\int_{R^n} \frac{2^{i/2}|\Omega(x-y)|}{|x-y|^{n-\alpha+1/2}} |a_i(y)|\mathrm{d}y\Big]^{q_2} \mathrm{d}x\Big\}^{1/q_2} \\
&\le C 2^{i/2} 2^{-k(n+1/2-\alpha)} \int_{B_i} \Big[\int_{C_k} |\Omega(x-y)|^{q_2} \mathrm{d}x\Big]^{1/q_2} |a_i(y)|\mathrm{d}y \\
&\le C 2^{i/2} 2^{-k(n+1/2-\alpha)} 2^{kn/q_2} \|\Omega\|_{L^r(S^{n-1})} \|a_i\|_{L^1} \\
&\le C 2^{(i-k)[n(1-1/q_1)+1/2]} 2^{-i\beta} \|\Omega\|_{L^r(S^{n-1})}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
I_{21} &\le C \sum_{k=-\infty}^{\infty} \Big(\sum_{i=-\infty}^{k-2} |\lambda_i| 2^{(i-k)[n(1-1/q_1)+1/2-\beta]}\Big)^p ||\Omega||_{L^r(S^{n-1})}^p \\
&\le C \sum_{i=-\infty}^{\infty} |\lambda_i|^p \|\Omega\|_{L^r(S^{n-1})}^p. \qquad (10)
\end{aligned}
$$

For I_{22}, noting that $y \in B_i,\ x \in R^n,\ i \le k-2,\ t \ge |x| + 2^{i+2} \ge |x| + |y| \ge |x-y|$, by the vanishing condition of a_i, we have

$$
\|D_i^2 X_k\|_{q_2} = \Big[\int_{|x|+2^{i+1}}^{\infty} \Big| \int_{|x-y|\le t} \frac{\Omega(x-y) a_i(y)}{|x-y|^{n-\alpha-1}} \mathrm{d}y\Big|^2 \frac{\mathrm{d}t}{t^3}\Big]^{1/2}
$$

$$
\begin{aligned}
&=\Big[\int_{|x|+2^{i+1}}^{\infty}\Big|\int_{|x-y|\le t}\Big(\frac{\Omega(x-y)}{|x-y|^{n-\alpha-1}}-\frac{\Omega(x)}{|x|^{n-\alpha-1}}\Big)a_i(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big]^{1/2}\\
&=\Big|\int_B\Big[\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-\alpha-1}}-\frac{\Omega(x)}{|x|^{n-\alpha-1}}\Big]\frac{a_i(y)}{|x|+2^{i+1}}\mathrm{d}y\Big|\\
&\le C2^{-k(1-\alpha)}\int_{B_i}\Big[\int_{C_k}\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x|^{n-1}}\Big|^{q_2}\mathrm{d}x\Big]^{1/q_2}a_i(y)\mathrm{d}y\\
&\le C2^{-k(1-\alpha)}2^{kn(1/q_2-1/q_1)}\int_{B_i}\Big[\int_{C_k}\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x|^{n-1}}\Big|^{q_1}\mathrm{d}x\Big]^{1/q_1}a_i(y)\mathrm{d}y\\
&\le C2^{-k(1-\alpha)}2^{kn(1/q_2-1/q_1)}(2^{k-1})^{-(n-1)}(\log 2^{k-i})^{-\rho}2^{kn/q_1}\times\\
&\quad\|\Omega\|_{L^r(S^{n-1})}\int_{B_i}|a_i(y)|\mathrm{d}y\\
&\le C2^{kn(1/q_1-1)}(k-i)^{-\rho}\|\Omega\|_{L^r(S^{n-1})}\|a_i\|_{L^1}\\
&\le C2^{-kn(1-1/q_1)}(k-i)^{-\rho}\|\Omega\|_{L^r(S^{n-1})}.
\end{aligned}\tag{11}
$$

If $0<p\le 1$, then $\rho p>1$, so we have

$$
\begin{aligned}
I_{22}&\le C\sum_{k=-\infty}^{\infty}2^{k\beta p}\Big\{\sum_{i=-\infty}^{k-2}|\lambda_i|2^{-kn(1-1/q_1)}(k-i)^{-\rho}\Big\}^p\|\Omega\|^p_{L^r(S^{n-1})}\\
&\le C\sum_{k=-\infty}^{\infty}\Big\{\sum_{i=-\infty}^{k-2}|\lambda_i|^p(k-i)^{-\rho p}\Big\}\|\Omega\|^p_{L^r(S^{n-1})}\\
&\le C\sum_{i=-\infty}^{\infty}|\lambda_i|^p\Big\{\sum_{k=i+2}^{+\infty}(k-i)^{-\rho p}\Big\}\|\Omega\|^p_{L^r(S^{n-1})}\\
&\le C\sum_{i=-\infty}^{\infty}|\lambda_i|^p\|\Omega\|^p_{L^r(S^{n-1})}.
\end{aligned}\tag{12}
$$

If $p>1$, by Hölder's inequality, we can get

$$
\begin{aligned}
I_{22}&\le C\sum_{k=-\infty}^{\infty}2^{k\beta p}\Big\{\sum_{i=-\infty}^{k-2}|\lambda_i|(k-i)^{-\rho(1/p+1/p')}\Big\}^p\|\Omega\|^p_{L^r(S^{n-1})}\\
&\le C\sum_{k=-\infty}^{\infty}\Big\{\sum_{i=-\infty}^{k-2}(k-i)^{-\rho}\Big\}\Big\{\sum_{i=-\infty}^{k-2}|\lambda_i|^p(k-i)^{-\rho}\Big\}^{p/p'}\|\Omega\|^p_{L^r(S^{n-1})}\\
&\le C\sum_{i=-\infty}^{\infty}|\lambda_i|^p\Big\{\sum_{k=i+2}^{+\infty}(k-i)^{-\rho}\Big\}\|\Omega\|^p_{L^r(S^{n-1})}\\
&\le C\sum_{i=-\infty}^{\infty}|\lambda_i|^p\|\Omega\|^p_{L^r(S^{n-1})}.
\end{aligned}\tag{13}
$$

Finally we get

$$
\|\mu_{\Omega,\alpha}(f)\|_{\dot K_{q_1}^{n(1-1/q_1),p}(R^n)}\le C\|\Omega\|^p_{L^r(S^{n-1})}\|f\|_{H\dot K_{q_1}^{n(1-1/q_1),p}(R^n)}.\tag{14}
$$

This via Lemma 4 completes the proof of Theorem 2. □

Acknowledgements The authors would like to thank the referees for their helpful suggestions

and comments.

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