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Some Properties of Solutions of Periodic Second Order Linear Differential Equations

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Abstract In this paper, the zeros of solutions of periodic second order linear differential equation $y''+Ay=0$, where $A(z)=B(e^z)$, $B(\zeta)=g(\zeta)+\sum_{j=1}^{p}b_{-j}\zeta^{-j}$, $g(\zeta)$ is a transcendental entire function of lower order no more than $1/2$, and p is an odd positive integer, are studied. It is shown that every non-trivial solution of above equation satisfies the exponent of convergence of zeros equals to infinity.

Keywords periodic differential equation; complex oscillation; regular order of growth.

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1. Introduction and main results

In this paper, we shall assume that the reader is familiar with the fundamental results and the stardard notations of the Nevanlinna's value distribution theory of meromorphic functions [12, 14, 16]. In addition, we will use the notation $\sigma(f)$, $\mu(f)$ and $\lambda(f)$ to denote respectively the order of growth, the lower order of growth and the exponent of convergence of the zeros of a meromorphic function f. $\sigma_e(f)$ (see [8]), the e-type order of $f(z)$, is defined to be

$$\sigma_e(f)=\overline{\lim_{r\to+\infty}}\frac{\log T(r,f)}{r}.$$

Similarly, $\lambda_e(f)$, the e-type exponent of convergence of the zeros of meromorphic function f, is defined to be

$$\lambda_e(f)=\overline{\lim_{r\to+\infty}}\frac{\log^+ N(r,1/f)}{r}.$$

We say that $f(z)$ has regular order of growth if a meromorphic function $f(z)$ satisfies

$$\sigma(f)=\lim_{r\to+\infty}\frac{\log T(r,f)}{\log r}.$$

We consider the second order linear differential equation

$$f''+Af=0, \tag{1.1}$$

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where $A(z)=B(e^{\alpha z})$ is a periodic entire function with period $\omega=2\pi i/\alpha$. The complex oscillation theory of (1.1) was first investigated by Bank and Laine [6]. Studies concerning (1.1) have been carried on and various oscillation theorems have been obtained [2–11, 13, 17–19]. When $A(z)$ is rational in $e^{\alpha z}$, Bank and Laine [6] proved the following theorem

Theorem A *Let $A(z)=B(e^{\alpha z})$ be a periodic entire function with period $\omega=2\pi i/\alpha$ and rational in $e^{\alpha z}$. If $B(\zeta)$ has poles of odd order at both $\zeta=\infty$ and $\zeta=0$, then for every solution $f(z)(\not\equiv 0)$ of (1.1), $\lambda(f)=+\infty$.*

Bank [5] generalized this result: The above conclusion still holds if we just suppose that both $\zeta=\infty$ and $\zeta=0$ are poles of $B(\zeta)$, and at least one is of odd order. In addition, the stronger conclusion

$$\log^+ N(r,1/f)\neq o(r) \tag{1.2}$$

holds. When $A(z)$ is transcendental in $e^{\alpha z}$, Gao [10] proved the following theorem

Theorem B *Let $B(\zeta)=g(1/\zeta)+\sum_{j=1}^{p} b_j\zeta^j$, where $g(t)$ is a transcendental entire function with $\sigma(g)<1$, p is an odd positive integer and $b_p\neq 0$. Let $A(z)=B(e^z)$. Then any non-trivial solution f of (1.1) must have $\lambda(f)=+\infty$. In fact, the stronger conclusion (1.2) holds.*

An example was given in [10] showing that Theorem B does not hold when $\sigma(g)$ is any positive integer. If the order $\sigma(g)>1$, but is not a positive integer, what can we say? Chiang and Gao [8] obtained the following theorems

Theorem C *Let $A(z)=B(e^z)$, where $B(\zeta)=g_1(1/\zeta)+g_2(\zeta)$, g_1 and g_2 are entire functions with g_2 transcendental and $\sigma(g_2)$ not equal to a positive integer or infinity, and g_1 arbitrary.*

(i) Suppose $\sigma(g_2)>1$. (a) If f is a non-trivial solution of (1.1) with $\lambda_e(f)<\sigma(g_2)$, then $f(z)$ and $f(z+2\pi i)$ are linearly dependent. (b) If f_1 and f_2 are any two linearly independent solutions of (1.1), then $\lambda_e(f_1f_2)\geq\sigma(g_2)$.

(ii) Suppose $\sigma(g_2)<1$. (a) If f is a non-trivial solution of (1.1) with $\lambda_e(f)<1$, then $f(z)$ and $f(z+2\pi i)$ are linearly dependent. (b) If f_1 and f_2 are any two linearly independent solutions of (1.1), then $\lambda_e(f_1f_2)\geq 1$.

Theorem D *Let $g(\zeta)$ be a transcendental entire function and its order be not a positive integer or infinity. Let $A(z)=B(e^z)$, where $B(\zeta)=g(1/\zeta)+\sum_{j=1}^{p} b_j\zeta^j$ and p is an odd positive integer. Then $\lambda(f)=+\infty$ for each non-trivial solution f to (1.1). In fact, the stronger conclusion (1.2) holds.*

Examples were also given in [8] showing that Theorem D is no longer valid when $\sigma(g)$ is infinity.

The main purpose of this paper is to improve above results in the case when $B(\zeta)$ is transcendental. Specially, we find a condition under which Theorem D still holds in the case when $\sigma(g)$ is a positive integer or infinity. We will prove the following results in Section 3.

Theorem 1 *Let $A(z)=B(e^z)$, where $B(\zeta)=g_1(1/\zeta)+g_2(\zeta)$, g_1 and g_2 are entire functions with g_2 transcendental and $\mu(g_2)$ not equal to a positive integer or infinity, and g_1 arbitrary. If*

$f(z)$ and $f(z+2\pi i)$ are two linearly independent solutions of (1.1), then

$$\lambda_e(f) = +\infty$$

or

$$\lambda_e(f)^{-1} + \mu(g_2)^{-1} \leq 2.$$

We remark that the conclusion of Theorem 1 remains valid if we assume $\mu(g_1)$ is not equal to a positive integer or infinity, and g_2 arbitrary and still assume $B(\zeta) = g_1(1/\zeta) + g_2(\zeta)$. In the case when g_1 is transcendental with its lower order not equal to an integer or infinity and g_2 is arbitrary, we need only to consider $B^*(\eta) = B(1/\eta) = g_1(\eta) + g_2(1/\eta)$ in $0 < |\eta| < +\infty$, $\eta = 1/\zeta$.

Corollary 1 *Let $A(z) = B(e^z)$, where $B(\zeta) = g_1(1/\zeta) + g_2(\zeta), g_1$ and g_2 are entire functions with g_2 transcendental and $\mu(g_2)$ no more than 1/2, and g_1 arbitrary.*

(a) If f is a non-trivial solution of (1.1) with $\lambda_e(f) < +\infty$, then $f(z)$ and $f(z+2\pi i)$ are linearly dependent.

(b) If f_1 and f_2 are any two linearly independent solutions of (1.1), then $\lambda_e(f_1 f_2) = +\infty$.

Theorem 2 *Let $g(\zeta)$ be a transcendental entire function and its lower order be no more than 1/2. Let $A(z) = B(e^z)$, where $B(\zeta) = g(1/\zeta) + \sum_{j=1}^{p} b_j \zeta^j$ and p is an odd positive integer, then $\lambda(f) = +\infty$ for each non-trivial solution f to (1.1). In fact, the stronger conclusion (1.2) holds.*

We remark that the above conclusion remains valid if

$$B(\zeta) = g(\zeta) + \sum_{j=1}^{p} b_{-j} \zeta^{-j}.$$

We note that Theorem 2 generalizes Theorem D when $\sigma(g)$ is a positive integer or infinity but $\mu(g) \leq 1/2$. Combining Theorem D with Theorem 2, we have

Corollary 2 *Let $g(\zeta)$ be a transcendental entire function. Let $A(z) = B(e^z)$, where $B(\zeta) = g(1/\zeta) + \sum_{j=1}^{p} b_j \zeta^j$ and p is an odd positive integer. Suppose that either (i) or (ii) below holds:*

(i) $\sigma(g)$ is not a positive integer or infinity;

(ii) $\mu(g) \leq 1/2$,

then $\lambda(f) = +\infty$ for each non-trivial solution f to (1.1). In fact, the stronger conclusion (1.2) holds.

2. Lemmas for the proofs of Theorems

Lemma 1 ([7]) *Suppose that $k \geq 2$ and that $A_0, \ldots, A_{k-2}$ are entire functions of period $2\pi i$, and that f is a non-trivial solution of*

$$y^{(k)} + \sum_{j=0}^{k-2} A_j(z) y^{(j)}(z) = 0.$$

Suppose further that f satisfies $\log^+ N(r, 1/f) = o(r)$, that A_0 is non-constant and rational in e^z, and that if $k \geq 3$, then $A_1, \ldots, A_{k-2}$ are constants. Then there exists an integer q with

$1 \le q \le k$ such that $f(z)$ and $f(z+q2\pi i)$ are linearly dependent. The same conclusion holds if A_0 is transcendental in e^z, and f satisfies $\log^+ N(r,1/f)=O(r)$, and if $k\ge 3$, then as $r\to+\infty$ through a set L_1 of infinite measure, we have $T(r,A_j)=o(T(r,A_0))$ for $j=1,\dots,k-2$.

Lemma 2 ([10]) *Let $A(z)=B(e^{\alpha z})$ be a periodic entire function with period $\omega=2\pi i\alpha^{-1}$ and be transcendental in $e^{\alpha z}$, i.e., $B(\zeta)$ is transcendental and analytic on $0<|\zeta|<+\infty$. If $B(\zeta)$ has a pole of odd order at $\zeta=\infty$ or $\zeta=0$ (including those which can be changed into this case by varying the period of $A(z)$), and Eq. (1.1) has a solution $f(z)\not\equiv 0$ which satisfies $\log^+ N(r,1/f)=o(r)$, then $f(z)$ and $f(z+\omega)$ are linearly independent.*

3. Proofs of main results

The proof of main results are based on [8] and [15].

Proof of Theorem 1 Let us assume $\lambda_e(f)<+\infty$. Since $f(z)$ and $f(z+2\pi i)$ are linearly independent, Lemma 1 implies that $f(z)$ and $f(z+4\pi i)$ must be linearly dependent. Let $E(z)=f(z)f(z+2\pi i)$. Then $E(z)$ satisfies the differential equation

$$4A(z)=(\frac{E'(z)}{E(z)})^2-2\frac{E''(z)}{E(z)}-\frac{c^2}{E(z)^2}, \tag{2.1}$$

where $c\neq 0$ is the Wronskian of f_1 and f_2 (see [12, p. 5] or [1, p. 354]), and $E(z+2\pi i)=c_1E(z)$ for some non-zero constant c_1. Clearly, E'/E and E''/E are both periodic functions with period $2\pi i$, while $A(z)$ is periodic by definition. Hence (2.1) shows that $E(z)^2$ is also periodic with period $2\pi i$. Thus we can find an analytic function $\Phi(\zeta)$ in $0<|\zeta|<+\infty$, so that $E(z)^2=\Phi(e^z)$. Substituting this expression into (2.1) yields

$$-4B(\zeta)=\frac{c^2}{\Phi}+\zeta\frac{\Phi'}{\Phi}-\frac{3}{4}\zeta^2(\frac{\Phi'}{\Phi})^2+\zeta^2\frac{\Phi''}{\Phi}. \tag{2.2}$$

Since both $B(\zeta)$ and $\Phi(\zeta)$ are analytic in $C^*=\{\zeta:1<|\zeta|<+\infty\}$, the Valiron theory [21, p. 15] gives their representations as

$$B(\zeta)=\zeta^nR(\zeta)b(\zeta),\quad \Phi(\zeta)=\zeta^{n_1}R_1(\zeta)\phi(\zeta), \tag{2.3}$$

where n, n_1 are some integers, $R(\zeta)$ and $R_1(\zeta)$ are functions that are analytic and non-vanishing on $C^*\bigcup\{\infty\}$, $b(\zeta)$ and $\phi(\zeta)$ are entire functions. Following the same arguments as used in [8], we have

$$T(\rho,\phi)=N(\rho,1/\phi)+T(\rho,b)+S(\rho,\phi), \tag{2.4}$$

where $S(\rho,\phi)=o(T(\rho,\phi))$. Furthermore, the following properties hold [8]

$$\lambda_e(f)=\lambda_e(E)=\lambda_e(E^2)=\max\{\lambda_{eR}(E^2),\lambda_{eL}(E^2)\},$$

$$\lambda_{eR}(E^2)=\lambda_1(\Phi)=\lambda(\phi),$$

where $\lambda_{eR}(E^2)$ (resp. $\lambda_{eL}(E^2)$) is defined to be

$$\overline{\lim_{r\to+\infty}}\frac{\log^+ N_R(r,1/E^2)}{r}\ (\text{ resp. }\overline{\lim_{r\to+\infty}}\frac{\log^+ N_L(r,1/E^2)}{r}),$$

where $N_R(r, 1/E^2)$ (resp. $N_L(r, 1/E^2)$) denotes a counting function that only counts the zeros of $E(z)^2$ in the right-half plane (resp. in the left-half plane), $\lambda_1(\Phi)$ is the exponent of convergence of the zeros of Φ in C^*, which is defined to be

$$\lambda_1(\Phi) = \overline{\lim_{\rho\to+\infty}} \frac{\log^+ N(\rho, 1/\Phi)}{\log \rho}.$$

Recall the condition $\lambda_e(f) < +\infty$, we obtain $\lambda(\phi) < +\infty$.

Now substituting (2.3) into (2.2) yields

$$\begin{aligned}-4\zeta^n R(\zeta)b(\zeta) =&\frac{c^2}{\zeta^{n_1}R_1(\zeta)\phi(\zeta)} + \zeta(\frac{n_1}{\zeta} + \frac{R_1'}{R_1} + \frac{\phi'}{\phi}) - \frac{3}{4}\zeta^2(\frac{n_1}{\zeta} + \frac{R_1'}{R_1} + \frac{\phi'}{\phi})^2 + \\ &\zeta^2(\frac{n_1(n_1-1)}{\zeta^2} + 2\frac{n_1R_1'}{\zeta R_1} + 2\frac{n_1\phi'}{\zeta\phi} + 2\frac{R_1'\phi'}{R_1\phi} + \frac{R_1''}{R_1} + \frac{\phi''}{\phi}).\end{aligned} \tag{2.5}$$

We assume $\sigma(\phi) < +\infty$. Since $R_1(\zeta)$, $R(\zeta)$ are analytic at ∞, we deduce $\frac{R_1^{(i)}(\zeta)}{R_1(\zeta)} = O(1)$ $(i = 1, 2)$, $\frac{1}{R(\zeta)} = O(1)$, $\frac{1}{R_1(\zeta)} = O(1)$, as $|\zeta| \to +\infty$. It follows from (2.5) and a standard estimate on the logarithmic derivative ([12, Section 3.6] or [16, Poposition 5.12]) that there exists a positive integer N such that

$$|b(\zeta)| \le |\zeta|^N, \tag{2.6}$$

for $|\phi(\zeta)| > 1$, $\zeta \notin V$, $\zeta \to \infty$ where V is an R-set ([12, Section 3.6] or [16, p. 84]).

By using the similar arguments as used in [8, p. 278], we can deduce that $\mu(g_2) = \mu(b)$. So $\mu(b)$ is not a positive integer or infinity. Thus $b(\zeta)$ must have infinitely many zeros. Let $a_1, a_2, \ldots, a_{N+1}$ be $N + 1$ zeros of $b(\zeta)$ with N as in (2.6). Define

$$H(\zeta) = b(\zeta) / \prod_{i=1}^{N+1} (\zeta - a_i), \tag{2.7}$$

then H is an entire function with $\mu(H) = \mu(b)$ not equal to a positive integer or infinity.

Next, we define $D_1^* = \{\zeta : |H(\zeta)| > 1\}$ and $D_2^* = \{\zeta : |\phi(\zeta)| > 1\}$. Clearly, D_1^* and D_2^* are open sets. We denote the boundary D_j^* by $\partial D_j^*, j = 1, 2$ and then we have $|H(\zeta)| = 1$ and $|\phi(\zeta)| = 1$ for ζ in ∂D_j^*, $j = 1, 2$. Since both $H(\zeta)$ and $\phi(\zeta)$ are transcendental, each D_j^* must contain an unbounded component D_j for $j = 1, 2$. Denote the boundary of D_j by $\partial D_j, j = 1, 2$. Let $E_j(\rho) = \{\theta : \rho e^{i\theta} \in D_j, 0 \le \theta < 2\pi\}$, $j = 1, 2$, and $E(\rho) = \{\theta : \rho e^{i\theta} \in V\}$. Clearly, $E_1(\rho) \bigcap E_2(\rho) \subset E(\rho)$, otherwise we will get

$$\frac{1}{2}|\zeta|^{N+1} \le |b(\zeta)| \le |\zeta|^N$$

from (2.6) and (2.7), where $|\phi(\zeta)| > 1$, $\zeta \notin V$, a contradiction for sufficiently large ζ.

We also let $\theta_j(\rho)$, $j = 1, 2$ and $\theta(\rho)$, respectively, be the angular measures of $E_j(\rho)$, $j = 1, 2$ and $E(\rho)$. We note that since V is an R-set, for given $\varepsilon > 0$, there exists $\rho_0 > 0$ such that $\theta(\rho) < \varepsilon$ for $\rho > \rho_0$. We also note that we can choose $\rho > \rho_0$ so that the circle $|\zeta| = \rho$ intersects D_j, $j = 1, 2$. By the Beurling-Tsuji inequality [20, Theorem III 68, p. 117], and the remark in [12, pp. 96-97] or [10, pp. 153-154], we have

$$\pi \int_{\rho_0}^{\rho/2} \frac{\mathrm{d}t}{t\theta_1(t)} < \log\log M(\rho, H) + \frac{\varepsilon}{2\pi} K_1 \log \rho + O(1), \tag{2.8}$$

where $M(\rho, H)$ denotes the usual maximum modulus of H on $|\zeta| = \rho$, $K_1 > \sigma(\phi)$, and ρ is sufficiently large. Let

$$\alpha = \varliminf_{\rho\to\infty} (\log\rho)^{-1}\pi \int_{\rho_0}^{\rho/2} \frac{\mathrm{d}t}{t\theta_1(t)}. \tag{2.9}$$

Then from (2.8), we have

$$\frac{1}{2} \le \alpha \le \mu(H). \tag{2.10}$$

Similarly, we have

$$\pi \int_{\rho_0}^{\rho/2} \frac{\mathrm{d}t}{t\theta_2(t)} < \log\log M(\rho,\phi) + \frac{\varepsilon}{2\pi}K_2 \log\rho + O(1), \tag{2.11}$$

where $K_2 > \sigma(H)$ and ρ is sufficiently large.

By Cauchy-Schwarz inequality

$$\int_{\rho_0}^{\rho/2} \frac{\theta_i(t)}{t}\mathrm{d}t \int_{\rho_0}^{\rho/2} \frac{\mathrm{d}t}{t\theta_i(t)} \ge (\int_{\rho_0}^{\rho/2} \frac{\mathrm{d}t}{t})^2, \quad i = 1, 2, \tag{2.12}$$

we obtain

$$\begin{aligned}\int_{\rho_0}^{\rho/2} \frac{\theta_2(t)}{t}\mathrm{d}t &\le \int_{\rho_0}^{\rho/2} \frac{(2\pi+\varepsilon)-\theta_1(t)}{t}\mathrm{d}t \\ &= (2\pi+\varepsilon)\int_{\rho_0}^{\rho/2} \frac{\mathrm{d}t}{t} - \int_{\rho_0}^{\rho/2} \frac{\theta_1(t)}{t}\mathrm{d}t \\ &\le (2\pi+\varepsilon)\int_{\rho_0}^{\rho/2} \frac{\mathrm{d}t}{t} - (\int_{\rho_0}^{\rho/2} \frac{\mathrm{d}t}{t})^2 / \int_{\rho_0}^{\rho/2} \frac{\mathrm{d}t}{t\theta_1(t)}\end{aligned}$$

and

$$\int_{\rho_0}^{\rho/2} \frac{\mathrm{d}t}{t\theta_2(t)} \ge \frac{(\int_{\rho_0}^{\rho/2} \frac{\mathrm{d}t}{t})^2}{(2\pi+\varepsilon)\int_{\rho_0}^{\rho/2} \frac{\mathrm{d}t}{t} - (\int_{\rho_0}^{\rho/2} \frac{\mathrm{d}t}{t})^2 / \int_{\rho_0}^{\rho/2} \frac{\mathrm{d}t}{t\theta_1(t)}}, \tag{2.13}$$

then from (2.9), (2.11) and (2.13), we have

$$\begin{aligned}\sigma(\phi) = \varlimsup_{\rho\to\infty} \frac{\log\log M(\rho,\phi)}{\log\rho} &\ge \varlimsup_{\rho\to\infty} (\log\rho)^{-1}\pi \int_{\rho_0}^{\rho/2} \frac{\mathrm{d}t}{t\theta_2(t)} - \frac{\varepsilon}{2\pi}K_2 \\ &\ge \frac{1}{\frac{2\pi+\varepsilon}{\pi} - \frac{1}{\alpha}} - \frac{\varepsilon}{2\pi}K_2.\end{aligned}$$

Since ε is arbitrary, we obtain

$$\sigma(\phi) \ge \frac{1}{2 - 1/\alpha} = \frac{\alpha}{2\alpha - 1}. \tag{2.14}$$

Inequalities (2.10) and (2.14) give

$$\sigma(\phi) \ge \frac{\mu(H)}{2\mu(H) - 1},$$

which implies

$$\sigma(\phi)^{-1} + \mu(H)^{-1} \le 2.$$

Recall that $\mu(H) = \mu(b) = \mu(g_2)$, we obtain

$$\sigma(\phi)^{-1} + \mu(g_2)^{-1} \le 2. \tag{2.15}$$

We assert $\lambda(\phi) = \sigma(\phi)$ (including $\sigma(\phi) = \infty$). If $\lambda(\phi) < \sigma(\phi)$, then the growth order of ϕ is regular [12, p. 216] and $\sigma(\phi)$ is a positive integer or infinity. From (2.4), we have $\sigma(\phi) = \mu(b)$, which means $\mu(b)$ is a positive integer or infinity, contradicting our assumption.

Finally, since

$$\lambda_{eR}(E^2) = \lambda(\phi) \leq \lambda_e(E^2) = \lambda_e(E) = \lambda_e(f),$$

we have

$$\lambda(\phi)^{-1} \geq \lambda_e(E^2)^{-1} = \lambda_e(E)^{-1} = \lambda_e(f)^{-1},$$

and can obtain

$$\lambda_e(f)^{-1} + \mu(g_2)^{-1} \leq 2.$$

This completes the proof of Theorem 1. □

Proof of Corollary 1 We can easily deduce Corollary 1 (a) from Theorem 1.

Proof of Corollary 1 (b). Suppose f_1 and f_2 are linearly independent and $\lambda_e(f_1 f_2) < +\infty$, then $\lambda_e(f_1) < +\infty$, and $\lambda_e(f_2) < +\infty$. We deduce from the conclusion of Corollary 1 (a) that $f_j(z)$ and $f_j(z + 2\pi i)$ are linearly dependent, $j = 1, 2$. Let $E(z) = f_1(z)f_2(z)$. Then we can find a non-zero constant c_2 such that $E(z + 2\pi i) = c_2 E(z)$. Repeating the same arguments as used in Theorem 1 by using the fact that $E(z)^2$ is also periodic, we obtain $\lambda_e(E)^{-1} + \mu(g_2)^{-1} \leq 2$, a contradiction since $\mu(g_2) \leq 1/2$. Hence $\lambda_e(f_1 f_2) = +\infty$. □

Proof of Theorem 2 Suppose there exists a non-trivial solution f of (1.1) that satisfies $\log^+ N(r, 1/f) = o(r)$. We deduce $\lambda_e(f) = 0$, so $f(z)$ and $f(z + 2\pi i)$ are linearly dependent by Corollary 1 (a). However, Lemma 2 implies that $f(z)$ and $f(z + 2\pi i)$ are linearly independent. This is a contradiction. Hence $\log^+ N(r, 1/f) \neq o(r)$ holds for each non-trivial solution f of (1.1). This completes the proof of Theorem 2. □

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