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Local Uniqueness of Weak Solutions for a Class of Quasilinear Subelliptic Equations

Xue Wei CUI*, Yong Zhong WANG
Department of Applied Mathematics, Northwestern Polytechnical University, Shaanxi 710072, *P. R. China*

Abstract In this note, we obtain some a-priori estimates for gradient of weak solutions to a class of subelliptic quasilinear equations constructed by Hörmander's vector fields, and then prove local uniqueness of weak solutions. A key ingredient is the estimated about kernel on metirc "annulus".

Keywords Hörmander's vector fields; subelliptic; weak solution; uniqueness.

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1. Introduction

Equations and systems constructed by Hörmander's vector fields [1] have been extensively investigated. Bony [2] studied existence and uniqueness of solutions of Dirichlet problem for elliptic equations of the form sum-of-squares. Xu [3] considered regularity of weak solutions for a class of quasilinear subelliptic systems. Capogna, Danielli and Garofalo [4] proved Hölder continuity of weak solutions for equations of the form:

$$\sum_{i=1}^{m} X_i^* A^i(x,u,Xu) = f(x,u,Xu), \tag{1}$$

where $X_1, X_2, \dots, X_m$ are Hörmander's vector fields, X_i^* denotes the formal adjoint of X_i $(i = 1,2,\dots,m)$, $Xu = (X_1u, X_2u, \dots, X_mu)$ is sub-elliptic gradient of u, $A^i(x,u,Xu)$, $f(x,u,Xu)$: $\Omega \times \mathrm{R} \times \mathrm{R}^m \to \mathrm{R}$ satisfy certain structure assumptions, $\Omega \subset \mathrm{R}^n$ is a bounded open set. Subsequently, Lu [5] extended the results of [4] under more general structure assumptions. In this note, we are concerned with local uniqueness of weak solutions for (1) under following structure conditions:

$$\sum_{i,j=1}^{m} A^i_{X_j u}\xi_i\xi_j \ge \gamma_0(|u|)|Xu|^{p-2}|\xi|^2, \quad \forall \xi \in \mathrm{R}^m, \tag{2}$$

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* Corresponding author
E-mail address: c88xw@163.com (X. W. CUI)

$$|A^i_{X_ju}| \le \gamma_1(|u|)|Xu|^{p-2}, \tag{3}$$

$$|A^i, A^i_u, f_{X_ju}| \le \gamma_1(|u|)|Xu|^{p-1}, \tag{4}$$

$$|f, f_u| \le \gamma_1(|u|)|Xu|^p \tag{5}$$

with $1 < p < \infty$. The functions γ_0 and γ_1 above are continuous in R^+, γ_0 is decreasing and strictly positive, γ_1 is increasing.

For describing the main result, we introduce some related knowledge.

Let $X_1, X_2, \ldots, X_m$ be C^∞ vector fields in R^n satisfying Hörmander's condition for hypoellipticity:

$$\mathrm{rankLie}[X_1, X_2, \ldots, X_m] = n$$

at every point $x \in \mathrm{R}^n$. A piecewise C^1 curve $\gamma : [0, T] \to \mathrm{R}^n$ is said to be sub-unitary, if for every $\xi \in \mathrm{R}^n$ and $t \in [0, T]$,

$$(\gamma'(t)\cdot\xi)^2 \le \sum_{i=1}^m (X_i(\gamma(t))\cdot\xi)^2.$$

Given two points $x, y \in \mathrm{R}^n$, the $(X_1, X_2, \ldots, X_m)$-control distance between x and y is defined as follows:

$$d(x, y) = \inf\{T > 0 \mid \gamma : [0, T] \to \mathrm{R}^n, \gamma(0) = x, \gamma(T) = y\},$$

where γ is any sub-unitary curve connecting x and y. For $x \in \mathrm{R}^n$ and $R > 0$, let $B(x, R) = \{y \mid d(x, y) < R\}$ denote a ball centered at x with radius R. Then for any bounded set $\Omega \subset \mathrm{R}^n$, there exist positive constants C, R_0 and Q, such that

$$|B(x, tR)| \ge Ct^Q\,|B(x, R)| \tag{6}$$

for every $x \in \Omega, R \le R_0$ and $0 < t < 1$. The number Q in (6) is called homogeneous dimension relative to Ω. Let $\partial B(R)$ denote boundary of $B(x, R)$. Subsequently, for convenience, we shall use short-hand notation $B(x, R) \equiv B(R)$.

The operator $\mathcal{L} = \sum_{i=1}^m X_i^* X_i$ is called sub-elliptic Laplace operator and let $\Gamma(x, y)$ denote positive fundamental solution of $\mathcal{L}$. Then familiar facts are

$$C\frac{d^2(x, y)}{|B(x, d(x, y))|} \le \Gamma(x, y) \le C^{-1}\frac{d^2(x, y)}{|B(x, d(x, y))|},$$

$$|X\Gamma(x, y)| \le C\frac{d(x, y)}{|B(x, d(x, y))|}, \tag{7}$$

where C denotes different positive constant [4].

For $\Omega \subset \mathrm{R}^n, u \in C_0^1(\Omega), x \in \Omega$, it is well known that [4]

$$u(x) = \int_\Omega X\Gamma(x, y)\cdot Xu(y)\mathrm{d}y. \tag{8}$$

Let k be a positive integer and $p > 1$. We define the non-isotropic Sobolev space by

$$S^{k,p}(\Omega) = \{u \in L^p(\Omega) | X^\alpha u \in L^p(\Omega), \forall |\alpha| \le k\},$$

where $X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_m^{\alpha_m}$, α is a multi-index: $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, α_i $(i = 1, 2, \ldots, m)$ are nonnegative integers. $S_0^{k,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $S^{k,p}(\Omega)$. The space

$$S^{k,p}_{\mathrm{loc}}(\Omega) = \{u : \Omega \to \mathrm{R} | \eta u \in S^{k,p}(\Omega), \forall \eta \in C_0^\infty(\Omega)\}$$

is called the local non-isotropic Sobolev space. For $0 < \alpha < 1$,

$$\Gamma^{\alpha}(\Omega) = \{u \in L^{\infty}(\Omega) | \sup_{\xi,\eta\in\Omega,\xi\neq\eta} \frac{|u(\xi) - u(\eta)|}{d^{\alpha}(\xi,\eta)} < +\infty\}$$

is the Folland-Stein space (or Hölder space) with the norm:

$$\|u\|_{\Gamma^{\alpha}(\Omega)} = \|u\|_{L^{\infty}(\Omega)} + \sup_{\xi,\eta\in\Omega,\xi\neq\eta} \frac{|u(\xi) - u(\eta)|}{d^{\alpha}(\xi,\eta)}.$$

For $0 < \lambda < 1$, $1 \leq p < \infty$, Morrey space $M^{p,\lambda}(\Omega)$ is the space of functions $f \in L^p_{\text{loc}}(\Omega)$ satisfying

$$\frac{1}{|B(r) \cap \Omega|} \int_{B(r)\cap\Omega} |f(x)|^p \,\mathrm{d}x < Cr^{p(\lambda-1)}$$

for every $x \in \Omega$ and $0 < r < \min\{R_0,\ \text{diam}(\Omega)\}$, where R_0 is as (6). Let Campanato space $\mathcal{L}^{p,\lambda}(\Omega)$ be the space of functions $f \in L^p_{\text{loc}}(\Omega)$ satisfying

$$\frac{1}{|B(r) \cap \Omega|} \int_{B(r)\cap\Omega} \left|f(x) - f_{B(r)}\right|^p \mathrm{d}x < Cr^{p\lambda}$$

for every $x \in \Omega$ and $0 < r < \min\{R_0, \text{diam}(\Omega)\}$, where $f_{B(r)} = \frac{1}{|B(r)|} \int_{B(r)} f(x)\mathrm{d}x$ is the average over the ball $B(r)$ of the function f. For ball $B(r)$, we have $\mathcal{L}^{p,\lambda}(B(r)) \subset \Gamma^{\lambda}_{\text{loc}}(B(r))$ (see [6]).

$u \in S^{1,p}_{\text{loc}}(\Omega)$ is called a local weak solution of equation (1), if for any $\varphi \in S^{1,p}_0(\Omega)$,

$$\int_{\Omega} \vec{A}(x,u,Xu) \cdot X\varphi \mathrm{d}x = \int_{\Omega} f(x,u,Xu)\varphi \mathrm{d}x. \tag{9}$$

Let

$$\vec{A}(x,u,Xu) = (A^1(x,u,Xu), A^2(x,u,Xu), \ldots, A^m(x,u,Xu)).$$

From (2), (4), we get

$$\begin{aligned}\sum_{i=1}^{m}(A^i(x,u,Xu) - A^i(x,u,0))X_i(u) &= \sum_{i,j=1}^{m} \int_0^1 A^i_{X_j u}(x,u,tXu)X_j u X_i(u)\mathrm{d}t \\ \geq \gamma_0 \int_0^1 |tXu|^{p-2}|Xu|^2 \mathrm{d}t &= \frac{\gamma_0}{p-1}|Xu|^p,\end{aligned}$$

i.e.,

$$\vec{A}(x,u,Xu) \cdot Xu \geq \frac{\gamma_0}{p-1}|Xu|^p = \gamma_0^{'}|Xu|^p. \tag{10}$$

Similarly, from (3), (4), we have

$$|\vec{A}(x,u,Xu)| \leq \frac{\gamma_1}{p-1}|Xu|^{p-1} = \gamma_1^{'}|Xu|^{p-1}. \tag{11}$$

Our main result is the following:

Theorem 1 *Suppose that $X_i^* = -X_i$ $(i = 1,\ldots,m)$, u_1, u_2 are local bounded weak solutions of (1) under structure conditions (2)–(5), and $u_1 = u_2$ on $\partial B(r)$ for sufficiently small $r > 0$. Then $u_1 = u_2$ on $B(r)$.*

The proof of Theorem 1 is based on estimates of fundamental solutions of $\mathcal{L}$ and some sub-elliptic estimates of weak solutions. We note that local uniqueness of weak solutions for quasi-linear elliptic equations in Euclidean space was studied in [7]. A main tool there is polar coordinates in Euclidean space. We know that polar coordinates in Heisenberg group and any Carnot group are established in [8] and [9], respectively. Unlike that, there is no "polar coordinate" in the setting of general Hörmander vector fields, which are not vector fields in any Carnot algebra. To avoid the difficulty, we establish desired estimates by cutting the kernel on metric "annulus".

With regard to research of Dirichlet problems of linear sub-elliptic equations, maximum principle implies global uniqueness of weak solutions [2, 3]. In general, for quasi-linear and nonlinear equations, uniqueness of weak solutions cannot be derived from maximum principles directly [10]. So, in this note, we use some subelliptic estimates to obtain local uniqueness of weak solutions for equations (1).

2. Some lemmas and proof of main result

The following lemma was inspired by [7] and its proof is essentially similar to one in Euclidean space case. For completeness, we write the proof with necessary changes.

Lemma 2 *If u is a local bounded weak solution of (1), then for sufficiently small $r > 0$, we have*

$$\int_{B(r)} |Xu|^p \mathrm{d}x \leq C r^{Q+p(\alpha-1)}, \tag{12}$$

where $B(2r) \subset \Omega$, α is the Hölder exponent of u (local Hölder continuity of u is infered from [4, 5].)

Proof Let $E_{b,2r} = \{x \in B(2r) \mid u(x) > b\}$. Taking test function

$$\varphi(x) = \eta^p \max\{u(x) - b, 0\}$$

in (9), where $\eta \in C_0^\infty(B(2r))$, $\eta \geq 0$, we have

$$\int_{E_{b,2r}} (\vec{A} \cdot Xu)\eta^p \mathrm{d}x + p\int_{E_{b,2r}} (\vec{A} \cdot X\eta)\eta^{p-1}(u-b)\mathrm{d}x = \int_{E_{b,2r}} f(u-b)\eta^p \mathrm{d}x. \tag{13}$$

From (5), (10), (11) and Young's inequality, we estimate the terms in (13), respectively, and get

$$\int_{E_{b,2r}} (\vec{A} \cdot Xu)\eta^p \mathrm{d}x \geq \gamma_0' \int_{E_{b,2r}} |Xu|^p \eta^p \mathrm{d}x;$$

$$\begin{aligned} p\Big|\int_{E_{b,2r}} (\vec{A} \cdot X\eta)\eta^{p-1}(u-b)\mathrm{d}x\Big| &\leq \gamma_1' p \int_{E_{b,2r}} |Xu|^{p-1}|X\eta|\eta^{p-1}|u-b|\mathrm{d}x \\ &\leq \epsilon p\gamma_1' \int_{E_{b,2r}} |Xu|^p\eta^p \mathrm{d}x + C_{\varepsilon,p} p\gamma_1' \int_{E_{b,2r}} |(u-b)X\eta|^p \mathrm{d}x; \end{aligned}$$

$$\Big|\int_{E_{b,2r}} f(u-b)\eta^p \mathrm{d}x\Big| \leq \gamma_1 \int_{E_{b,2r}} |Xu|^p\eta^p|u-b|\mathrm{d}x.$$

Since u is local Hölder continuous [4, 5], we can take sufficiently small $r > 0$ such that $b = \min\{u(x) \mid x \in B(2r)\}$ and $\max_{B(2r)} |u-b| \leq \frac{\gamma_0'}{4\gamma_1}$. Choosing $\varepsilon < \frac{\gamma_0'}{4p\gamma_1'}$, and combining previous estimates with (13), we obtain

$$\int_{B(2r)} |Xu|^p\eta^p \mathrm{d}x \leq C\int_{B(2r)} |(u-b)X\eta|^p \mathrm{d}x.$$

Now let $\eta(x) \in C_0^\infty(B(2r))$ satisfy $\eta = 1$ $(x \in B(r))$, $0 \le \eta \le 1$ $(x \in B(2r))$, $|X\eta| \le \frac{C}{r}$. By Hölder continuity of u, inequality (12) follows.

Remark 3 Inequality (12) implies $Xu \in M^{p,\alpha}(B(r))$. By Poincaré inequality [5], we have $u \in \mathcal{L}^{p,\alpha}(B(r))$. Hence, if u is a local bounded weak solution of (1), then $u \in \Gamma^\alpha_{\mathrm{loc}}(\Omega)$ if and only if u satisfies (12).

The following two lemmas rely on estimates on metric "annulus".

Lemma 4 *If u is a local bounded weak solution of (1), then for sufficiently small $r > 0$, we have*

$$\int_{B(r)} d(x,y)^{-Q+p-\frac{\alpha p}{2}} |Xu|^p \mathrm{d}x \le C r^{\frac{\alpha p}{2}},$$

where $B(r) = B(y,r) \subset B(y,2r) \subset \Omega, \alpha$ is the Hölder exponent of u.

Proof If $-Q + p - \frac{\alpha p}{2} < 0$, using Lemma 2, for sufficiently small $r > 0$, we obtain

$$\begin{aligned}
&\int_{B(r)} d(x,y)^{-Q+p-\frac{\alpha p}{2}} |Xu|^p \mathrm{d}x \le \sum_{k=0}^{\infty} \int_{B(2^{-k}r)\setminus B(2^{-k-1}r)} d(x,y)^{-Q+p-\frac{\alpha p}{2}} |Xu|^p \mathrm{d}x \\
&\le \sum_{k=0}^{\infty} \int_{B(2^{-k}r)} (2^{-k-1}r)^{-Q+p-\frac{\alpha p}{2}} |Xu|^p \mathrm{d}x \le C \sum_{k=0}^{\infty} (2^{-k-1}r)^{-Q+p-\frac{\alpha p}{2}} (2^{-k}r)^{Q+\alpha p-p} \\
&= C \sum_{k=0}^{\infty} (2^{-k})^{\frac{\alpha p}{2}} 2^{Q+\frac{\alpha p}{2}-p} r^{\frac{\alpha p}{2}} \le C r^{\frac{\alpha p}{2}}.
\end{aligned}$$

If $-Q + p - \frac{\alpha p}{2} \ge 0$, for sufficiently small $r > 0$, we get

$$\begin{aligned}
&\int_{B(r)} d(x,y)^{-Q+p-\frac{\alpha p}{2}} |Xu|^p \mathrm{d}x \le \sum_{k=0}^{\infty} \int_{B(2^{-k}r)\setminus B(2^{-k-1}r)} d(x,y)^{-Q+p-\frac{\alpha p}{2}} |Xu|^p \mathrm{d}x \\
&\le \sum_{k=0}^{\infty} \int_{B(2^{-k}r)} (2^{-k}r)^{-Q+p-\frac{\alpha p}{2}} |Xu|^p \mathrm{d}x \le C \sum_{k=0}^{\infty} (2^{-k}r)^{-Q+p-\frac{\alpha p}{2}} (2^{-k}r)^{Q+\alpha p-p} \\
&= C \sum_{k=0}^{\infty} (2^{-k})^{\frac{\alpha p}{2}} r^{\frac{\alpha p}{2}} \le C r^{\frac{\alpha p}{2}}.
\end{aligned}$$

So, the conclusion is proved. □

Lemma 5 *For $1 < p < 2$, if u is a local bounded weak solution of (1), then for sufficiently small $r > 0$, $\eta \in C_0^\infty(B(r))$, we have*

$$\int_{B(r)} |Xu(x)|^p \eta^2(x) \mathrm{d}x \le C r^\alpha \int_{B(r)} |Xu(x)|^{p-2} |X\eta(x)|^2 \mathrm{d}x,$$

where $B(2r) \subset \Omega$, α is the Hölder exponent of u.

Proof From (7),(8) and Hölder's inequality, one obtains

$$\begin{aligned}
&\int_{B(r)} |Xu(x)|^p \eta^2(x) \mathrm{d}x = \int_{B(r)} |Xu(x)|^p \Big(\int_{B(r)} |X\eta(y)||X\Gamma(x,y)| \mathrm{d}y \Big)^2 \mathrm{d}x \\
&\le C \int_{B(r)} |Xu(x)|^p \Big(\int_{B(r)} \frac{|X\eta(y)|}{d^{Q-1}(x,y)} \mathrm{d}y \Big)^2 \mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\le C\int_{B(r)}|Xu(x)|^p\Big(\int_{B(r)}\frac{|X\eta(y)|^2}{|Xu(y)|^{2-p}d^{Q-p+\frac{\alpha p}{2}}(x,y)}\mathrm{d}y\Big)\\
&\quad\Big(\int_{B(r)}\frac{|Xu(y)|^{2-p}}{d^{Q+p-\frac{\alpha p}{2}-2}(x,y)}\mathrm{d}y\Big)\mathrm{d}x. \qquad (14)
\end{aligned}
$$

If $Q-\frac{\alpha p}{2p-2}>0$, we have

$$
\begin{aligned}
&\int_{B(r)}d(x,y)^{-Q+\frac{\alpha p}{2p-2}}\mathrm{d}x\le\sum_{k=0}^{\infty}\int_{B(2^{-k}r)\setminus B(2^{-k-1}r)}d(x,y)^{-Q+\frac{\alpha p}{2p-2}}\mathrm{d}x\\
&\le\sum_{k=0}^{\infty}\int_{B(2^{-k}r)}(2^{-k-1}r)^{-Q+\frac{\alpha p}{2p-2}}\mathrm{d}x\le C\sum_{k=0}^{\infty}(2^{-k-1}r)^{-Q+\frac{\alpha p}{2p-2}}(2^{-k}r)^Q\\
&=C\sum_{k=0}^{\infty}(2^{-k})^{\frac{\alpha p}{2p-2}}2^{Q-\frac{\alpha p}{2p-2}}r^{\frac{\alpha p}{2p-2}}\le Cr^{\frac{\alpha p}{2p-2}}. \qquad (15)
\end{aligned}
$$

If $Q-\frac{\alpha p}{2p-2}\le 0$, we get

$$
\begin{aligned}
&\int_{B(r)}d(x,y)^{-Q+\frac{\alpha p}{2p-2}}\mathrm{d}x\le\sum_{k=0}^{\infty}\int_{B(2^{-k}r)\setminus B(2^{-k-1}r)}d(x,y)^{-Q+\frac{\alpha p}{2p-2}}\mathrm{d}x\\
&\le\sum_{k=0}^{\infty}\int_{B(2^{-k}r)}(2^{-k}r)^{-Q+\frac{\alpha p}{2p-2}}\mathrm{d}x\le C\sum_{k=0}^{\infty}(2^{-k}r)^{-Q+\frac{\alpha p}{2p-2}}(2^{-k}r)^Q\\
&=C\sum_{k=0}^{\infty}(2^{-k})^{\frac{\alpha p}{2p-2}}r^{\frac{\alpha p}{2p-2}}\le Cr^{\frac{\alpha p}{2p-2}}. \qquad (16)
\end{aligned}
$$

On account of (15) and (16), for sufficiently small $r>0$, Lemma 4 and Hölder's inequality imply

$$
\begin{aligned}
\int_{B(r)}\frac{|Xu(y)|^{2-p}}{d^{Q+p-\frac{\alpha p}{2}-2}(x,y)}\mathrm{d}y&\le C\Big(\int_{B(r)}\frac{|Xu(y)|^p}{d^{Q-p+\frac{\alpha p}{2}}(x,y)}\mathrm{d}y\Big)^{\frac{2-p}{p}}\Big(\int_{B(r)}\frac{1}{d^{Q-\frac{\alpha p}{2p-2}}(x,y)}\mathrm{d}y\Big)^{\frac{2p-2}{p}}\\
&\le Cr^{2\alpha-\frac{\alpha p}{2}}. \qquad (17)
\end{aligned}
$$

Hence, by virtue of (17), Lemma 4 and Fubini's theorem, we estimate (14), and get

$$
\begin{aligned}
&\int_{B(r)}|Xu(x)|^p\eta^2(x)\mathrm{d}x\le Cr^{2\alpha-\frac{\alpha p}{2}}\int_{B(r)}|Xu(x)|^p\Big(\int_{B(r)}\frac{|X\eta(y)|^2|Xu(y)|^{p-2}}{d^{Q-p+\frac{\alpha p}{2}}(x,y)}\mathrm{d}y\Big)\mathrm{d}x\\
&=Cr^{2\alpha-\frac{\alpha p}{2}}\int_{B(r)}|X\eta(y)|^2|Xu(y)|^{p-2}\int_{B(r)}\frac{|Xu(x)|^p}{d^{Q-p+\frac{\alpha p}{2}}(x,y)}\mathrm{d}x\mathrm{d}y\\
&\le Cr^{2\alpha}\int_{B(r)}|X\eta(y)|^2|Xu(y)|^{p-2}\mathrm{d}y\le Cr^{\alpha}\int_{B(r)}|X\eta(y)|^2|Xu(y)|^{p-2}\mathrm{d}y.
\end{aligned}
$$

Lemma 6 *For $p\ge 2$, if u is a local bounded weak solution of (1), then for sufficiently small $r>0$ and $\eta\in C_0^\infty(B(r))$, we have*

$$
\int_{B(r)}|Xu(x)|^p\eta^2(x)\mathrm{d}x\le Cr^\alpha\int_{B(r)}|Xu(x)|^{p-2}|X\eta(x)|^2\mathrm{d}x,
$$

where $B(2r)\subset\Omega$, α is the Hölder exponent.

Proof For $x_0\in B(r)$, taking $\varphi=(u(x)-u(x_0))\eta^2$ in (9), we obtain

$$
\int_{B(r)}\vec{A}(x,u,Xu)\cdot(Xu\eta^2+(u(x)-u(x_0))2\eta X\eta)\mathrm{d}x
$$

$$= \int_{B(r)} f(x,u,Xu)(u(x)-u(x_0))\eta^2 \mathrm{d}x. \tag{18}$$

Using (5), (10), (11), Young's inequality and Hölder continuity of u, we can estimate the terms in (18), and get

$$\int_{B(r)} (\vec{A}(x,u,Xu)\cdot Xu)\eta^2 \mathrm{d}x \ge \gamma_0' \int_{B(r)} |Xu|^p \eta^2 \mathrm{d}x;$$

$$\begin{aligned} \Big| \int_{B(r)} \vec{A}(x,u,Xu)\cdot(u(x)-u(x_0)2\eta X\eta \mathrm{d}x\Big| &\le C\gamma_1' r^\alpha \int_{B(r)} |Xu|^{p-1}|\eta||X\eta| \mathrm{d}x \\ \le C\varepsilon\gamma_1' r^\alpha \int_{B(r)} |Xu|^p|\eta|^2 \mathrm{d}x + C_\varepsilon \gamma_1' r^\alpha &\int_{B(r)} |Xu|^{p-2}|X\eta|^2 \mathrm{d}x; \end{aligned}$$

$$\Big| \int_{B(r)} f(x,u,Xu)(u(x)-u(x_0))\eta^2 \mathrm{d}x\Big| \le C\gamma_1 r^\alpha \int_{B(r)} |Xu|^p \eta^2 \mathrm{d}x.$$

Replacing these estimates into (18) leads to the conclusion.

Proof of Theorem 1 Since u_1 and u_2 are local bounded weak solutions of (1), for any $\varphi \in S_0^{1,p}(B(r))$, we obtain

$$\int_{B(r)} (\vec{A}(x,u_1,Xu_1) - \vec{A}(x,u_2,Xu_2))\cdot X\varphi \mathrm{d}x = \int_{B(r)} (f(x,u_1,Xu_1) - f(x,u_2,Xu_2))\varphi \mathrm{d}x. \tag{19}$$

Setting $\varphi = u_1 - u_2$, we have $\varphi \in S_0^{1,p}(B(r))$ and it follows

$$\begin{aligned} &(A^k(x,u_1,Xu_1) - A^k(x,u_2,Xu_2))X_k\varphi \\ &= \int_0^1 A_u^k(x, tu_1+(1-t)u_2, tXu_1+(1-t)Xu_2)\mathrm{d}t\varphi X_k\varphi + \\ &\quad \sum_{j=1}^m \int_0^1 A_{X_j u}^k(x, tu_1+(1-t)u_2, tXu_1+(1-t)Xu_2)\mathrm{d}t X_j\varphi X_k\varphi. \end{aligned}$$

Hence from (2),(4) and Young's inequality, we get

$$\begin{aligned} &|(\vec{A}(x,u_1,Xu_1) - \vec{A}(x,u_2,Xu_2))\cdot X\varphi| \\ &\ge \gamma_0 \int_0^1 |tXu_1+(1-t)Xu_2|^{p-2}\mathrm{d}t|X\varphi|^2 - \\ &\quad \gamma_1 \int_0^1 |tXu_1+(1-t)Xu_2|^{p-1}\mathrm{d}t|X\varphi||\varphi| \\ &\ge (\gamma_0 - \varepsilon\gamma_1)\int_0^1 |tXu_1+(1-t)Xu_2|^{p-2}\mathrm{d}t|X\varphi|^2 - \\ &\quad C_\varepsilon\gamma_1 \int_0^1 |tXu_1+(1-t)Xu_2|^{p}\mathrm{d}t|\varphi|^2. \end{aligned} \tag{20}$$

Similarly, from (4) and (5), we have

$$\begin{aligned} &|(f(x,u_1,Xu_1) - f(x,u_2,Xu_2))\varphi| \\ &\le \Big| \int_0^1 f_u(x, tu_1+(1-t)u_2, tXu_1+(1-t)Xu_2)\mathrm{d}t\varphi^2 \Big| + \end{aligned}$$

$$
\begin{aligned}
&\Big|\sum_{j=1}^{m}\int_0^1 f_{X_ju}(x,tu_1+(1-t)u_2,tXu_1+(1-t)Xu_2)\mathrm{d}tX_j\varphi\varphi\Big|\\
\le{}&\gamma_1\int_0^1|tXu_1+(1-t)Xu_2|^p\mathrm{d}t|\varphi|^2+\\
&\gamma_1\int_0^1|tXu_1+(1-t)Xu_2|^{p-1}\mathrm{d}t|X\varphi||\varphi|\\
\le{}&\varepsilon\gamma_1\int_0^1|tXu_1+(1-t)Xu_2|^{p-2}\mathrm{d}t|X\varphi|^2+\\
&(C_\varepsilon+1)\gamma_1\int_0^1|tXu_1+(1-t)Xu_2|^p\mathrm{d}t|\varphi|^2.
\end{aligned}
$$

Combining (20), (21) with (19), standard calculations show

$$\int_{B(r)}(|Xu_1|+|Xu_2|)^{p-2}|X\varphi|^2\mathrm{d}x\le C\int_{B(r)}(|Xu_1|+|Xu_2|)^p|\varphi|^2\mathrm{d}x. \tag{21}$$

For sufficiently small r, we maximize the integral on the right hand side of (22) by applying Lemmas 5 and 6, and get

$$\int_{B(r)}(|Xu_1|+|Xu_2|)^{p-2}|X\varphi|^2\mathrm{d}x\le Cr^\alpha\int_{B(r)}(|Xu_1|+|Xu_2|)^{p-2}|X\varphi|^2\mathrm{d}x. \tag{22}$$

For sufficiently small r satisfying $Cr^\alpha<1$, from (23), we have $u_1=u_2$ on $B(r)$.

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