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# On Skew McCoy Rings

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**Abstract** For a ring endomorphism $\alpha$, we introduce $\alpha$-skew McCoy rings which are generalizations of $\alpha$-rigid rings and McCoy rings, and investigate their properties. We show that if $\alpha^t = I_R$ for some positive integer $t$ and $R$ is an $\alpha$-skew McCoy ring, then the skew polynomial ring $R[x;\alpha]$ is $\alpha$-skew McCoy. We also prove that if $\alpha(1) = 1$ and $R$ is $\alpha$-rigid, then $R[x;\alpha]/\langle x^2\rangle$ is $\bar{\alpha}$-skew McCoy.

**Keywords** McCoy ring; skew McCoy ring; skew polynomial ring; rigid ring; skew Armendariz ring; upper triangular matrix ring.

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## 1. Introduction

All rings considered here are associative with identity. According to Nielsen [10], a ring $R$ is called a left McCoy ring if whenever $f(x),\ g(x) \in R[x]\backslash\{0\}$ satisfy $f(x)g(x) = 0$, then there exists a nonzero element $r \in R$ with $rg(x) = 0$. Similarly, right McCoy rings can be defined. If a ring is both left and right McCoy, then we say that the ring is a McCoy ring. Some properties of McCoy rings have been studied in Camillo and Nielsen [2, 9], Yang et al. [11, 12].

According to Krempa [7], an endomorphism $\alpha$ of a ring $R$ is called rigid if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. We call a ring $R$ $\alpha$-rigid if there exists a rigid endomorphism $\alpha$ of $R$. Note that any rigid endomorphism of a ring is a monomorphism and $\alpha$-rigid rings are reduced rings by Hong et al. [3, Proposition 5]. For an endomorphism $\alpha$ of a ring $R$, $R[x;\alpha]$ is reduced if and only if $R$ is $\alpha$-rigid by Hong et al. [4, Proposition 3]. Recall that for a ring $R$ with a ring endomorphism $\alpha : R \to R$, a skew polynomial ring (also called an Ore extension of endomorphism type) $R[x;\alpha]$ of $R$ is the ring obtained by giving the polynomial ring over $R$ with the new multiplication $xr = \alpha(r)x$ for all $r \in R$.

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Motivated by results in Hong et al. [3, 4], Nielsen [10] and so on, we investigate a generalization of $\alpha$-rigid rings and McCoy rings which we call an $\alpha$-skew McCoy ring.

## 2. Skew McCoy rings

**Definition 2.1** *Let $\alpha$ be an endomorphism of a ring $R$. Assume that $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\alpha]\backslash\{0\}$ satisfy $f(x)g(x) = 0$. We say that $R$ is a left $\alpha$-skew McCoy ring if there exists a nonzero element $r \in R$ with $rb_j = 0$ for all $0 \leq j \leq m$, and say that $R$ is a right $\alpha$-skew McCoy ring if there exists a nonzero element $s \in R$ with $a_i\alpha^i(s) = 0$ for all $0 \leq i \leq n$. If a ring is both left $\alpha$-skew McCoy and right $\alpha$-skew McCoy, then we say that the ring is an $\alpha$-skew McCoy ring.*

It can be easily checked that if $R$ is a McCoy ring, then it is an $I_R$-skew McCoy ring, where $I_R$ is an identity endomorphism of $R$, and thus every reversible ring (or reduced ring ) $R$ is $I_R$-skew McCoy since reversible rings are McCoy by Nielsen [10, Theorem 2]. However, the following example shows that there exists an $I_R$-skew McCoy ring $R$ which is not reversible.

**Example 2.2** Suppose that $R$ is a McCoy ring. Let

$$aUT_3(R) = \left\{ \begin{pmatrix} a & b & d \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix} \mid a,\ b,\ c,\ d \in R \right\}.$$

Then $aUT_3(R)$ is an $I_R$-skew McCoy ring since $aUT_3(R)$ is McCoy by Yang and Song [11, Proposition 2.5 and Corollary 2.8]. Let $A = E_{23}$, $B = E_{12}$, where $E_{ij}$, a $3 \times 3$ matrix, is the matrix unit with 1 in the $(i, j)$th position and 0 elsewhere. Then $AB = 0$. But $BA = E_{13} \neq 0$. Thus $aUT_3(R)$ is not reversible.

Recall that a ring is called an Armendariz ring if $a_ib_j = 0$ for all $i,\ j$ whenever polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$. For a monoid $M$, a ring $R$ is called an $M$-Armendariz ring if whenever elements $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n$, $\beta = b_1h_1 + b_2h_2 + \cdots + b_mh_m \in R[M]$ satisfy $\alpha\beta = 0$, then $a_ib_j = 0$ for each $i,\ j$. A ring $R$ is called a left $M$-McCoy ring if whenever elements $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n$, $\beta = b_1h_1 + b_2h_2 + \cdots + b_mh_m \in R[M]\backslash\{0\}$ satisfy $\alpha\beta = 0$, then there exists a nonzero element $r \in R$ with $r\beta = 0$, the right $M$-McCoy rings can be defined similarly. If a ring is both left and right $M$-McCoy, then we say that the ring is an $M$-McCoy ring. Armendariz rings are clearly McCoy. $M$-Armendariz rings are $M$-McCoy for any monoid $M$ by Yang and Song [11, Theorem 2.2]. Power-serieswise Armendariz rings are power-serieswise McCoy by Yang et al. [12, Theorem 2.2]. Some properties of these rings were studied in Anderson and Camillo [1], Hong et al. [4], Huh et al. [5], Kim et al. [6], Liu [8], Yang and Song [11], and Yang et al. [12]. Now let $\alpha$ be an endomorphism of a ring $R$, one may conjecture that if $R$ is $\alpha$-skew Armendariz, then $R$ is $\alpha$-skew McCoy. However, the following example eliminates the possibility.

**Example 2.3** ([4, Example 5]) Let $R = \mathbb{Z}_2[x]$, $\alpha : R \to R$ be an endomorphism defined by $\alpha(f(x)) = f(0)$. Then $R$ is $\alpha$-skew Armendariz by Hong et al. [4, Example 5]. However,

$p = y - xy^2, q = xy \in R[y,\alpha]\backslash\{0\}$ satisfy $pq = 0$, but for any nonzero element $f(x) \in R$, $f(x)x \neq 0$. Thus $R$ is not left $\alpha$-skew McCoy. Hence $R$ is not $\alpha$-skew McCoy.

**Proposition 2.4** *Let $\alpha$ be an endomorphism of a ring $R$. If $R$ is $\alpha$-skew Armendariz, then $R$ is right $\alpha$-skew McCoy.*

**Proof** Let $f(x) = \sum_{i=0}^n a_i x^i$, $g(x) = \sum_{j=0}^m b_j x^j \in R[x;\alpha]\backslash\{0\}$ satisfy $f(x)g(x) = 0$. Then $a_i\alpha^i(b_j) = 0$ for all $i,\ j$ since $R$ is $\alpha$-skew Armendariz. Since $g(x) \neq 0$, there exists $j_0$ such that $b_{j_0} \in R\backslash\{0\}$. Hence $a_i\alpha^i(b_{j_0}) = 0$ for all $i$. Therefore $R$ is right $\alpha$-skew McCoy. □

**Theorem 2.5** *Let $\alpha$ be an endomorphism of a ring $R$. If $R$ is $\alpha$-rigid, then $R$ is $\alpha$-skew McCoy.*

**Proof** Let $f(x) = \sum_{i=0}^n a_i x^i$, $g(x) = \sum_{j=0}^m b_j x^j \in R[x;\alpha]\backslash\{0\}$ with $f(x)g(x) = 0$. Then $R$ is $\alpha$-skew Armendariz by Hong et al. [4, Corollary 4]. Thus $a_i\alpha^i(b_j) = 0$ for all $0 \leq i \leq n,\ 0 \leq j \leq m$. Since $f(x) \neq 0$, there exists $i_0$ such that $a_{i_0} \neq 0$. Hence $a_{i_0}\alpha^{i_0}(b_j) = 0$ implies $a_{i_0}b_j = 0$ for all $0 \leq j \leq m$ by Hong et al. [3, Lemma 4(iii)]. Therefore $R$ is left $\alpha$-skew McCoy. Moreover, $R$ is right $\alpha$-skew McCoy by Proposition 2.4. The proof is completed. □

The following example shows that the converse of Theorem 2.5 is not true.

**Example 2.6** Let $R = \left\{\begin{pmatrix} r & a & b \\ 0 & r & a \\ 0 & 0 & r \end{pmatrix} \mid r \in \mathbb{Z}, a, b \in \mathbb{Q}\right\}$, where $\mathbb{Z}$ and $\mathbb{Q}$ are the sets of all integers and all rational numbers, respectively. Let $\alpha : R \to R$ be an automorphism defined by $\alpha\left(\begin{pmatrix} r & a & b \\ 0 & r & a \\ 0 & 0 & r \end{pmatrix}\right) = \begin{pmatrix} r & a/2 & b/4 \\ 0 & r & a/2 \\ 0 & 0 & r \end{pmatrix}$. Then

(1) $R$ is not $\alpha$-rigid since $\begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\alpha\left(\begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = 0$, but $\begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$ if $b \neq 0$.

(2) $R$ is $\alpha$-skew McCoy.

Let $f(x) = A_0 + A_1x + \cdots + A_nx^n,\ g(x) = B_0 + B_1x + \cdots + B_mx^m \in R[x;\alpha]\backslash\{0\}$ with $f(x)g(x) = 0$, where $A_i = \begin{pmatrix} r_i & a_i & b_i \\ 0 & r_i & a_i \\ 0 & 0 & r_i \end{pmatrix}$ and $B_j = \begin{pmatrix} s_j & c_j & d_j \\ 0 & s_j & c_j \\ 0 & 0 & s_j \end{pmatrix}$ for $0 \leq i \leq n,\ 0 \leq j \leq m$. Since $f(x), g(x) \neq 0$, by a similar proof to Hong et al. [4, Example 1] we have that $A_i = \begin{pmatrix} 0 & a_i & b_i \\ 0 & 0 & a_i \\ 0 & 0 & 0 \end{pmatrix}$ and $B_j = \begin{pmatrix} 0 & c_j & d_j \\ 0 & 0 & c_j \\ 0 & 0 & 0 \end{pmatrix}$ for $0 \leq i \leq n,\ 0 \leq j \leq m$. Take $C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. We have $Cb_j = 0$ for all $j$, and $A_i\alpha^i(C) = 0$ for all $i$. Thus $R$ is $\alpha$-skew McCoy.

The following example shows that there exists an endomorphism $\alpha$ of a McCoy ring $R$ such that $R$ is not $\alpha$-skew McCoy.

**Example 2.7** ([4, Example 2]) Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then $R$ is a commutative reduced ring. Thus it is McCoy. Let $\alpha : R \to R$ be an endomorphism defined by $\alpha((a,\ b)) = (b,\ a)$. Then for $f(x) = (1,0) + (1,0)x,\ g(x) = (0,\ 1) + (1,\ 0)x \in R[x;\alpha]\backslash\{0\}$, $f(x)g(x) = 0$. But for $(a,\ b) \in R$, if $(a,\ b)g(x) = 0$, then $a = b = 0$. Thus $R$ is not left McCoy. Similarly, if $f(x)(a,\ b) = 0$, then

$(a,\ b) = 0$. So $R$ is not right $\alpha$-skew McCoy.

Recall that if $\alpha$ is an endomorphism of a ring $R$, then the map $R[x] \to R[x]$ defined by $\sum_{i=0}^n a_i x^i \mapsto \sum_{i=0}^n \alpha(a_i) x^i$ is an endomorphism of the polynomial ring $R[x]$ and clearly this map extends $\alpha$. We shall also denote the extended map $R[x] \to R[x]$ by $\alpha$ and the image of $f \in R[x]$ by $\alpha(f)$.

**Theorem 2.8** *Let $\alpha$ be an endomorphism of a ring $R$ and $\alpha^t = I_R$ for some positive integer $t$. If $R$ is $\alpha$-skew McCoy, then $R[x;\alpha]$ is $\alpha$-skew McCoy.*

**Proof** Let $p(y) = f_0 + f_1 y + \cdots + f_n y^n$, $q(y) = g_0 + g_1 y + \cdots + g_m y^m \in R[x;\alpha][y;\alpha]\backslash\{0\}$ with $p(y)q(y) = 0$. Assume that $f_i = a_{i0} + a_{i1}x + \cdots + a_{iu_i}x^{u_i}$, $g_j = b_{j0} + b_{j1}x + \cdots + b_{jv_j}x^{v_j}$ for each $0 \le i \le n$, and $0 \le j \le m$, where $a_{i0}, a_{i1}, \ldots, a_{iu_i}, b_{j0}, b_{j1}, \ldots, b_{jv_j} \in R$. Take a positive integer $k$ such that $k > \max\{\deg(f_i), \deg(g_j)\}$ for any $0 \le i \le n$, and $0 \le j \le m$, where the degree is as polynomial in $R[x;\alpha]$ and the degree of zero polynomial is taken to be 0. Suppose that $p(x^{tk}) = f_0 + f_1 x^{tk+1} + \cdots + f_n x^{ntk+n}$, $q(x^{tk}) = g_0 + g_1 x^{tk+1} + \cdots + g_m x^{mtk+m}$. Then $p(x^{tk})$, $q(x^{tk}) \in R[x;\alpha]\backslash\{0\}$, and the set of coefficients of $f_i$'s (resp., $g_j$'s) equals the set of coefficients of $p(x^{tk})$ (resp., $q(x^{tk})$). It is easy to check that $p(x^{tk})q(x^{tk}) = 0$ in $R[x;\alpha]$ since $p(y)q(y) = 0$ in $R[x;\alpha][y;\alpha]$ and $\alpha^{tk} = I_R$. Since $R$ is $\alpha$-skew McCoy, there exist $r, s \in R\backslash\{0\}$ such that $rq(x^{tk}) = 0$, and $p(x^{tk})s = 0$. $rq(x^{tk}) = 0$ implies $rb_{jk} = 0$ for any $0 \le j \le m$, and $0 \le k \le v_j$. Hence $rg_j = 0$ for any $0 \le j \le m$. Therefore $R[x;\alpha]$ is left $\alpha$-skew McCoy. $p(x^{tk})s = 0$ implies $a_{il}\alpha^{itk+i+l}(s) = 0$ for any $0 \le i \le n$, and $0 \le l \le u_i$. Thus $a_{il}\alpha^{i+l}(s) = 0$ for any $0 \le i \le n$, and $0 \le l \le u_i$ since $\alpha^{itk} = I_R$. Hence we have

$$\begin{aligned} f_i\alpha^i(s) &= (a_{i0} + a_{i1}x + \cdots + a_{iu_i}x^{u_i})\alpha^i(s) \\ &= a_{i0}\alpha^{i+0}(s) + a_{i1}\alpha^{i+1}(s)x + \cdots + a_{iu_i}\alpha^{i+u_i}(s)x^{u_i} = 0 \end{aligned}$$

for any $0 \le i \le n$. Therefore $R[x;\alpha]$ is right $\alpha$-skew McCoy. The proof is completed. □

Recall that an element $a$ in $R$ is called regular if $r_R(a) = 0 = l_R(a)$, i.e., $a$ is not a zero divisor. For subrings of an $\alpha$-skew McCoy ring, we have the following.

**Proposition 2.9** *Let $\alpha$ be an endomorphism of a ring $R$ and $I$ be an ideal of $R$ satisfying that every nonzero element in $I$ is regular. If $R$ is $\alpha$-skew McCoy, then $I$ is $\alpha$-skew McCoy (without identity).*

**Proof** Let $f(x) = \sum_{i=0}^n a_i x^i$, $g(x) = \sum_{j=0}^m b_j x^j \in I[x;\alpha]\backslash\{0\}$ with $f(x)g(x) = 0$. Since $I$ is an ideal of $R$ and $R$ is $\alpha$-skew McCoy, there exist nonzero elements $r, s \in R$ satisfying $rb_j = 0$ for any $0 \le j \le m$, and $a_i\alpha^i(s) = 0$ for any $0 \le i \le n$. Therefore $tr$, $st \in I\backslash\{0\}$ for any nonzero element $t \in I$ (Otherwise, if $tr = 0$ (resp., $st = 0$) for a element $t \in I\backslash\{0\}$, then $r \in r_R(t)$ (resp., $s \in l_R(t)$). Hence $r = 0$ (resp., $s = 0$) since every nonzero element in $I$ is regular. This is a contradiction). Consequently, we have

$$0 = t(rb_j) = (tr)b_j,\ 0 = (a_i\alpha^i(s))\alpha^i(t) = a_i\alpha^i(st)$$

for any $0 \le i \le n$, and $0 \le j \le m$. Thus $I$ is $\alpha$-skew McCoy. □

Let $R_i$ be a ring and $\alpha_i$ an endomorphism of $R_i$ for each $i \in I$. For the product $\prod_{i\in I} R_i$ of $R_i$, the endomorphism $\bar{\alpha} : \prod_{i\in I} R_i \to \prod_{i\in I} R_i$ defined by $\bar{\alpha}((a_i)) = (\alpha(a_i))$. Yang et al. [12, Theorem 2.12] have shown that $\prod_{i\in I} R_i$ is a power-serieswise McCoy ring if and only if each $R_i$ is.

**Proposition 2.10** *Let $\alpha_i$ be an endomorphism of $R_i, i \in I$. Then $\prod_{i\in I} R_i$ is $\bar{\alpha}$-skew McCoy if and only if each $R_i$ is $\alpha_i$-skew McCoy.*

**Proof** The proof is similar to Yang et al. [12, Theorem 2.12]. $\Box$

**Corollary 2.11** *Let $R$ be an abelian ring, $\alpha$ an endomorphism of $R$, and $e^2 = e \in R$. If $eR$ and $(1-e)R$ are $\alpha$-skew McCoy, then $R$ is $\alpha$-skew McCoy.*

**Proof** Since $R$ is an abelian ring and $e^2 = e \in R$, $R = eR \times (1-e)R$. Hence the conclusion follows from Proposition 2.10. $\Box$

**Lemma 2.12** *Let $\alpha$ be an endomorphism and $R$ an $\alpha$-rigid ring. If $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\alpha]$ satisfy $f(x)g(x) = 0$, then $f(x)\alpha(g(x)) = 0$ and $\alpha(f(x))g(x) = 0$.*

**Proof** Since $R$ in $\alpha$-skew Armendariz by Hong et al. [4, Corollary 4], $a_i\alpha^i(b_j) = 0$ for each $i, j$. Thus $a_i\alpha^{i+1}(b_j) = 0$ for each $i, j$ by Hong et al. [3, Lemma 4(i)]. Hence $f(x)\alpha(g(x)) = 0$. Since $R[x;\alpha]$ is reduced by Hong et al. [4, Proposition 3], $\alpha(f(x))g(x) = 0$. $\Box$

For an ideal $I$ of a ring $R$, if $\alpha(I) \subseteq I$, then $\bar{\alpha} : R/I \to R/I$ defined by $\bar{\alpha}(a + I) = \alpha(a) + I$ is an endomorphism of a factor ring $R/I$.

**Theorem 2.13** *Let $\alpha$ be an endomorphism of $R$ and $\alpha(1) = 1$. If $R$ is $\alpha$-rigid, then $R[x;\alpha]/\langle x^2\rangle$ is $\bar{\alpha}$-skew McCoy.*

**Proof** Suppose that $p(y) = \sum_{i=0}^{n} \bar{f}_i y^i$, $q(y) = \sum_{j=0}^{m} \bar{g}_j y^j \in (R[x]/\langle x^2\rangle)[y;\bar{\alpha}]\backslash\{0\}$ with $p(y)q(y) = 0$. Let $\bar{f}_i = a_{i0} + a_{i1}\bar{x}$, $\bar{g}_j = b_{j0} + b_{j1}\bar{x}$, where $a_{i0}, a_{i1}, b_{j0}, b_{j1} \in R, \bar{x} = x + \langle x^2\rangle$. Note that $\bar{x}y = y\bar{x}$ since $\alpha(1) = 1$. Thus $p(y) = h_0 + h_1\bar{x}$ and $q(y) = k_0 + k_1\bar{x}$, where $h_0 = \sum_{i=0}^{n} a_{i0}y^i$, $h_1 = \sum_{i=0}^{n} a_{i1}y^i$, $k_0 = \sum_{j=0}^{m} b_{j0}y^j$ and $k_1 = \sum_{j=0}^{m} b_{j1}y^j$. Since $\bar{x}^2 = 0$ and $\bar{x}a = \alpha(a)\bar{x}$ for any $a \in R$, we have

$$0 = p(y)q(y) = (h_0 + h_1\bar{x})(k_0 + k_1\bar{x}) = h_0k_0 + (h_0k_1 + h_1\alpha(k_0))\bar{x}.$$

Hence in $R[y;\alpha]$ we have $h_0k_0 = 0$ and $h_0k_1 + h_1\alpha(k_0) = 0$. Thus $h_0k_0 = 0$ implies $h_0\alpha(k_0) = 0$ by Lemma 2.12. Since $R[y;\alpha](\cong R[x;\alpha])$ is reduced, $\alpha(k_0)h_0 = 0$, and so $0 = \alpha(k_0)(h_0k_1 + h_1\alpha(k_0)) = \alpha(k_0)h_1\alpha(k_0) = (h_1\alpha(k_0))^2$. Thus $h_1\alpha(k_0) = 0$, and hence $h_0k_1 = 0$.

If $h_0 \neq 0$, then the equation $0 = h_0(k_0 + k_1)$ implies that

$$0 = h_0(k_0 + k_1y^{m+1}) = h_0\Big(\sum_{j=0}^{m} b_{j0}y^j + \sum_{j=0}^{m} b_{j1}y^{j+m+1}\Big).$$

Since $R$ is (left) $\alpha$-skew McCoy by Theorem 2.5, there exists $r \in R\backslash\{0\}$ such that $rb_{j0} = 0$ and $rb_{j1} = 0$ for any $j$. Hence $r\bar{g}_j = 0$ for any $0 \leq j \leq m$.

Otherwise, if $h_0 = 0$, then $h_1 \neq 0$, and $0 = p(y)q(y) = (h_1\bar{x})(k_0 + k_1\bar{x}) = (h_1\alpha(k_0))\bar{x}$. Thus

$h_1\alpha(k_0)=0$. If $\alpha(k_0)\neq 0$, then there exists $s\in R\backslash\{0\}$ such that $s\alpha(b_{j0})=0$ for any $j$ since $R$ is (left) $\alpha$-skew McCoy. Let $r=s\bar{x}$. Then $r\in(R[x;\alpha]/\langle x^2\rangle)\backslash\{0\}$ and $r\bar{g}_j=s\bar{x}(b_{j0}+b_{j1}\bar{x})=s\alpha(b_{j0})\bar{x}=0$ for any $0\le j\le m$. If $\alpha(k_0)=0$, then let $r=\bar{x}$, and $r\bar{g}_j=0$ for any $0\le j\le m$.

So $R[x;\alpha]/\langle x^2\rangle$ is left $\bar{\alpha}$-skew McCoy.

Moreover, the equations $h_0k_0=0$ and $h_0k_1+h_1\alpha(k_0)=0$ yield that $h_0\alpha(k_0)=0$ and $h_1\alpha(k_0)=0$. Thus $h_1y^{n+1}\alpha(k_0)=h_1\alpha^{n+2}(k_0)y^{n+1}=0$ by Lemma 2.12. Hence we have

$$0=(h_0+h_1y^{n+1})\alpha(k_0)=(\sum\nolimits_{i=0}^{n}a_{i0}y^i+\sum\nolimits_{i=0}^{n}a_{i1}y^{i+n+1})(\sum\nolimits_{j=0}^{m}\alpha(b_{j0})y^j).$$

Then the right case can be proved similarly as above. □

**Proposition 2.14** *Let $\alpha$ be an endomorphism of a ring $R$ and $I$ an ideal of $R$ with $\alpha(I)\subseteq I$. If $a\alpha(a)\in I$ implies $a\in I$ for $a\in R$, then $R/I$ is $\bar{\alpha}$-skew McCoy.*

**Proof** By the proof of Hong et al. [4, Proposition 9], $R/I$ is $\bar{\alpha}$-rigid. Thus $R/I$ is $\bar{\alpha}$-skew McCoy by Theorem 2.5. □

**Lemma 2.15** *Let $\alpha$ be a monomorphism of a ring $R$, $I$ an $\alpha$-rigid ideal(without identity) of $R$ with $\alpha(I)\subseteq I$, $r\in I$ and $s\in R$. Then we have the following:*

*(1) If $rs=0$, then $r\alpha^k(s)=\alpha^k(r)s=0$ for any positive integer $k$.*

*(2) If $r\alpha^k(s)=0$ (or $\alpha^k(r)s=0$) for some positive integer $k$, then $rs=0$.*

**Proof** The proof is similar to Hong et al. [3, Lemma 4]. □

**Proposition 2.16** *Let $\alpha$ be a monomorphism of a ring $R$, $I$ an $\alpha$-rigid ideal (without identity) of $R$ with $\alpha(I)\subseteq I$ and every nonzero element in $I$ regular. Suppose $br\in I\backslash\{0\}$ implies $b\alpha(r)\in I\backslash\{0\}$ for $b,r\in R$. If $R/I$ is reversible and $\bar{\alpha}$-skew McCoy, then $R$ is $\alpha$-skew McCoy.*

**Proof** Let $f(x)=\sum_{i=0}^{n}a_ix^i$, $g(x)=\sum_{j=0}^{m}b_jx^j\in R[x;\alpha]\backslash\{0\}$ with $f(x)g(x)=0$. Consider the following three cases.

**Case 1** Both $f(x)$ and $g(x)$ are in $I[x;\alpha]$. By Theorem 2.5, $I$ is $\alpha$-skew McCoy. Thus there exist nonzero $r,\ s\in I\subseteq R$ such that $rb_j=0$, $a_i\alpha^i(s)=0$ for all $i$ and $j$.

**Case 2** One and only one of $f(x)$, $g(x)$ is in $I[x;\alpha]$. Without loss of generality, assume that $f(x)\in I[x;\alpha]$, but $g(x)\notin I[x;\alpha]$. Using Lemma 2.15 and $I$ is $\alpha$-rigid repeatedly, similar to the proof of Hong et al. [3, Proposition 6], we have that $a_ib_j=0$ for all $i,j$. Since $f(x),g(x)\neq 0$, there are $i_0,j_0$ such that $a_{i_0},b_{j_0}\in R\backslash\{0\}$. Take $r=a_{i_0},s=b_{j_0}$, and hence $rb_j=0$, and $a_i\alpha^i(s)=0$ for all $i$ and $j$.

**Case 3** Neither $f(x)$ nor $g(x)$ is in $I[x;\alpha]$. Then $\sum_{i=0}^{n}\overline{a_i}x^i,\sum_{j=0}^{m}\overline{b_j}x^j\in(R/I)[x;\bar{\alpha}]\backslash\{\overline{0}\}$. Since $R/I$ is $\bar{\alpha}$-skew McCoy, there exist $\overline{r},\overline{s}\in R/I\backslash\{\overline{0}\}$ such that $\overline{r}\overline{b_j}=\overline{0}$, $\overline{a_i}\bar{\alpha}^i(\overline{s})=\overline{0}$. Since $R/I$ is reversible, $\overline{b_j}\overline{r}=\overline{0}$, $\bar{\alpha}^i(\overline{s})\overline{a_i}=\overline{0}$. So $rb_j,b_jr,a_i\alpha^i(s),\alpha^i(s)a_i\in I$ for all $i,j$. We claim that $\alpha^i(s)a_i=0$ for all $i$, and $b_jr=0$ for all $j$.

Assume that there exists some $i$ such that $\alpha^i(s)a_i\neq 0$. Let $t$ be the smallest one relation to the property. Then $0=\alpha^t(s)f(x)g(x)=(\sum_{i=t}^n \alpha^t(s)a_ix^i)(\sum_{j=0}^m b_jx^j)$ implies $g(x)=0$ since $\alpha^t(s)a_t(\in I\backslash\{0\})$ is regular and $\alpha$ is monomorphism, a contradiction. Similarly, if there exists some $j$ such that $b_jr\neq 0$. Let $l$ be the smallest one relation to the property. Since $I$ is reduced, $b_jr=0$ yields $b_j\alpha^j(r)=0$ for $0\leq j\leq l-1$. Thus $0=f(x)g(x)r=(\sum_{i=0}^n a_ix^i)(\sum_{j=l}^m b_j\alpha^j(r)x^j)$. Since $b_lr\in I\backslash\{0\}$, we have $b_l\alpha^l(r)\in I\backslash\{0\}$. Hence $0=f(x)g(x)r$ implies $f(x)=0$ since $b_l\alpha^l(r)\in I\backslash\{0\}$ is regular and $\alpha$ is monomorphism, this is a contradiction. Thus $\alpha^i(s)a_i=0$ for all $i$, and $b_jr=0$ for all $j$.

Hence $R$ is $\alpha$-skew McCoy. □

In the last part of this section, we consider the $n\times n$ upper triangular matrix ring $T_n(R)$ over a ring $R$. Let $aUTn(R)$ be the ring consisting of $n\times n$ upper triangular matrices with equal diagonal entries over $R$, where $n\geq 2$ is a positive integer. Hong et al. [4, Proposition 17] proved that if $R$ is an $\alpha$-rigid ring, then $aUT_3(R)$ is $\bar{\alpha}$-skew Armendariz, but $aUTn(R)$ is not $\bar{\alpha}$-skew Armendariz for $n\geq 4$ (see [4, Example 18]), where $\alpha$ is an endomorphism of a ring $R$ and $\bar{\alpha}$ is the endomorphism of $aUTn(R)$ defined by $\bar{\alpha}((a_{ij}))=(\alpha(a_{ij}))$.

By Camillo and Nielsen [2, Proposition 10.2], the full matrix ring $M_n(R)$ and $T_n(R)$ over a nonzero ring $R$ need not to be $I_R$-skew McCoy.

**Proposition 2.17** *Let $n\geq 2$. Then a ring $R$ is $\alpha$-skew McCoy if and only if $aUT_n(R)$ is $\bar{\alpha}$-skew McCoy.*

**Proof** The proof is similar to Yang and Song [11, Theorem 2.7] □

**Corollary 2.18** *A ring $R$ is $\alpha$-skew McCoy if and only if the trivial extension*

$$T(R,R)=\left\{\begin{pmatrix} a & b\\ 0 & a\end{pmatrix}|a,b\in R\right\}$$

*of $R$ is $\bar{\alpha}$-skew McCoy.*

## References

[1] ANDERSON D D, CAMILLO V. *Armendariz rings and Gaussian rings* [J]. Comm. Algebra, 1998, **26**(7): 2265–2272.
[2] CAMILLO V, NIELSEN P P. *McCoy rings and zero-divisors* [J]. J. Pure Appl. Algebra, 2008, **212**(3): 599–615.
[3] HONG C Y, KIM N K, KWAK T K. *Ore extensions of Baer and p.p.-rings* [J]. J. Pure Appl. Algebra, 2000, **151**(3): 215–226.
[4] HONG C Y, KIM N K, KWAK T K. *On skew Armendariz rings* [J]. Comm. Algebra, 2003, **31**(1): 103–122.
[5] HUH C, LEE Y, SMOKTUNOWICZ A. *Armendariz rings and semicommutative rings* [J]. Comm. Algebra, 2002, **30**(2): 751–761.
[6] KIM N K, LEE K H, LEE Y. *Power series rings satisfying a zero divisor property* [J]. Comm. Algebra, 2006, **34**(6): 2205–2218.
[7] KREMPA J. *Some examples of reduced rings* [J]. Algebra Colloq., 1996, **3**(4): 289–300.
[8] LIU Zhongkui. *Armendariz rings relative to a monoid* [J]. Comm. Algebra, 2005, **33**(3): 649–661.
[9] MCCOY N H. *Remarks on divisors of zero* [J]. Amer. Math. Monthly, 1942, **49**: 286–295.
[10] NIELSEN P P. *Semi-commutativity and the McCoy condition* [J]. J. Algebra, 2006, **298**(1): 134–141.
[11] YANG Shizhou, SONG Xuemei. *Extensions of McCoy rings relative to a monoid* [J]. J. Math. Res. Exposition, 2008, **28**(3): 659–665.
[12] YANG Shizhou, SONG Xuemei, LIU Zhongkui. *Power series McCoy rings* [J]. Algebra Colloq. to appear.