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A New Fixed Point Theorem in Noncompact L-Convex Metric Spaces with Applications to Minimax Inequalities and Saddle Points

Kai Ting WEN
Department of Mathematics, Bijie University, Guizhou 551700*, P. R. China*

Abstract In this paper, a new fixed point theorem is established in noncompact complete L-convex metric spaces. As applications, a maximal element theorem, a minimax inequality and a saddle point theorem are obtained.

Keywords L-convex metric space; fixed point; maximal element; minimax inequality; saddle point.

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1. Introduction

In 1956, Aronszajn and Panitchpakdi [1] introduced the notion of hyperconvex metric spaces. Recently, Khamsi [2] established a hyperconvex version of the famous KKM-Fan principle. Park [3] obtained a Ky Fan matching theorem for open covers, a fixed point theorem and other results in hyperconvex spaces. Kirk et al. [4] established KKM theory in hyperconvex spaces and as applications of their results, a fixed point theorem, maximal element theorem and the other results were given. In [5–12], we established fixed point theorems, Ky Fan matching theorem and other results in noncompact hyperconvex spaces.

In 2001, Ding and Xia [13] introduced H-metric spaces and established some generalized H-KKM theorems in H-metric spaces. In 2005, Meng et al. [14] introduced G-convex metric spaces and established some generalized KKM theorems and fixed point theorems in G-convex metric spaces. In [15–17], we introduced L-convex metric spaces, and established some GLKKM theorems, fixed point theorems, Ky Fan matching theorems and equilibrium existence theorems for abstract economies and qualitative games in complete L-convex metric spaces.

In this paper, a new fixed point theorem is established in noncompact complete L-convex metric spaces. As applications, a maximal element theorem, a minimax inequality and a saddle point theorem are obtained. Our results unify, improve and generalize some recent known results in several aspects.

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E-mail address: wenkaiting_2004@sina.com.cn

2. Preliminaries

Let X be a nonempty set. We denote by $\mathcal{F}(X)$ and 2^X the family of all nonempty finite subsets of X and the family of all subsets of X, respectively, by $|A|$ the cardinality of A for each $A \in \mathcal{F}(X)$, and by $\triangle_n$ the standard n-dimensional simplex with vertices $e_0, e_1, \ldots, e_n$. Let X and Y be two topological spaces. We denote by $\mathcal{C}(X, Y)$ the class of single-valued continuous maps of X into Y. Let X be a nonempty set and Y a topological space. A mapping $G : X \to 2^Y$ is said to be transfer compactly open (resp., closed) valued if for each $x \in X$ and for each compact set $K \subset Y$, $y \in G(x) \cap K$ (resp., $y \notin G(x) \cap K$) implies that there exists $x' \in X$ such that $y \in \mathrm{int}_K(G(x') \cap K)$ (resp., $y \notin \mathrm{cl}_K(G(x') \cap K)$) (see [15, 17]).

Following Wen [15–17], an L-convexity structure on a topological space X is given by a mapping $\Gamma : \mathcal{F}(X) \to 2^X$ satisfying the following condition: for each $A \in \mathcal{F}(X)$ with $|A| = n+1$, there exists a continuous mapping $\phi_A : \Delta_n \to \Gamma(A)$ such that $B \in \mathcal{F}(A)$ with $|B| = J + 1$, implies $\phi_A(\Delta_J) \subset \Gamma(B)$, where Δ_J denotes the face of Δ_n corresponding to $B \in \mathcal{F}(A)$. The pair (X, Γ) is then called an L-convex space. A set $D \subset X$ is said to be L-convex if for each $A \in \mathcal{F}(D)$, $\Gamma(A) \subset D$. Let X be a nonempty set and (Y, Γ) be an L-convex space. A mapping $G : X \to 2^Y$ is said to be a GLKKM mapping if for each $\{x_1, \ldots, x_n\} \in \mathcal{F}(X)$, there exists $\{y_1, \ldots, y_n\} \in \mathcal{F}(Y)$ such that for any nonempty subset $\{y_{i_1}, \ldots, y_{i_k}\} \subset \{y_1, \ldots, y_n\}$, we have $\Gamma(\{y_{i_j} : j = 1, \ldots, k\}) \subset \bigcup_{j=1}^k G(x_{i_j})$. (M, d, Γ) is said to be an L-convex metric space if (M, d) is a metric space and (M, Γ) is an L-convex space such that $\Gamma(A) \subset \mathrm{co}(A)$ for each $A \in \mathcal{F}(M)$.

The following result, in which Y need not be a topological space, is Lemma 1.3 of Wen [17], which is the improving version of Lemma 1.1 of Wen [6] and Lemma 2.1 of Ding [18].

Lemma 2.1 ([17]) *Let X be a topological space, Y a nonempty set, K a nonempty compact subset of X and $G : X \to 2^Y$ a mapping such that $G(x) \neq \emptyset$ for each $x \in K$. Then the following conditions are equivalent:*

(a) G has the compactly local intersection property;

(b) For each $y \in Y$, there exists an open subset O_y of X such that $O_y \cap K \subset G^{-1}(y)$ and $K = \bigcup_{y \in Y}(O_y \cap K)$;

(c) There exists a mapping $F : X \to 2^Y$ such that for each $y \in Y$, $F^{-1}(y)$ is open in X, $F^{-1}(y) \cap K \subset G^{-1}(y)$, and $K = \bigcup_{y \in Y}(F^{-1}(y) \cap K)$;

(d) For each $x \in K$, there exists $y \in Y$ such that $x \in \mathrm{cint}G^{-1}(y) \cap K$ and $K = \bigcup_{y \in Y}(\mathrm{cint}G^{-1}(y) \cap K) = \bigcup_{y \in Y}(G^{-1}(y) \cap K)$;

(e) G^{-1} is transfer compactly open valued on X.

Now, we introduce the following definitions and lemmas.

Definition 2.1 *Let X be a nonempty set, (Y, Γ) an L-convex space and $A, B : X \to 2^Y$ two mappings. A is said to be relatively L-convex valued in B if for each $x \in X$ and for each $\{y_1, \ldots, y_n\} \in \mathcal{F}(B(x))$, $\Gamma(\{y_1, \ldots, y_n\}) \subset A(x)$.*

Remark 2.1 Obviously, A is relatively L-convex valued in A if A is L-convex valued, but A need not be L-convex valued if A is relatively L-convex valued in B. A is relatively L-convex valued in B if A is L-convex valued and for each $x \in X, B(x) \subset A(x)$, but the inverse is not true. A is nonempty valued if A is relatively L-convex valued in B and B is nonempty valued.

Definition 2.2 *Let X be an nonempty set, (Y, Γ) an L-convex space, $\gamma \in R$ a real number and $f, g : X \times Y \to \bar{R} := R \cup \{\pm\infty\}$ two functions. f is said to be relatively L-γ-quasiconcave (resp., L-γ-quasiconvex) in g on y if for each $x \in X$ and for each $\{y_1, \ldots, y_n\} \in \mathcal{F}(\{y \in Y : g(x,y) > \gamma\})$ (resp., $\{y_1, \ldots, y_n\} \in \mathcal{F}(\{y \in Y : g(x,y) < \gamma\})$), $\Gamma(\{y_1, \ldots, y_n\}) \subset \{y \in Y : f(x,y) > \gamma\}$ (resp., $\Gamma(\{y_1, \ldots, y_n\}) \subset \{y \in Y : f(x,y) < \gamma\}$). Where the strict inequality $>$ (resp., $<$) can be replaced equivalently by the inequality $\geq$ (resp., $\leq$).*

Remark 2.2 Definition 2.2 unifies and generalizes the definition of L-quasiconcave (resp., L-quasiconvex) of Lu et al. [19] and Ding et al. [20], Definition 2.5(1) of Kirk et al. [4], Definition 1.1 of Wen [6], Definition 1.4 of Zhang [21], Definition 5 of Liu [22] and Definition 1.2(2) of Tan [23].

Clearly, we have the following lemmas.

Lemma 2.2 *Let X be a nonempty set, (Y, Γ) an L-convex space, $\gamma \in R$ a real number, $f, g : X \times Y \to \bar{R}$ two functions and $A, B : X \to 2^Y$ two mappings defined by $A(x) := \{y \in Y : f(x,y) > \gamma\}$ and $B(x) := \{y \in Y : g(x,y) > \gamma\}$ for each $x \in X$, respectively. Then f is relatively L-γ-quasiconcave in g on y if and only if A is relatively L-convex valued in B.*

Lemma 2.3 *Let X be a nonempty set, (Y, Γ) an L-convex space, $\gamma \in R$ a real number, $f, g : X \times Y \to \bar{R}$ two functions. Then f is relatively L-γ-quasiconcave in g on y if and only if $-f$ is relatively L-γ-quasiconvex in $-g$ on y.*

Definition 2.3 ([17]) *Let X be a nonempty set, Y a topological space and $\gamma \in R$ a real number. A function $g : X \times Y \to \bar{R}$ is said to be γ-transfer compactly lower semicontinuous (in short, γ-t.c.l.s.c.) (resp., γ-transfer compactly upper semicontinuous (in short, γ-t.c.u.s.c.)) in y if for each nonempty compactly subset K of Y and for each $x \in X$ and $y \in K$, $g(x,y) > \gamma$ (resp., $g(x,y) < \gamma$) implies that there exist $x' \in X$ and a relatively open neighborhood $\mathcal{N}(y)$ of y in K such that $g(x', z) > \gamma$ (resp., $g(x', z) < \gamma$) for all $z \in \mathcal{N}(y)$.*

Lemma 2.4 ([17]) *Let X be a nonempty set, Y a topological space and $\gamma \in R$ a real number. Then a function $g : X \times Y \to \bar{R}$ is γ-t.c.l.s.c. (resp., γ-t.c.u.s.c.) in y if and only if the mapping $G : X \to 2^Y$ defined by $G(x) := \{y \in Y : g(x,y) \leq \gamma\}$ (resp., $G(x) := \{y \in Y : g(x,y) \geq \gamma\}$) for each $x \in X$ is transfer compactly closed valued.*

3. Main results

Theorem 3.1 *Let X be a topological space, Y a nonempty subset of a complete L-convex metric space (M, d, Γ). Suppose $s \in \mathcal{C}(M, X)$ is a continuous map and $A, B : X \to 2^Y \setminus \{\emptyset\}$ are two nonempty valued mappings satisfying*

(i) $\inf_{y \in Y} \mu(s^{-1}(X \setminus B^{-1}(y))) = 0$;

(ii) B satisfies one of the conditions (a)$\sim$(e) in Lemma 2.1;

(iii) A is relatively L-convex valued in B.

Then, there exists $y_0 \in Y$ such that $y_0 \in A(s(y_0))$.

Proof Since B is nonempty valued, we have:

(a) $X = \bigcup_{y\in Y} B^{-1}(y)$.

We claim that

(b) There exist $\{y_1, \ldots, y_n\} \in \mathcal{F}(Y)$ and $y_0 \in \Gamma(\{y_1, \ldots, y_n\})$ such that $s(y_0) \in \bigcap_{i=1}^n B^{-1}(y_i)$.

Suppose the conclusion of (b) is false, which implies that for each $\{y_1, \ldots, y_n\} \in \mathcal{F}(Y)$, $s(\Gamma(\{y_1, \ldots, y_n\})) \subset X \setminus \bigcap_{i=1}^n B^{-1}(y_i)$. Define the mapping $B^* : Y \to 2^X$ by

$$B^*(y) := X \setminus B^{-1}(y) \ \text{ for each } \ y \in Y.$$

Then $s(\Gamma(\{y_1, \ldots, y_n\})) \subset \bigcup_{i=1}^n B^*(y_i)$, and hence, $\Gamma(\{y_1, \ldots, y_n\}) \subset \bigcup_{i=1}^n (s^{-1}B^*)(y_i)$. Define $G : Y \to 2^M$ by

$$G(y) := (s^{-1}B^*)(y) \ \text{ for each } \ y \in Y.$$

Then, $\Gamma(\{y_1, \ldots, y_n\}) \subset \bigcup_{i=1}^n G(y_i)$, thus, G is a GLKKM mapping, moreover, by (i), $\inf_{y\in Y} \mu(G(y)) = 0$. By (ii), B^{-1} is transfer compactly open valued, which implies that B^* is transfer compactly closed valued. By the continuity of s, G is also transfer compactly closed valued. In virtue of Theorem 2.1 of Wen [17], $\bigcap_{y\in Y} G(y) = \bigcap_{y\in Y} (s^{-1}B^*)(y)$ is nonempty and compact, thus, $\bigcap_{y\in Y} B^*(y) = X \setminus \bigcup_{y\in Y} B^{-1}(y) \neq \emptyset$, which contradicts (a).

Finally, by (b), there exist $\{y_1, \ldots, y_n\} \in \mathcal{F}(Y)$ and $y_0 \in \Gamma(\{y_1, \ldots, y_n\})$ such that $s(y_0) \in \bigcap_{i=1}^n B^{-1}(y_i)$, which results in that $\{y_1, \ldots, y_n\} \in \mathcal{F}(B(s(y_0)))$. By (iii), we have $\Gamma(\{y_1, \ldots, y_n\}) \subset A(s(y_0))$. Therefore, $y_0 \in \Gamma(\{y_1, \ldots, y_n\}) \subset A(s(y_0))$.

Remark 3.1 Note that a metric space (M, d) is complete if (M, d) is hyperconvex by Proposition 1 of Khamsi [2]. If $X = Y$ is a nonempty L-convex subset of (M, d, Γ) and $s = I_X$, $s \in \mathcal{C}(X, X)$, certainly. Let $X = Y = M$ be a hyperconvex space, $A = B$ and $s = I_X$. If there exists a compact subset K of X and $y_0 \in X$ such that $X \backslash K \subset \text{int}A^{-1}(y_0)$, then $X \backslash A^{-1}(y_0) \subset X \backslash \text{int}A^{-1}(y_0) \subset K$. And hence, $\mu(s^{-1}(X \setminus A^{-1}(y_0))) = \mu(X \setminus A^{-1}(y_0)) = 0$. Thus, the condition (i) holds. Suppose $X = Y$ is compact L-convex subset of (M, d, Γ). Then the condition (i) is also satisfied trivially. If B^{-1} is open valued or transfer open valued, the condition (ii) is satisfied. If $A = B$ is L-convex valued, the condition (iii) holds, of course. Therefore, Theorem 3.1 unifies, improves and generalizes Theorem 3 of Park [3], Theorem 3.1 of Kirk et al. [4], Theorem 3.1 of Wen [5], Theorem 2.5 of Wen [16], Theorem 2.6 of Wen [17], Lemma 2.2 of Zhang [21], Corollaries 2 and 3 of Chen and Shen [24], Theorem 8 of Park [25], Theorem 3.6 of Yuan [26], Theorems 2.11, 2.22 of Yuan [27] and Theorem 3.3 of Wen [28].

As an immediate consequence of Theorem 3.1, we have the following maximal element theorem in L-convex metric spaces.

Theorem 3.2 *Let X be a topological space, Y a nonempty subset of a complete L-convex metric space (M, d, Γ). Suppose $s \in \mathcal{C}(M, X)$ is a continuous map and $A, B : X \to 2^Y$ are two* *nonempty valued mappings satisfying*

(i) $\inf_{y\in Y}\mu(s^{-1}(X\setminus B^{-1}(y)))=0$;

(ii) *B satisfies one of the conditions (a)$\sim$(e) in Lemma 2.1;*

(iii) *A is relatively L-convex valued in B;*

(iv) *For each $y\in Y$, $y\notin A(s(y))$.*

Then, there exists $x_0\in X$ such that $B(x_0)=\emptyset$.

Remark 3.2 Let $X=Y=M$ be a hyperconvex metric space. Then M is a complete L-convex metric space. Moreover, if $X=Y=M$ is compact, the condition (i) is satisfied trivially. If A^{-1} is transfer open valued, the condition (ii) is satisfied, certainly. If $A(x)=B(x)$ is either empty or admissible for each $x\in X$, the condition (iii) holds. If $s=I_X$, of course, $s\in\mathcal{C}(X,X)$. Therefore, Theorem 3.2 improves and generalizes Theorem 3.4 of Kirk et al. [4] in several aspects. Meanwhile, Theorem 3.2 improves and generalizes Theorem 3.4 of Wen [28].

Theorem 3.3 *Let X be a topological space, Y a nonempty subset of a complete L-convex metric space (M,d,Γ), $s\in\mathcal{C}(M,X)$ a continuous map, $\gamma\in R$ a real number. Suppose $f,g:X\times Y\to\bar{R}$ are two functions satisfying*

(i) $\inf_{y\in Y}\mu(s^{-1}\{x\in X: g(x,y)\le\gamma\})=0$;

(ii) *$g(x,y)$ is γ-t.c.l.s.c. in x;*

(iii) *f is relatively L-γ-quasiconcave in g on y;*

(iv) *For each $y\in Y$, $f(s(y),y)\le\gamma$.*

Then, there exists $x_0\in X$ such that $\sup_{y\in Y} g(x_0,y)\le\gamma$.

Proof Define $A,B:X\to 2^Y$ by $A(x):=\{y\in Y: f(x,y)>\gamma\}$ and $B(x):=\{y\in Y: g(x,y)>\gamma\}$ for each $x\in X$. Then, $B^*(y):=X\setminus B^{-1}(y)=\{x\in X: g(x,y)\le\gamma\}$ for each $y\in Y$. By (i), $\inf_{y\in Y}\mu(s^{-1}(X\setminus B^{-1}(y)))=0$. By (ii) and Lemma 2.4, B^* is transfer compactly closed valued, and hence, B^{-1} is transfer compactly open valued. i.e., B satisfies the condition (e) in Lemma 2.1. By (iii) and Lemma 2.2, A is relatively L-convex valued in B. By (iv), for each $y\in Y$, $y\notin A(s(y))$. In virtue of Theorem 3.2, there exists $x_0\in X$ such that $B(x_0)=\emptyset$, i.e., $g(x_0,y)\le\gamma$ for all $y\in Y$, and hence $\sup_{y\in Y} g(x_0,y)\le\gamma$.

Theorem 3.4 *Let (X,d,Γ) be a complete L-convex metric space, $s\in\mathcal{C}(X,X)$ a continuous map. Suppose $f,g:X\times X\to\bar{R}$ are two functions satisfying*

(i) $\inf_{y\in X}\mu(s^{-1}\{x\in X: g(x,y)\le 0\})=\inf_{x\in X}\mu(s^{-1}\{y\in X: g(x,y)\ge 0\})=0$;

(ii) *$g(x,y)$ is 0-t.c.l.s.c. in x and 0-t.c.u.s.c. in y;*

(iii) *$f(x,y)$ is relatively L-0-quasiconcave in g on y and $-f(x,y)$ is relatively L-0-quasiconvex in $-g$ on x;*

(iv) *For each $y\in X$, $f(s(y),y)=f(y,s(y))=0$.*

Then g has a saddle point in $X\times X$, i.e., there exists $(x_0,y_0)\in X\times X$ such that

$$\sup_{y\in X}\inf_{x\in X} g(x,y)=g(x_0,y_0)=\inf_{x\in X}\sup_{y\in X} g(x,y).$$

Proof By conditions (i)–(iv), in virtue of Theorem 3.3, there exists $x_0 \in X$ such that

$$\sup_{y\in X} g(x_0, y) \le 0. \tag{1}$$

Define h: $X \times X \to \bar{R}$ by $h(x,y) = -g(y,x)$ for each $(x,y) \in X \times X$. Then, by condition (i), $\inf_{y\in X} \mu(s^{-1}\{x \in X : h(x,y) \le 0\}) = 0$. By condition (ii), $h(x,y)$ is 0-t.c.l.s.c. in x. By condition (iii) and Lemma 2.3, $f(x,y)$ is relatively L-0-quasiconcave in h on y. By condition (iv) and in virtue of Theorem 3.3, there exists $y_0 \in X$ such that $\sup_{x\in X} h(y_0, x) \le 0$, i.e.,

$$\inf_{x\in X} g(x, y_0) \ge 0. \tag{2}$$

By inequalities (1) and (2), we have

$$g(x_0, y_0) = 0. \tag{3}$$

Moreover, inequalities (1)–(3) imply

$$\inf_{x\in X} g(x,y) \le \sup_{y\in X} g(x_0,y) \le g(x_0,y_0) \le \inf_{x\in X} g(x,y_0) \le \sup_{y\in X} g(x,y). \tag{4}$$

In turn inequality (4) implies

$$\sup_{y\in X}\inf_{x\in X} g(x,y) \le g(x_0,y_0) \le \inf_{x\in X}\sup_{y\in X} g(x,y), \tag{5}$$

$$\sup_{y\in X}\inf_{x\in X} g(x,y) \ge \inf_{x\in X} g(x,y_0) \ge g(x_0,y_0), \tag{6}$$

$$\inf_{x\in X}\sup_{y\in X} g(x,y) \le \sup_{y\in X} g(x_0,y) \le g(x_0,y_0). \tag{7}$$

Therefore,

$$\sup_{y\in X}\inf_{x\in X} g(x,y) = g(x_0,y_0) = \inf_{x\in X}\sup_{y\in X} g(x,y),$$

i.e., (x_0, y_0) is a saddle point of g. The proof is completed. □

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