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On w-Linked Overrings

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Abstract Let $R \subseteq T$ be an extension of commutative rings. T is called w-linked over R if T as an R-module is a w-module. In the case of $R \subseteq T \subseteq Q_0(R)$, T is called a w-linked overring of R. As a generalization of Wang-McCsland-Park-Chang Theorem, we show that if R is a reduced ring, then R is a w-Noetherian ring with $w\text{-}\dim(R) \leqslant 1$ if and only if each w-linked overring T of R is a w-Noetherian ring with $w\text{-}\dim(T) \leqslant 1$. In particular, R is a w-Noetherian ring with $w\text{-}\dim(R) = 0$ if and only if R is an Artinian ring.

Keywords GV-ideal; w-module; w-linked; w-Noetherian ring.

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1. Introduction

There has been considerable amount of research on w-theory over domains. Recently, by virtue of homological algebra, Yin [1] constructed w-module over arbitrary commutative rings. Let R be a commutative ring and J a finitely generated ideal of R. J is called a GV-ideal, denoted by $J \in GV(R)$, if the natural homomorphism $R \to \mathrm{Hom}_R(J, R)$ is an isomorphism. An R-module M is called a GV-torsion-free module if whenever $Jx = 0$ for some $J \in GV(R)$ and $x \in M$, then $x = 0$. A GV-torsion-free module M is called a w-module if $\mathrm{Ext}_R^1(R/J, M) = 0$ for any $J \in GV(R)$, and the w-envelope of M is the set given by

$$M_w = \{x \in E(M) \mid Jx \subseteq M \ \text{ for some } J \in GV(R)\},$$

where $E(M)$ is the injective hull of M. Therefore, M is a w-module if and only if $M_w = M$. For w-modules, readers are refereed to literature [1, 2].

Throughout this paper R denotes a commutative ring with identity, $T(R)$ denotes the total quotient ring of R, and $Q_0(R)$ denotes the ring of finite fractions over R. In this paper, we introduce the notion of w-linked. Let $R \subseteq T$ be an extension of commutative rings. T is called w-linked over R if T as an R-module is a w-module. In the case of $R \subseteq T \subseteq Q_0(R)$, T is called a w-linked overring of R. In particular, $T(R)$ and $Q_0(R)$ are w-linked overrings of R.

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The Krull-Akizuki Theorem states that if R is a Noetherian domain with $\dim(R) = 1$, then each overring T of R is a Noetherian domain with $\dim(T) \leqslant 1$. This was generalized to reduced Noetherian rings by Matijevic [3]. Let R be a reduced Noetherian ring. He introduced the transform ring R^g, and proved that every ring between R and R^g is a Noetherian ring. Wang and McCsland [4] generalized Krull-Akizuki Theorem to strong Mori domains. Let R be a strong Mori domain with w-$\dim(R) \leqslant 1$. They showed that every t-linked overring T of R is a strong Mori domain with w-$\dim(T) \leqslant 1$. Park [5] introduced the w-transform ring R^{wg}, and proved that every t-linked overring between R and R^{wg} is a strong Mori domain. As a corollary, she obtained Wang and McCsland's theorem again. By considering valuation overrings of strong Mori ring, Chang [6] showed that Wang and McCsland's theorem is necessary and sufficient. By introducing the concept of w-Noetherian ring, Yin [1] generalized Matijevic's result to w-Noetherian rings. R is called a w-Noetherian ring if it has the ascending chain condition on w-ideals. Let R be a w-Noetherian ring with w-$\dim(R) \leqslant 1$, T a w-linked overring of R, and $T \subseteq T(R)$. She showed that T has the ascending chain condition on regular w-ideals with w-$\dim(T) \leqslant 1$. By considering the ring $R\{X\}$ of fractions over $R[X]$, we obtain a similar Wang-McCsland-Park-Chang Theorem: If R is a reduced ring, then R is a w-Noetherian ring with w-$\dim(R) \leqslant 1$ if and only if each w-linked overring T of R is a w-Noetherian ring with w-$\dim(T) \leqslant 1$. In particular, R is a w-Noetherian ring with w-$\dim(R) = 0$ if and only if R is an Artinian ring.

2. Some results on w-module

Let M be an R-module and $M[X] = R[X]\bigotimes\limits_R M = \{\sum_i u_i X^i \mid u_i \in M\}$. For any $\alpha \in M[X]$, $c(\alpha)$ is a submodule of M generated by coefficients of α. For any $R[X]$-module $N \subseteq M[X]$, $c(N)$ is a submodule of M generated by coefficients of elements in N.

Lemma 2.1 ([1]) *(1) $R \in GV(R)$.*

(2) Let J_1, J_2 be finitely generated ideals of R, and $J_1 \subseteq J_2$. If $J_1 \in GV(R)$, then $J_2 \in GV(R)$.

(3) Let J_1 and J_2 be GV-ideals of R. Then $J_1 J_2 \in GV(R)$.

(4) Let R_1 and R_2 be rings. Set $R = R_1 \times R_2$. Then $J = J_1 \times J_2 \in GV(R)$ if and only if $J_i \in GV(R_i)$ for $i = 1, 2$.

Remark 2.2 Let J be a finitely generated ideal of R. When R is a domain, then $J \in GV(R)$ if and only if $(R : J) = \{x \in T(R) \mid xJ \subseteq R\} = R$, since $J \in GV(R)$ if and only if $J_w = R$. For any commutative ring R, $J \in GV(R)$ implies that $(R : J) = J^{-1} = \{x \in Q_0(R) \mid xJ \subseteq R\} = R$, since $(R : J_w) = R$ and $(J_w)^{-1} = J^{-1}$. For the converse, we show it is false by constructing an example.

Example 2.3 Let $R = \mathbb{Q} \times \mathbb{Z}$ and $A = 0 \times \mathbb{Z}$ a finitely generated ideal of R. Then $T(R) = \mathbb{Q} \times \mathbb{Q}$ and $(R : A) = A^{-1} = R$. However, A is not a GV-ideal of R by Lemma 2.1.

The consideration to establish the following lemma thanks to [1]. Here we show a better result.

Lemma 2.4 *Let J be a finitely generated ideal of R. Then $J \in GV(R)$ if and only if $J[X] \in GV(R[X])$.*

Proof It is clear that J is a GV-ideal if and only if $\mathrm{Hom}_R(R/J, R) = 0$ and $\mathrm{Ext}^1_R(R/J, R) = 0$. Since $R[X]$ is a faithfully flat R-module, it suffices to show that $R[X]\bigotimes_R \mathrm{Hom}_R(R/J, R) \cong \mathrm{Hom}_{R[X]}(R[X]/J[X], R[X])$ and $R[X]\bigotimes_R \mathrm{Ext}^1_R(R/J, R) \cong \mathrm{Ext}^1_{R[X]}(R[X]/J[X], R[X])$. By [7, Theorem 4.10.1], the former holds. Since $0 \to J \to R \to R/J \to 0$ and $0 \to J[X] \to R[X] \to R[X]/J[X] \to 0$ are short exact sequences, we have commutative diagram with exact rows:

$$\begin{tikzcd}
R[X] \bigotimes_R \mathrm{Hom}_R(R,R) \arrow[r] \arrow[d, "\cong"'] & R[X] \bigotimes_R \mathrm{Hom}_R(J,R) \arrow[r] \arrow[d, "\theta_1"'] & R[X] \bigotimes_R \mathrm{Ext}^1_R(R/J,R) \arrow[r] \arrow[d, "\theta_2"'] & 0 \\
\mathrm{Hom}_{R[X]}(R[X], R[X]) \arrow[r] & \mathrm{Hom}_{R[X]}(J[X], R[X]) \arrow[r] & \mathrm{Ext}^1_{R[X]}(R[X]/J[X], R[X]) \arrow[r] & 0
\end{tikzcd}$$

By [7, Theorem 4.10.1], θ_1 is isomorphism. Therefore, θ_2 is isomorphism. □

Corollary 2.5 *Let $B \in GV(R[X])$. Then $c(B) \in GV(R)$, and hence there exists a non-zero-divisor $f \in B$ of $R[X]$ such that $c(f)_w = R$.*

Proof Since B is a finitely generated ideal of $R[X]$, $c(B)[X]$ is a finitely generated ideal of $R[X]$ and $B \subseteq c(B)[X]$. By Lemma 2.1, $c(B)[X] \in GV(R[X])$, and hence $c(B) \in GV(R)$ by Lemma 2.4. Note that $c(B)$ is a finitely generated ideal of R, there exists $f \in B$ such that $c(f)_w = R$. If f is a zero-divisor of $R[X]$, then $bf = 0$ for some nonzero element $b \in R$. Thus $b \in bc(f)_w \subseteq (bc(f))_w = c(bf)_w = 0$, a contradiction. □

Remark 2.6 By Corollary 2.5, $\mathcal{N}_w = \{f \in R[X] \mid c(f)_w = R\}$. Then $\mathcal{N}_w$ is a multiplicatively closed set of non-zero-divisors of $R[X]$, and hence $R\{X\} = R[X]_{\mathcal{N}_w}$ is a ring of fractions over $R[X]$.

Lemma 2.7 *Let M be a GV-torsion-free R-module and N a submodule of M, $x \in M$. If $Jx \subseteq N$ for some $J \in GV(R)$, then $x \in N_w$.*

Proof Assume that $x \neq 0$. Since M is a GV-torsion-free module, $0 \neq rx \in N$ for some $0 \neq r \in J$. Thus M is an essential extension of N, and hence $x \in E(N)$. Therefore, $x \in N_w$. □

Lemma 2.8 *Let M be a GV-torsion-free R-module and $\{A_i\}$ a collection w-submodules of M. Then $\bigcap A_i$ is a w-module.*

Proof It is straightforward. □

Lemma 2.9 *Let M be a GV-torsion-free R-module with submodules A and B. Then the following hold.*

(1) $cA_w \subseteq (cA)_w$ for all $c \in R$.

(2) $A \subseteq A_w$, and $A \subseteq B \Rightarrow A_w \subseteq B_w$.

(3) $(A_w)_w = A_w$.

(4) $(u)_w = (u)$ *for a non-zero-divisor* u *of* R.

(5) $M_w = \bigcup N_w$ *where* N *runs over finitely generated* R*-submodule of* M.

(6) $(IM)_w = (I_w M_w)_w$ *for any ideal* I *of* R.

(7) $(A \bigcap B)_w = A_w \bigcap B_w$.

Proof (1)–(5) see [1].

(6) Clearly, $(IM)_w \subseteq (I_w M_w)_w$. On the other hand, suppose $x \in (I_w M_w)_w$. Then $J_1 x \subseteq I_w M_w$ for some $J_1 \in GV(R)$. Since J_1 is finitely generated, $J_1 J_2 x \subseteq IM$ for some $J_2 \in GV(R)$. By Lemma 2.7, $x \in (IM)_w$, and hence $(IM)_w = (I_w M_w)_w$.

(7) By Lemma 2.8, $(A \bigcap B)_w \subseteq A_w \bigcap B_w$. Let $x \in A_w \bigcap B_w$. Then $J_1 x \in A, J_2 x \in B$ for $J_1, J_2 \in GV(R)$, and hence $J_1 J_2 x \in A \bigcap B$. By Lemma 2.7, $x \in (A \bigcap B)_w$. Therefore, $(A \bigcap B)_w = A_w \bigcap B_w$. □

Lemma 2.10 *Let* M *be a* GV*-torsion-free* R*-module,* $\alpha \in M[X], g \in \mathcal{N}_w$. *Then* $c(\alpha)_w = c(g\alpha)_w$.

Proof Since $c(g)^{n+1} c(\alpha) = c(g)^n c(g\alpha)$ for some integer n, $c(g\alpha)_w = ((c(g)_w)^n c(g\alpha)_w)_w = (c(g)^n c(g\alpha))_w = (c(g)^{n+1} c(\alpha)_w)_w = c(\alpha)_w$ by Lemma 2.9. □

It is easy to see that, M is a GV-torsion-free R-module if and only if $\mathrm{Hom}_R(R/J, M) = 0$ for any $J \in GV(R)$. Here we have

Proposition 2.11 *Let* S *be a multiplicatively closed set of non-zero-divisors of* R *and* N *a* w*-module. If natural homomorphism* $N \to N_S$ *is monomorphism, then* N_S *as an* R*-module is* w*-module.*

Proof Since $\mathrm{Hom}_R(R/J, N_S) \cong R_S \bigotimes_R \mathrm{Hom}_R(R/J, N)$ for any $J \in GV(R)$, N_S is a GV-torsion-free R-module. Since N_S is an essential extension of N, $E(N_S) = E(N)$. Let $Jx \subseteq N_S$ for some $J \in GV(R)$ where $x \in E(N)$. Then $Jsx \subseteq N$ for some $s \in S$, since J is finitely generated. Thus $sx \in N$, and hence $x \in N_S$. □

It is well-known that M is a torsion-free R-module if $M[X]$ is a torsion-free $R[X]$-module, the converse is false. However, we have

Theorem 2.12 M *is a* GV*-torsion-free* R*-module if and only if* $M[X]$ *is a* GV*-torsion-free* $R[X]$*-module.*

Proof Let M be a GV-torsion-free R-module and $\alpha \in M[X]$. If $B\alpha = 0$ for some $B \in GV(R[X])$, then there exists $g \in B$ such that $c(g) \in GV(R)$ and $g\alpha = 0$. By Lemma 2.10, $c(\alpha) \subseteq c(\alpha)_w = (c(g\alpha)_w) = 0$, and so $\alpha = 0$. Therefore, $M[X]$ is a GV-torsion-free $R[X]$-module.
Suppose $M[X]$ is a GV-torsion-free $R[X]$-module and $\alpha \in M$. If $J\alpha = 0$ for some $J \in GV(R)$, then $J[X]\alpha = 0$. Since $J[X] \in GV(R[X])$, $\alpha = 0$. Therefore, M is a GV-torsion-free R-module. □

Theorem 2.13 *Let* M *be a* GV*-torsion-free module. Then the following hold.*

(1) $(M[X]_W)_{\mathcal{N}_w} = M\{X\}$.

(2) $M\{X\}$ as an $R[X]$-module is a w-module, and hence $M_w \subseteq M_w[X] \subseteq M[X]_W \subseteq M\{X\}$.

Proof (1) By Theorem 2.12, $M[X]$ is a GV-torsion-free $R[X]$-module, and hence $M[X]_W$ is a GV-torsion-free $R[X]$-module. For $0 \neq \alpha \in M[X]_W$, we have $f\alpha \neq 0$ for any $f \in \mathcal{N}_w$. Suppose $u \in M[X]_W$. Then $Bu \subseteq M[X]$ for some $B \in GV(R[X])$, and so $fu \in M[X]$ for some $f \in B \bigcap \mathcal{N}_w$ by Corollary 2.5. So $u \in M\{X\}$, and hence $(M[X]_W)_{\mathcal{N}_w} = M\{X\}$.

(2) Since $(M[X]_W)_{\mathcal{N}_w}$ is a GV-torsion-free $R[X]$-module, $M\{X\} = (M[X]_W)_{\mathcal{N}_w}$ as an $R[X]$-module is a w-module by Proposition 2.11. Let $\beta = \sum_{i=0}^{n} a_i X^i \in M_w[X], a_i \in M_w$. Then there exists $J \in GV(R)$ such that $Ja_i \subseteq M$ for any i, and hence $J[X]\beta \in M[X]$. Since $J[X] \in GV(R[X])$, $M_w[X] \subseteq M[X]_W$ by Lemma 2.7, and so $M_w \subseteq M_w[X] \subseteq M[X]_W \subseteq M\{X\}$. □

3. w-linked

Following Lucas [8], $Q_0(R)$ consists of those elements $\frac{a(X)}{b(X)}$, where $a(X), b(X) \in R[X]$ and $b(X)$ is a non-zero-divisor of $R[X]$ with the coefficient relations $a_i b_j = a_j b_i$ for each i and j. For any commutative ring R, $T(R) \subseteq Q_0(R)$. When R is a domain, then $T(R) = Q_0(R)$ is the quotient field of R.

Let $R \subseteq T$ be an extension of domains. Following Dobbs [9], T is called t-linked over R if $(R : J) = R$ (i.e., $J_t = R$) implies that $(T : JT) = T$ (i.e., $(JT)_t = T$) for any non-zero finitely generated ideal J. Wang [10] proved that T is t-linked over R if and only if $T_w = T$. Following the result, we extend t-linked to the extension of commutative rings.

Definition 3.1 *Let $R \subseteq T$ be an extension of commutative rings. T is called w-linked over R if T as an R-module is a w-module. In the case of $R \subseteq T \subseteq Q_0(R)$, T is called a w-linked overring of R.*

Remark 3.2 When $R \subseteq T$ is an extension of domains, then w-linked coincides with t-linked. For any commutative ring R, $R[X]$ is w-linked over R. In the case of $R = T(R) = Q_0(R)$ and $nilradical(R) \neq 0$, we have $(R : A) = R$ but $(R[X] : A[X]) \neq R[X]$ for any finitely generated nilpotent ideal A of R. Therefore, we use w-linked instead of t-linked.

Lemma 3.3 ([1]) *Let $R \subseteq T$ be an extension of commutative rings and T a GV-torsion-free R-module. The following are equivalent.*

(1) T is w-linked over R.

(2) $A \bigcap R$ is a w-ideal of R for any w-ideal A of T.

(3) Let P be a prime w-ideal of T. Then $P \cap R$ is a prime w-ideal of R.

(4) If $J \in GV(R)$, then $JT \in GV(R)$.

(5) Let L be a T-module. If L as a T-module is a w-module, then L as an R-module is a w-module.

Proposition 3.4 *$T(R)$ and $Q_0(R)$ are w-linked overrings of R.*

Proof Set $T = T(R[X])$. By Proposition 2.11, $T(R)$ is a w-linked overring of R and T is w-

linked over $R[X]$. Let $J \in GV(R)$. Then $J[X] \in GV(R[X])$, and hence $JT \in GV(T)$ by Lemma 3.3. Thus T is w-linked over R. Let $\frac{a(X)}{b(X)} \in T$ such that $J\frac{a(X)}{b(X)} \subseteq Q_0(R)$ for some $J \in GV(R)$. Then for any $u \in J$, we have the coefficient relations $ua_ib_j = ua_jb_i$ for each i and j, and so $J(a_ib_j - a_jb_i) = 0$. Thus $a_ib_j = a_jb_i$, which implies that $\frac{a(X)}{b(X)} \in Q_0(R)$. Therefore, $Q_0(R)$ as an R-module is a w-module. □

Lemma 3.5 *Let $R \subseteq T$ be an extension of commutative rings, $x \in T$, and N an R-submodule of T. If T is a GV-torsion-free R-module, then the following hold.*

(1) $xN_w \subseteq (xN)_w$.

(2) If x is a non-zero-divisor of N, then $xN_w = (xN)_w$.

Proof (1) Suppose $u \in N_w$. Then $Ju \subseteq N$ for some $J \in GV(R)$, and hence $Jxu \subseteq xN$. By Theorem 2.13, $T\{X\}$ is a GV-torsion-free R-module and $xu \in xN_w \subseteq T\{X\}$. By Lemma 2.7, $xN_w \subseteq (xN)_w$.

(2) It is clear that x is a non-zero-divisor of N_w. Since $xN_w \cong N_w$, xN_w is a w-module, and hence $xN_w = (xN)_w$. □

Proposition 3.6 *Let $R \subseteq T$ be an extension of commutative rings and T a GV-torsion-free R-module. Then the following hold*

(1) T_w is w-linked over R.

(2) A_w is an ideal of T_w for any ideal A of T.

(3) Let P be a prime ideal of T and $P \cap R$ a w-ideal of R. Then $P_w \neq T_w$.

(4) Let P be a prime ideal of T and $P_w \neq T_w$. Then P_w is a prime ideal of T_w and $P_w \cap T = P$.

(5) Let P be a prime ideal of T, $P_w \neq T_w$, P_1 a prime ideal of T_w such that $P_1 \subseteq P_w$, and $P_1 \bigcap T = P$. Then $P_1 = P_w$.

(6) Let P be a prime ideal of T. If $\operatorname{ht} P_w = 0$, then $\operatorname{ht} P = 0$.

Proof (1) It suffices to show that T_w is a ring. Suppose $a, b \in T_w, ab \neq 0$. Then $J_1a \subseteq T, J_2b \subseteq T$ for $J_1, J_2 \in GV(R)$, and hence $J_1J_2ab \subseteq T$. By Theorem 2.13, $T\{X\}$ is a GV-torsion-free R-module and $ab \in T\{X\}$. Therefore, $ab \in T_w$.

(2) It is similar to (1).

(3) If $P_w = T_w$, then $J \subseteq P$ for some $J \in GV(R)$. Thus $J \subseteq P \bigcap R$, a contradiction.

(4) Suppose $x \in P_w \bigcap T$. Then $Jx \subseteq P$ for some $J \in GV(R)$. Since $J \nsubseteq P$, $P_w \bigcap T = P$.

(5) Suppose $x, y \in T_w, xy \in P_w$. Then $J_1x \subseteq T, J_2y \subseteq T$ for $J_1, J_2 \in GV(R)$. Thus $Jxy \subseteq T$ for some $J = J_1J_2 \in GV(R)$, and hence $Jx \subseteq P$ or $Jy \subseteq P$. Therefore, $x \in P_w$ or $y \in P_w$.

(6) It follows from (4). □

For a T-module Y, we denote by Y_w the w-envelope of Y as an R-module and by Y_W the w-envelope of Y as a T-module.

Theorem 3.7 *Let $R \subseteq T$ be an extension of commutative rings and T a GV-torsion-free R-module. The following are equivalent.*

(1) $I_w \subseteq (IT)_W$ for any ideal I of R;

(2) $(I_wT)_W = (IT)_W$ *for any ideal* I *of* R*;*

(3) $(IT)_W \bigcap R$ *is a* w*-ideal of* R *for any ideal* I *of* R*;*

(4) $(IT)_W \bigcap R$ *is a* w*-ideal of* R *for any finitely generated ideal* I *of* R*;*

(5) T *is* w*-linked over* R*.*

Proof (1)⇒(2). Since $(I_wT)_W \subseteq ((IT)_W)_W = (IT)_W$, $I_w \subseteq (IT)_W$.

(2)⇒(3). Set $J = (IT)_W \bigcap R$. Then $J_w \subseteq (I_wT)_W \bigcap R = (IT)_W \bigcap R = J$, and hence $J = J_w$.

(3)⇒(4). Clearly.

(4)⇒(1). Let B be a finitely generated subideal of I. Since $B_w \subseteq (BT)_W \bigcap R$, $I_w = \bigcup B_w \subseteq (IT)_W$ by Lemma 2.9.

(1)⇒(5). Since $R = J_w \subseteq (JT)_W$ for any $J \in GV(R)$, $T = (JT)_W$ by Proposition 3.5. Thus $JT \in GV(T)$, and hence T is w-linked over R by Lemma 3.3.

(5)⇒(2). Clearly $(IT)_W \subseteq (I_wT)_W$. On the other hand, suppose $x = \sum_{i=1}^{n} a_it_i \in I_wT$ where $a_i \in I_w$ and $t_i \in T$. Thus $Jx \subseteq IT$ for some $J \in GV(R)$, and hence $JTx \subseteq IT$. By Lemma 3.3, $(I_wT)_W \subseteq (IT)_W$. □

Let T be w-linked over R and A an ideal of T. A is called a w_R-ideal (or w-linked ideal) if A as an R-module is a w-module.

Theorem 3.8 *Let* T *be* w*-linked over* R *and* P *a prime ideal of* T*. Then the following hold.*

(1) $P \bigcap R$ *is a* w*-ideal of* R *for any* w_R*-ideal* P *of* T*.*

(2) P *is a proper* w_R*-ideal of* T *if and only if* $P_w \neq T$*.*

(3) If $P \bigcap R$ *is a proper* w*-ideal of* R*, then* P *is a* w_R*-ideal of* T*.*

(4) Let A *be a* w_R*-ideal of* T *and* P *a minimal prime ideal over* A*. Then* P *is a* w_R*-ideal of* T*.*

(5) Let P *be a* w_R*-ideal of* T*, and* Q *a prime ideal of* T *such that* $Q \subseteq P$*. Then* Q *is a* w_R*-ideal of* T*.*

Proof (1) It follows from Lemma 2.8.

(2) If P is a w_R-ideal of T, then $P_w = P \subset T$. On the other hand, let $x \in T$, $Jx \subseteq P$ for some $J \in GV(R)$. If $J \subseteq P$, then $P_w = T$, a contradiction. Thus $J \not\subseteq P$, and hence $x \in P$.

(3) By Lemma 2.9, $P_w \neq T$.

(4) Let B be a finitely generated R-submodule of P. Since PT_P is a minimal prime ideal over A_P, there exists some $s \in T \backslash P$ such that $sB^n \subseteq A$ for some integer n. By Lemma 3.6, $s(B_w)^n \subseteq s((B_w)^n)_w \subseteq (sB^n)_w \subseteq A_w = A \subseteq P$. Therefore, $P_w = P$.

(5) Since $Q_w \subseteq P_w = P \neq T$, Q is a w_R-ideal of T. □

4. The generalization of Wang-McCsland-Park-Chang Theorem

Following Yin [1], R is called a w-Noetherian ring if R has the ascending chain condition on w-ideals, which contains Noetherian ring, strong Mori domain and so on. Note that each maximal w-ideal of R is a prime ideal, we use w-Max(R) to denote the set of all maximal w-ideals of R. Since a prime ideal P of R is a w-ideal if and only if $P_w \neq R$, each prime ideal contained in a proper w-ideal of R is also a w-ideal. Following Wang [11], w-dim$(R) = \mathrm{Sup}\{\mathrm{ht}P \mid P \in w\text{-Max}(R)\}$. Following Matijevic [3], $R^g = \{x \in T(R) \mid xM_1M_2\cdots M_n \subseteq R$ where $M_i \in \mathrm{Max}(R)\}$. Then R^g is a ring and $R^g \subseteq T(R)$. Following Park [5], $R^{wg} = \{x \in T(R) \mid xM_1M_2\cdots M_n \subseteq R$ where $M_i \in w\text{-Max}(R)\}$. Then R^{wg} is a ring and $R^{wg} \subseteq T(R)$.

Lemma 4.1 $\mathrm{Max}(R\{X\}) = \{M\{X\}\}$ *where M runs over all maximal w-ideals of R.*

Proof See [12, Proposition 2.1]. □

Lemma 4.2 *For any ring R, we have $R\{X\}^g \bigcap T(R) = R^{wg}$, and $R^{wg}[X]_{\mathcal{N}_w} \subseteq R\{X\}^g$.*

Proof Let $u \in T(R)$. Then $u \in R\{X\}^g$ if and only if $uM_1\cdots M_n \subseteq R\{X\}$ where $M_i \in \mathrm{Max}(R\{X\})$. By Lemma 4.1, there exists $\mathrm{m}_i \in w\text{-Max}(R)$ such that $M_1 = \mathrm{m}_i\{X\}$ for all i. Thus $u \in R\{X\}^g \bigcap T(R)$ if and only if $u\mathrm{m}_1\cdots \mathrm{m}_n \subseteq R$, if and only if $u \in R^{wg}$. Thus $R\{X\}^g \bigcap T(R) = R^{wg}$, and hence $R^{wg} \subseteq R\{X\}^g$. Since $R\{X\}^g$ is an $R\{X\} = R[X]_{\mathcal{N}_w}$-module, $R^{wg}[X]_{\mathcal{N}_w} \subseteq R\{X\}^g$. □

It is easy to see that, R is a w-Noetherian ring if and only if for each ideal I of R, $I_w = A_w$ for some finitely generated subideal A of I. Here we have

Proposition 4.3 *The following are equivalent for a ring R.*

(1) R is a w-Noetherian ring;

(2) $R[X]$ is a w-Noetherian ring;

(3) $R\{X\}$ is a Noetherian ring.

Proof (1)⇒(2). See [1].

(2)⇒(3). Let A be an ideal of $R\{X\}$. Then $A = B_{\mathcal{N}_w}$ for some ideal B of $R[X]$. Since $R[X]$ is a w-Noetherian ring, $B_w = C_w$ for some finitely generated subideal C of B. For any $f \in B$, $Jf \subseteq C$ for some $J \in GV(R[X])$. Note that $J_{\mathcal{N}_w} = R\{X\}$ and $fR\{X\} \subseteq C_{\mathcal{N}_w}$, we have $f \in C_{\mathcal{N}_w}$, and $B \subseteq C_{\mathcal{N}_w} \subseteq B_{\mathcal{N}_w}$. Therefore, $A = C_{\mathcal{N}_w}$ is a finitely generated ideal of $R\{X\}$, and hence $R\{X\}$ is a Noetherian ring.

(3)⇒(1). Let I be a ideal of R. Then $I\{X\}$ is a finitely generated ideal of $R\{X\}$, and $I\{X\} = A_{\mathcal{N}_w}$ for some finitely generated ideal A of $R[X]$. Since $A \subseteq c(A)[X]_{\mathcal{N}_w} \subseteq I\{X\}$, $I\{X\} = c(A)[X]_{\mathcal{N}_w}$. For any $u = \frac{\alpha}{g} \in I \subseteq I\{X\}$, where $g \in \mathcal{N}_w, \alpha \in c(A)[X]$, we have $uc(g) = c(ug) = c(\alpha) \subseteq c(A)$, and hence $u \in uc(g)_w \subseteq c(ug)_w = c(\alpha)_w \subseteq c(A)_w$. Thus $I_w = c(A)_w$, and hence R is a w-Noetherian ring. □

Lemma 4.4 *Let R be a w-Noetherian ring. Then* w-dim$(R) = \dim(R\{X\})$.

Proof Let P be a prime w-ideal of R. Then $P\{X\}$ is a prime ideal of $R\{X\}$, and hence w- $\dim(R) \leqslant \dim(R\{X\})$. Assume that Q be a maximal ideal of $R\{X\}$, then $Q = M\{X\}$ for some $M \in w\text{-Max}(R)$ by Lemma 4.1. Since $R[X]\backslash M[X] \supseteq \mathcal{N}_w$, $R\{X\}_{M\{X\}} = (R[X]_{\mathcal{N}_w})_{M[X]_{\mathcal{N}_w}} = R[X]_{M[X]} = R_M[X]_{MR_M[X]}$. Since R_M is a Noetherian ring, $\mathrm{ht}Q = \dim(R\{X\}_{M\{X\}}) =$

$\mathrm{ht}MR_M[X] = \mathrm{ht}MR_M = \mathrm{ht}M \leqslant w\text{-}\dim(R)$. □

Proposition 4.5 *Let R be a reduced w-Noetherian ring and $R \subseteq T \subseteq R^{wg}$. If T is a w-linked overring of R, then T is a w-Noetherian ring.*

Proof Note that T is also a reduced ring. By Proposition 4.3, $R\{X\}$ is a reduced Noetherian ring. Since T is w-linked over R, $\mathcal{N}_w \subseteq \mathcal{N}_w(T) = \{f \in T[X] \mid c(f) \in GV(T)\}$ by Lemma 3.3, and hence $T\{X\} = (T[X]_{\mathcal{N}_w})_{\mathcal{N}_{w(T)}}$. Since $R \subseteq T \subseteq R^{wg}$, $R\{X\} \subseteq T[X]_{\mathcal{N}_w} \subseteq (R\{X\})^g$ by Lemma 4.2. By [3, Corollary], $T[X]_{\mathcal{N}_w}$ is a Noetherian ring. Thus $T\{X\}$ is a Noetherian ring, and hence T is a w-Noetherian ring. □

Corollary 4.6 *Let R be a reduced w-Noetherian ring. Then R^{wg} is a w-Noetherian ring.*

Proof Let $x \in (R^{wg})_w \subseteq T(R)$. Then $Jx \subseteq R^{wg}$ for some $J \in GV(R)$. Since J is a finitely generated ideal of R, there exist $M_1, \ldots, M_n \in w\text{-Max}(R)$ such that $M_1 \cdots M_n Jx \subseteq R$, and hence $M_1 \cdots M_n x \subseteq R$, which implies that $x \in R^{wg}$, and R^{wg} is a w-liked overring of R. By Lemma 4.5, R^{wg} is a w-Noetherian ring. □

Lemma 4.7 *Let $R \subseteq T \subseteq T(R)$ be rings. If R is a Noetherian ring with* $\dim(R) \leqslant 1$, *then* $\dim(T) \leqslant 1$.

Proof Let P be a minimal prime of T. Then $\mathrm{p} = P \bigcap R$ is a prime of R, and hence R/p is a Noetherian domain with $\dim(R/\mathrm{p}) \leqslant \dim(R) \leqslant 1$. For any $x \in T$, there exists a non-zero-divisor s of R such that $sx \in R$, and $\bar{s}\bar{x} \in R/\mathrm{p}$. Since P is a minimal prime of T, $s \notin P$, and hence $s \notin \mathrm{p}$. Thus T/P is contained in the quotient field of R/p. If $\dim(R/\mathrm{p}) = 0$, then $\dim(T/P) = 0$ and so $\dim(T) = 0$. If $\dim(R/\mathrm{p}) = 1$, then T/P is a Noetherian domain and $\dim(T/P) \leqslant 1$ by the Krull-Akizuki Theorem. Therefore, $\dim(T) \leqslant 1$. □

Following Lucas [8], I is called a *semi-regular* ideal of R if it contains a finitely generated ideal A of R such that $\mathrm{ann}(A) = 0$. If every semi-regular ideal I contains a non-zero-divisor of R, then $T(R) = Q_0(R)$. When R is a w-Noetherian ring, then $T(R) = Q_0(R)$ by [2, Theorem 3.19]. Combining with [1], we have

Proposition 4.8 *Let R be a w-Noetherian ring with* $w\text{-}\dim(R) \leqslant 1$. *Then* $R^{wg} = Q_0(R)$.

Theorem 4.9 *Let R be a reduced ring. Then R is a w-Noetherian ring with* $w\text{-}\dim(R) \leqslant 1$ *if and only if each w-linked overring T of R is w-Noetherian ring with* $w\text{-}\dim(T) \leqslant 1$.

Proof Necessity. By Proposition 4.5 and Proposition 4.8, T is a w-Noetherian ring, and hence $T\{X\}$ is a Noetherian ring. By Proposition 4.3, $R\{X\}$ is a Noetherian ring. By Lemma 4.4, $\dim(R\{X\}) = w\text{-}\dim(R) \leqslant 1$. Since $T\{X\}$ is contained in the total quotient ring of $R\{X\}$, $w\text{-}\dim(T) = \dim(T\{X\}) \leqslant 1$ by Lemmas 4.4 and 4.7.

Sufficiency. Note that each T is also a reduced ring. Set $R = T$. □

Theorem 4.10 *R is a w-Noetherian ring with* $w\text{-}\dim(R) = 0$ *if and only if R is an Artinian* *ring.*

Proof Sufficiency is immediate, since R is an Artinian ring if and only if R is a Noetherian ring with $\dim(R) = 0$.

Necessity. Let u be a non-zero-divisor of R. Then $\mathrm{ht}P \geqslant 1$ for a prime ideal P of R minimal over (u), since each minimal prime ideal of R consists of zero-divisors. Repeating the way of Theorem 3.8(4), P is a w-ideal of R. Since w-$\dim(R) = 0$, $P = R$, and so u is a unit of R. Thus $GV(R) = \{R\}$ by [2, Corollary 3.20], and hence each ideal of R is a w-ideal. Therefore, $\dim(R) = 0$. For any ideal I of R, $I = I_w = B_w = B$ for some finitely generated subideal B of I, since R is a w-Noetherian ring. Therefore, R is a Noetherian ring. □

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