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A Class of Generalized Nilpotent Groups

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Abstract This paper considers such a group G which possesses nontrivial proper subgroups H_1, H_2 such that any proper subgroup of G not contained in $H_1 \cup H_2$ is p-closed and obtains that if G is soluble, then the number of prime divisors contained in $|G|$ is $2, 3$ or 4; if not, then it has a form $\langle x\rangle \ltimes N$ where $N/\Phi(N)$ is a non-abelian simple group. Then the structure of such a group is determined for $p = 2$, $H_1 = H_2$ under some conditions.

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1. Introduction

There has been a fair amount of interest for generalized nilpotent groups, such as, local nilpotent groups, Engel groups, nilpotent-by-(finite exponent) groups, etc. [1–3]. On the other hand, finite nilpotent groups possess nice properties. A well-known result is that finite groups are nilpotent if and only if all of their Sylow subgroups are normal.

To generalize nilpotency of finite groups, one considered p-closed groups and inner-p-closed groups (A finite group G is said to be p-closed, if its Sylow p-subgroup is normal. Particularly, if $p \nmid |G|$, G is p-closed. If G is non-p-closed but all of its proper subgroups are p-closed, then it is called an inner-p-closed group [4]). Chen showed that if a group G is inner-p-closed, then it is a q-basic group of order $p^\alpha q^\beta$ or $G/\Phi(G)$ is a non-abelian simple group [5, Theorem 4.1].

Li [6] further considered a generalized inner-p-closed group, i.e., a group G which has a proper subgroup H such that any proper subgroup of G not contained in H is p-closed. He proved that if such a group G is soluble, then the number of prime divisors contained in the order of G is 2 or 3; if G is not soluble, then it has a form $\langle x\rangle \ltimes N$, where $N/\Phi(N)$ is a non-abelian simple group [6, Theorem 1].

The aim of this paper is further to generalize the work of [6]. We introduce

Definition 1 *Let G be a non-p-closed group. G is called an almost p-closed group, if there exist*

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nontrivial proper subgroups H_1, H_2 of G such that for any proper subgroup R of G, if $R \nleq H_1$ and $R \nleq H_2$, then R is p-closed.

Remark 1 Observe that if $H_1 \leq H_2$ (resp. $H_2 \leq H_1$), or one of H_1, H_2 is p-closed, then G is the group of [6]. For convenience, let (G, H) denote the group of [6] for which H is non-p-closed and let (G, H_1, H_2) denote the almost p-closed group G for which none of H_1, H_2 is p-closed, $H_1 \nleq H_2$ and $H_2 \nleq H_1$.

This paper uses standard notations. $|G|$ denotes the order of a finite group G, $\pi(G)$ the set of prime divisors contained in $|G|$ and G_p a Sylow p-subgroup of G. By $H \lessdot G$ we mean that H is a maximal subgroup of G.

2. Properties

In this section we consider some properties of almost p-closed groups.

Proposition 1 *Let (G, H_1, H_2) be an almost p-closed group. Then H_1, H_2 are normal maximal subgroups of G.*

Proof It suffices to show that H_1 is a normal maximal subgroup of G.

If there exists a subgroup L such that $H_1 < L < G$, then $L \leq H_2$ since otherwise, L is p-closed, so is H_1, a contradiction. Thus $H_1 < L \leq H_2$, a contrary to $H_2 \nleq H_1$. Hence H_1 is a maximal subgroup of G.

Assume that there is $g \in G$ such that $H_1^g \neq H_1$. If $H_1^g \neq H_2$, then H_1^g is p-closed, and so is H_1, a contradiction. Thus $H_1^g = H_2$. This deduces that $|G : H_1| = |G : N_G(H_1)| = 2$ and it follows that H_1 is normal in G, a contrary to the assumption. Hence $H_1 \trianglelefteq G$. □

Remark 2 In (G, H), it is easy to know that $H \lessdot G$ and $H \trianglelefteq G$.

Proposition 2 *Let (G, H_1, H_2) be an almost p-closed group. If $N \trianglelefteq G$, $N \leq \Phi(G)$, then G/N is still almost p-closed.*

Proof H_1, H_2 are maximal subgroups of G by Proposition 1. Thus $N \leq H_1$ and $N \leq H_2$. Let $P \in \mathrm{Syl}_p G$. Then $PN/N \in \mathrm{Syl}_p(G/N)$. If G/N is p-closed, then $PN \trianglelefteq G$. By Frattini argument $G = PNN_G(P) = NN_G(P) = N_G(P)$, in contradiction to G non-p-closed. Hence G/N is non-p-closed. For any proper subgroup $\overline{R} = R/N$ of G/N, if $\overline{R} \nleq H_1/N$ and $\overline{R} \nleq H_2/N$, then $R \nleq H_1$ and $R \nleq H_2$, and thus R is p-closed, so is $\overline{R}$, as required. □

Proposition 3 *Let (G, H_1, H_2) be an almost p-closed group. Then $H_1 \cap H_2 \neq 1$.*

Proof By Proposition 1, H_1, H_2 are normal maximal subgroups of G. Set $|G : H_1| = r_1$, $|G : H_2| = r_2$, then r_1, r_2 are primes. If $H_1 \cap H_2 = 1$, then $G = H_1 \times H_2$. Thus $|G| = r_1 r_2$ and $p = r_1$ or $p = r_2$. This deduces that both of H_1 and H_2 are p-closed, a contradiction. Hence $H_1 \cap H_2 \neq 1$. □

Proposition 4 *A group G has a non-p-closed proper subgroup H such that G is almost p-closed iff there exists a minimal subgroup K in the set of all non-p-closed normal proper subgroups of G such that G/K is a cyclic group of order prime power and for any $L \lessdot G$ if only $K \nleq L$, then L is p-closed.*

Proof Suppose that (G, H) is almost p-closed. By Remark 2, H is a normal maximal subgroup of G. Suppose that K is a minimal one of non-p-closed normal subgroups of G contained in H. Set $|G : H| = r$ (a prime), and then choose x in $G_r \setminus H$. Thus $\langle x\rangle K \nleq H$ and it follows that $G = \langle x\rangle K$ because $\langle x\rangle K$ is non-p-closed, where $|x| = r^s$. Hence G/K is a cyclic group of order prime power. Thus H is the solely maximal subgroup of G containing K. For any $L \lessdot G$ if $K \nleq L$, then $L \neq H$ and thus L is p-closed.

Conversely, let K be a non-p-closed proper normal subgroup of G satisfying the assumption. Suppose that H is a maximal subgroup of G containing K. Since G/K is a cyclic group of order a power of a prime, then $H/K \trianglelefteq G/K$ and H is the solely maximal subgroup of G containing K. Suppose that $R < G$ such that $R \nleq H$ and $R \leq L \lessdot G$. If $K \leq L$, then $L = H$, in contradiction to $R \nleq H$. Thus $K \nleq L$. It follows that L is p-closed, so is R. Hence G is almost p-closed about H. □

Definition 2 *Let (G, H) be an almost p-closed group. The subgroup K of G in the sense of Proposition 4 is called a non-p-closed kernel of G.*

3. Results

Now we have the following result.

Theorem 1 *Let (G, H_1, H_2) be an almost p-closed group. Then*

(1) If G is soluble, then $2 \leq |\pi(G)| \leq 4$;

(2) If G is insoluble, then $G = \langle a\rangle \ltimes N$,

where $N/\Phi(N)$ is a non-abelian simple group.

Note that the statement is true for inner-p-closed groups and almost p-closed groups like (G, H).

Proof By Proposition 1, H_1, H_2 are normal maximal subgroups of G. Set $|G : H_1| = r_1$, $|G : H_2| = r_2$, then r_1, r_2 are primes. By Proposition 3, $H_1 \cap H_2 \neq 1$. Now we consider the situation:

I. $\Phi(G) = 1$.

Let N be the minimal normal subgroup of G contained in $H_1 \cap H_2$ and let M be a minimal supplement of N in G. Then by [7, p. 271, 618], $G = MN, M \cap N \leq \Phi(M)$. Notice that $M \nleq H_1$, $M \nleq H_2$ and $M \neq G$. Thus M is p-closed, i.e., $M_p \trianglelefteq M$. Thus $M_pN \trianglelefteq G$. If N is a p-group, then $M_pN \in \mathrm{Syl}_pG$, in contradiction to G non-p-closed. Hence N is not a p-group.

(1) If N is soluble, then $M \cap N = 1$ and N is a group of order q^α where $q \neq p$ is a prime. If M_pN is p-closed, then M_p char $M_pN \trianglelefteq G$, where $M_p \in \mathrm{Syl}_pG$, a contradiction. Hence M_pN is

non-p-closed. If one of r_1, r_2 equals p and without loss of generality suppose that $r_2 = p$, then $NM_pM_{r_1} \nleq H_1$ and $NM_pM_{r_1} \nleq H_2$. Since $NM_pM_{r_1}$ is non-p-closed, $G = NM_pM_{r_1}$ is soluble with $2 \leq |\pi(G)| \leq 3$. Thus suppose that $r_1 \neq p$ and $r_2 \neq p$. If $r_1 = r_2$, then similarly, we have $G = NM_pM_{r_1}$. Now suppose that $r_1 \neq r_2$, say $r_2 > r_1$. Then $M_{r_1} \leq H_2$, $M_{r_2} \leq H_1$ and $M_p \leq H_1 \cap H_2$. If $M_{r_2} \trianglelefteq G$, then $NM_pM_{r_1}M_{r_2}$ is a group. Observe that $NM_pM_{r_1}M_{r_2} \nleq H_1, H_2$. Hence $G = NM_pM_{r_1}M_{r_2}$ because NM_p is non-p-closed. Thus G is soluble with $3 \leq |\pi(G)| \leq 4$. Let $M_{r_2} \ntrianglelefteq G$. If $r_2 \neq q$, then M_{r_2} is a Sylow r_2-subgroup of G; if $r_2 = q$, then NM_{r_2} is a Sylow r_2-subgroup of G. For the later case, if $NM_{r_2} \trianglelefteq G$, then $G = NM_pM_{r_1}NM_{r_2} = G = NM_pM_{r_1}M_{r_2}$; if not, then set $G_{r_2} = M_{r_2}(q \neq r_2)$ or $G_{r_2} = NM_{r_2}(q = r_2)$. Therefore, $N_G(G_{r_2}) < G$ and $N_G(G_{r_2}) \neq H_1$ since $N_G(G_{r_2})$ is self-normalizing. If $N_G(G_{r_2}) < H_1$, then $1 + k_1r_2 = |G : N_G(G_{r_2})| = |G : H_1||H_1 : N_G(G_{r_2})|$. Observe that $N_{H_1}(G_{r_2}) = N_G(G_{r_2}) \cap H_1 = N_G(G_{r_2})$. Hence $|H_1 : N_G(G_{r_2})| = 1+k_2r_2$ and thus $r_1 = |G : H_1| = 1+(k_1-r_1k_2)r_2 > r_1$, a contradiction. Hence $N_G(G_{r_2}) \nleq H_1$.

If $NM_pN_G(G_{r_2}) < G$, since $N_G(G_{r_2}) \nleq H_2$, it follows that $NM_pN_G(G_{r_2})$ is p-closed, in contradiction to NM_p non-p-closed. Hence $NM_pN_G(G_{r_2}) = G$. Since G_{r_2} is a normal Sylow r_2-subgroup of $N_G(G_{r_2})$, there exists a minimal complement Q of G_{r_2} in $N_G(G_{r_2})$ such that $N_G(G_{r_2}) = G_{r_2}Q$, $G_{r_2} \cap Q = 1$. Thus $G = NM_pG_{r_2}Q = NM_pM_{r_2}Q$. Set $R = NM_pM_{r_2}$, then $G = RQ$. Note that Q normalizes G_{r_2} and $NM_p \trianglelefteq G$, thus Q normalizes R. Hence $G = N_G(R)$, that is $R \trianglelefteq G$. Since $RM_{r_1} \nleq H_1, H_2$, if $RM_{r_1} < G$, then RM_{r_1} is p-closed, so is NM_p, a contradiction. Hence $G = RM_{r_1} = NM_pM_{r_2}M_{r_1}$ is soluble and $3 \leq |\pi(G)| \leq 4$.

(2) N is insoluble.

If N is a p'-group, then by Frattini argument $G = NN_G(N_{p_i})$, where $p_i \in \pi(N)$. Thus some conjugate of M_p is contained in $N_G(N_{p_i})$, without loss of generality, assuming $M_p \leq N_G(N_{p_i})$. Note that $N_G(N_{p_i}) < G$ and $N_G(N_{p_i}) \nleq H_1, H_2$ since $N \leq H_1 \cap H_2$, thus $N_G(N_{p_i})$ is p-closed, i.e., $M_p \trianglelefteq N_G(N_{p_i})$. Set $U = \langle N_{p_i} \mid p_i \in \pi(N)\rangle$, then U normalizes M_p. Note that $|N_{p_i}|||U|$ and $U \leq N$, then $U = N$ and thus N normalizes M_p. Thus $M_p \trianglelefteq MN = G$ and $M_p \in Syl_pG$. By this contradiction $p \mid |N|$. But N is not a p-group, thus it is non-p-closed. Since $H_1 \nleq H_2, H_2 \nleq H_1$, there exists $x_1 \in H_1$ but $x_1 \notin H_2$, $x_2 \in H_2$ but $x_2 \notin H_1$. Thus $x = x_1x_2 \notin H_1, H_2$. Hence $G = \langle x\rangle N$ in view of N non-p-closed.

Suppose that $N = N_1 \times N_2 \times \cdots \times N_t$ where $N_i \cong N_j$ are non-abelian simple groups. Then $\langle x\rangle$ acts transitively on $N_1, N_2, \ldots, N_t$ by conjugates. Let $N_1^{x^i} = N_{i+1}$ $(i = 1, 2, \ldots, t-1)$, $N_t^x = N_1$. Set $D = \{\prod_{i=1}^t \alpha^{x^i} \mid \alpha \in N_1\}$, then D is a subgroup of N and $D^x = D$, $\langle x, D\rangle = \langle x\rangle D$. Consider $\psi : \alpha \to \prod_{i=1}^t \alpha^{x^i}$, $\alpha \in N_1$, then ψ is an automorphism from N_1 into D. Hence D is not p-closed. If $t > 1$, then $G \neq \langle x\rangle D$. However, $\langle x\rangle D \nleq H_1$, $\langle x\rangle D \nleq H_2$, it deduces that $\langle x\rangle D$ is p-closed, in contradiction to D non-p-closed. Hence $t = 1$, N is simple.

II. $\Phi(G) \neq 1$.

Set $\overline{G} = G/\Phi(G)$, then by Proposition 2, $\overline{G}$ is also almost p-closed. By induction on the order of groups we have:

(i) If $\overline{G}$ is soluble, then $2 \leq |\pi(\overline{G})| \leq 4$, and thus $2 \leq |\pi(G)| \leq 4$ since $\pi(G) = \pi(\overline{G})$;

(ii) If $\overline{G}$ is not soluble, then $\overline{G} = \langle \overline{x}\rangle \ltimes \overline{N}$, where $\overline{N} = N/\Phi(G)$ is a non-abelian simple group and $\overline{x} = x\Phi(G)$. Hence $G = \langle x\rangle \ltimes N, \Phi(G) = \Phi(N)$. This completes the proof. $\square$

To deduce the structures of almost 2-closed groups, we recall the following result due to Chen.

Lemma 1 ([5, Theorem 4.5]) *An inner-2-closed group G is a dihedral group: $\langle a, b | a^{2^\alpha} = b^q = 1, b^a = b^{-1}\rangle$, where q is an odd prime.*

Proof In the first place, we show that G is soluble.

If there exists an element t of order 2 in G such that for each g in G, $\langle t, t^g\rangle$ is a 2-group, then by [8, Theorem 5.1], $Q_2(G) \neq 1$. Since G is non-2-closed, $Q_2(G)$ is not a Sylow 2-subgroup of G. Hence $G/Q_2(G)$ is inner-2-closed and thus it is soluble on induction. Therefore G is soluble. If there exists an element t of order 2 in G such that for some $g \in G$, $\langle t, t^g\rangle$ is not a 2-group, then $\langle t, t^g\rangle$ is not 2-closed following that $G = \langle t, t^g\rangle$ is also soluble.

Thus G is a q-basic group of order $2^\alpha q^\beta$. Note that $q \equiv 1 \pmod 2$, thus the Sylow q-subgroup Q of G is a cyclic group of order q. The result follows. $\square$

Theorem 2 *Suppose that (G, H) is an almost 2-closed group and G has an inner-2-closed kernel K. Then G is a supersoluble group generated by two elements and is as follows:* $(q \neq 2)$

I. If $|G : H| = r \neq 2$, then

1) $G = \langle x\rangle\langle a\rangle,\ x^{r^\beta} = a^{2^\alpha} = 1$;

2) $G = \langle xa\rangle \times \langle b\rangle, x^{r^\beta} = a^{2^\alpha} = b^q = 1,\ a^{-1}ba = b^{-1},\ a^x = a,\ b^x = b^k,\ 2 \le k \le q-1.$

II. If $r = 2$, then

3) $G = \langle x\rangle \times \langle b\rangle,\ x^{r^\beta} = b^q = 1,\ b^x = b^k,\ 2 \le k \le q-1$;

4) $G = \langle a\rangle\langle xb\rangle, x^{r^\beta} = a^{2^\alpha} = b^q = 1,\ a^{-1}ba = b^{-1},\ b^x = b,\ a^x = a^i,\ 1 \le i \le 2^\alpha - 1.$

Proof Suppose that K is an inner 2-closed kernel of G. By the lemma above

$$K = \langle a, b\rangle = \langle a\rangle \times \langle b\rangle$$

such that $a^{2^\alpha} = b^q = 1$, $q \neq 2$ and $a^{-1}ba = b^{-1}$. By $|G : H| = r$, $G_r \nleq H$. Choose $x \in G_r \setminus H$, then $\langle x\rangle K \nleq H$ and $\langle x\rangle K$ is non-2-closed. Thus

$$G = \langle x\rangle K = \langle x\rangle\langle a\rangle\langle b\rangle,\ |x| = r^s.$$

Note that $\langle b\rangle \operatorname{char} K \trianglelefteq G$, thus $\langle b\rangle \trianglelefteq G$. Since $\langle a\rangle \in \mathrm{Syl}_2 K$, by Frattini argument $G = KN_G(\langle a\rangle) = \langle b\rangle N_G(\langle a\rangle)$.

I. $r \neq 2$.

1) If $\langle b\rangle \le \langle x\rangle$, then $r = q$ and $G = \langle x\rangle\langle a\rangle\langle b\rangle = \langle x\rangle\langle a\rangle$ satisfying $x^{r^\beta} = a^{2^\alpha} = 1$.

2) If $\langle b\rangle \nleq \langle x\rangle$, then $\langle b\rangle \cap \langle x\rangle = 1$. Thus some conjugate of $\langle x\rangle$ is contained in $N_G(\langle a\rangle)$, and without loss of generality suppose that $\langle x\rangle \le N_G(\langle a\rangle)$. Set $a^x = a^j$, then $a^{x^n} = a^{j^n}$, where n is a natural number. Write $t = r^s$, then t is odd. Note that

$$(xa)^t = x^t a^{x^{t-1}} \cdots a^{x^2} a^x a = a^{j^{t-1}} \cdots a^{j^2} a^j a = a^{j^{t-1} + \cdots + j^2 + j + 1}.$$

All j^k have the same odd-even, where $k = 1, 2, \ldots, t-1$ and $t-1$ is even. So $2 \mid (j^{t-1} + \cdots + j^2 + j)$. It follows that $j^{t-1} + \cdots + j^2 + j + 1$ is prime to $|a|$. Thus $\langle a\rangle = \langle a^{j^{t-1} + \cdots + j^2 + j + 1}\rangle = \langle (xa)^t\rangle \le \langle xa\rangle$. It is deduced that $K = \langle a\rangle\langle b\rangle \le \langle xa\rangle\langle b\rangle = \langle xa\rangle \times \langle b\rangle$ and thus $\langle xa\rangle\langle b\rangle$ is non-p-closed. Note that $a \in H$, but $x \notin H$, then $xa \notin H$ and thus $\langle xa\rangle\langle b\rangle \notin H$. It is deduced that $G = \langle xa\rangle \times \langle b\rangle$, $\langle xa\rangle = \langle x\rangle \times \langle a\rangle$. Thus $x^{r^\beta} = a^{2^\alpha} = b^q = 1$, $a^{-1}ba = b^{-1}$, $a^x = a$, $b^x = b^k$, $2 \le k \le q-1$.

II. $r = 2$.

3) If $\langle x\rangle\langle b\rangle$ is non-2-closed, then $G = \langle x\rangle\langle b\rangle = \langle x\rangle \times \langle b\rangle$ satisfying $x^{r^\beta} = b^q = 1$, $b^x = b^k$, where $2 \le k \le q-1$.

4) If $\langle x\rangle\langle b\rangle$ is 2-closed, then $\langle x\rangle\langle b\rangle = \langle x\rangle \times \langle b\rangle = \langle xb\rangle$. Thus $G = \langle a\rangle\langle xb\rangle$ satisfying $x^{r^\beta} = a^{2^\alpha} = b^q = 1$, $a^{-1}ba = b^{-1}$, $b^x = b$, $a^x = a^i$, $1 \le i \le 2^\alpha - 1$. $\square$

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References

[1] BURNS R G, MEDVEDEV Y. *A note on Engel groups and local nilpotence* [J]. J. Austral. Math. Soc. Ser. A, 2003, **74**(1): 295–312.
[2] TRAUSTASON G. *Milnor groups and (virtual) nilpotence* [J]. J. Group Theory, 2005, **8**(2): 203–221.
[3] LI Qianlu. *Words and almost nilpotent varieties of groups* [J]. Comm. Algebra, 2005, **33**(10): 3569–3582.
[4] CHEN Zhongmu. *Inner-p-closed groups* [J]. Adv. in Math. (Beijing), 1986, **15**(4): 385–388. (in Chinese)
[5] CHEN Zhongmu. *Inner and Outer $\sum$-Groups and Minimal Non-$\sum$-Groups* [M]. West-South Normal University Press, Chongqing, 1988. (in Chinese)
[6] LI Xianhua. *Inner p-closed groups and their generalization* [J]. J. Math. Res. Exposition, 1994, **14**(2): 285–288. (in Chinese)
[7] ROSE J S. *A Course on Group Theory* [M]. Cambridge University Press, Cambridge-New York-Melbourne, 1978.
[8] KURZWEIL H. *Endliche Grupprn* [M]. Springe-Verlag, New York/Heidelberg/Berlin, 1977.