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A Nontrivial Product in the May Spectral Sequence

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Abstract In this paper, we prove the non-triviality of the product $h_0k_o\tilde{\delta}_{s+4}\in \mathrm{Ext}_A^{s+6,t(s)}(Z_p,Z_p)$ in the classical Adams spectral sequence, where $p\geq 11, 0\leq s<p-4, t(s)=(s+4)p^3q+(s+3)p^2q+(s+4)pq+(s+3)q+s$ with $q=2(p-1)$. The elementary method of proof is by explicit combinatorial analysis of the (modified) May spectral sequence.

Keywords stable homotopy groups of spheres; Adams spectral sequence; May spectral sequence.

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1. Introduction and statement of results

Let A be the mod p Steenrod algebra and S be the sphere spectrum localized at the prime number p. To determine the stable homotopy groups of spheres $\pi_*(S)$ is one of the central problems in homotopy theory. The Adams spectral sequence $E_2^{s,t}=\mathrm{Ext}_A^{s,t}(Z_p,Z_p)\Longrightarrow \pi_{t-s}(S)$ has been an invaluable tool in studying the stable homotopy groups of spheres, where Z denotes the integral and the $E_2^{s,t}$-term is the cohomology of A.

If a family of homotopy generators x_i in $E_2^{s,*}$ converges nontrivially to the Adams spectral sequence, then we get a family of homotopy elements f_i in $\pi_*(S)$ and we say that f_i is represented by $x_i\in E_2^{s,*}$ and has filtration s in the ASS. So far, not so many families of homotopy elements in $\pi_*(S)$ have been detected. Recently, Lin got a series of results and detected some new families in $\pi_*(S)$ (see [1–4]).

Throughout this paper, p denotes an odd prime and $q=2(p-1)$.

The known results on $\mathrm{Ext}_A^{*,*}(Z_p,Z_p)$ are as follows. $\mathrm{Ext}_A^{0,*}(Z_p,Z_p)=Z_p$ by its definition from [5]. $\mathrm{Ext}_A^{1,*}(Z_p,Z_p)$ has Z_p-basis consisting of $a_0\in \mathrm{Ext}_A^{1,1}(Z_p,Z_p)$. $h_i\in \mathrm{Ext}_A^{1,p^iq}(Z_p,Z_p)$ for all $i\geq 0$ and $\mathrm{Ext}_A^{2,*}(Z_p,Z_p)$ has Z_p-basis consisting of α_2, a_0^2, a_0h_i $(i>0)$, g_i $(i\geqslant 0)$, k_i $(i\geqslant 0)$, b_i $(i\geqslant 0)$, and h_ih_j $(j\geqslant i+2, i\geqslant 0)$ whose internal degrees are $2q+1$, 2, p^iq+1, $q(p^{i+1}+2p^i)$, $q(2p^{i+1}+p^i)$, $p^{i+1}q$ and $q(p^i+p^j)$, respectively. In 1980, Aikawa [6] determined $\mathrm{Ext}_A^{3,*}(Z_p,Z_p)$ by λ-algebra.

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Let M denote the Moore spectrum modulo the prime p given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S.$$

Let $\alpha : \Sigma^q M \to M$ be the Adams map and $V(1)$ be its cofibre given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} V(1) \xrightarrow{j'} \Sigma^{q+1} M. \tag{2}$$

Let $\beta : \Sigma^{(p+1)q} V(1) \to V(1)$ be the v_2-map.

In 1998, Wang and Zheng [7] proved the following theorem.

Theorem 1.1 ([7]) *For $p \geq 11$ and $4 \leq s < p$, there exists the fourth Greek letter family element $\tilde{\delta}_s \neq 0 \in \mathrm{Ext}_A^{s,t_1(s)}(Z_p, Z_p)$, where $t_1(s) = q[sp^3 + (s-1)p^2 + (s-2)p + (s-3)] + (s-4)$.*

Note that we write $\tilde{\delta}_{s+4}$ for $\tilde{\alpha}_s^{(4)}$ which is described in [7].

In this note, our main result can be stated as follows.

Theorem 1.2 *For $p \geq 11$, $0 \leq s < p-4$, then the product*

$$h_0 k_0 \tilde{\delta}_{s+4} \neq 0 \in \mathrm{Ext}_A^{s+7,q[(s+4)p^3+(s+3)p^2+(s+4)p+(s+3)]+s,*}(Z_p, Z_p).$$

The method of proof is by explicit combinatorial analysis of the MSS.

The paper is arranged as follows. After recalling some knowledge on the MSS in Section 2, we give the proof of Theorem 1.2 in Section 3.

2. The May spectral sequence

From [8], there is a May Spectral Sequence (MSS) $\{E_r^{s,t,*}, d_r\}$ which converges to $\mathrm{Ext}_A^{s,t}(Z_p, Z_p)$ with E_1-term

$$E_1^{*,*,*} = E(h_{m,i} | m > 0, i \geqslant 0) \otimes P(b_{m,i} | m > 0, i \geqslant 0) \otimes P(a_n | n \geqslant 0), \tag{3}$$

where E is the exterior algebra, P is the polynomial algebra, and

$$h_{m,i} \in E_1^{1,2(p^m-1)p^i,2m-1}, b_{m,i} \in E_1^{2,2(p^m-1)p^{i+1},p(2m-1)}, a_n \in E_1^{1,2p^n-1,2n+1}.$$

The r-th May differential is

$$d_r : E_r^{s,t,u} \to E_r^{s+1,t,u-r}, \tag{4}$$

and if $x \in E_r^{s,t,*}$ and $y \in E_r^{s',t',*}$, then

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^s x \cdot d_r(y). \tag{5}$$

There exists a graded commutativity in the E_1-term of MSS:

$$x \cdot y = (-1)^{(s+t)(s'+t')} y \cdot x$$

for x, $y = h_{m,i}$, $b_{m,i}$ or a_n. The first May differential d_1 is given by

$$\begin{cases} d_1(h_{i,j}) = \sum\limits_{0<k<i} h_{i-k,k+j} h_{k,j}, \\ d_1(a_i) = \sum\limits_{0 \leqslant k<i} h_{i-k,k} a_k, \\ d_1(b_{i,j}) = 0. \end{cases} \tag{6}$$

For each element $x \in E_1^{s,t,\mu}$, we define $\dim x = s$, $\deg x = t$. Then we have that

$$\begin{cases} \dim h_{i,j} = \dim a_i = 1, \dim b_{i,j} = 2, \\ \deg h_{i,j} = 2(p^i - 1)p^j = q(p^{i+j-1} + \cdots + p^j), \\ \deg b_{i,j} = 2(p^i - 1)p^{j+1} = q(p^{i+j} + \cdots + p^{j+1}), \\ \deg a_i = 2p^i - 1 = q(p^{i-1} + \cdots + 1) + 1, \\ \deg a_0 = 1, \end{cases} \tag{7}$$

where $i \geqslant 1$, $j \geqslant 0$.

Note that by the knowledge on p-adic expression in number theory, we have that each integer $t \geqslant 0$ can be always expressed uniquely as

$$t = q(c_n p^n + c_{n-1} p^{n-1} + \cdots + c_1 p + c_0) + c_{-1},$$

where $0 \leqslant c_i < p$ $(0 \leqslant i < n)$, $0 < c_n < p$, $0 \leqslant c_{-1} < q$.

3. Proof of Theorem 1.2

Lemma 3.1 ([9]) *Let $p \geq 11$ and $0 \leq s < p-4$. Then $\tilde{\delta}_{s+4}$ is represented by $a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3}$ in the May spectral sequence.*

Lemma 3.2 *Let $p \geq 11$ and $0 \leq s < p-4$. Then the May E_1-term*

$$E_1^{s+6,(s+4)p^3q+(s+3)p^2q+(s+4)pq+(s+3)q+s,*} = 0.$$

Proof Consider $h = x_1 x_2 \cdots x_m \in E_1^{s+6,t,*}$ in the MSS, where x_i is one of $a_k, h_{l,j}$ or $b_{u,z}$, $0 \leq k \leq 4, 0 \leq l+j \leq 4, 0 \leq u+z \leq 4, l > 0, j \geq 0, u > 0, z \geq 0$. By (7) we can assume that $\deg x_i = q(c_{i,3}p^3 + c_{i,2}p^2 + c_{i,1}p + c_{i,0}) + e_i$, where $c_{i,j} = 0$ or 1 if $x_i = a_{k_i}$ or $e_i = 0$.

$$\dim h = \sum_{i=1}^{m} \deg x_i = s + 6$$

and

$$\begin{aligned} \deg \ &= \sum_{i=1}^{m} \deg x_i = q[(\sum_{i=1}^{m} c_{i,3})p^3 + (\sum_{i=1}^{m} c_{i,2})p^2 + (\sum_{i=1}^{m} c_{i,1})p + (\sum_{i=1}^{m} c_{i,0})] + \sum_{i=1}^{m} e_i \\ &= q[(s+4)p^3 + (s+3)p^2 + (s+4)p + (s+3)] + s. \end{aligned} \tag{8}$$

By virtue of $0 \leq s, s+3, s+4 < p$ and the knowledge on the p-adic expression in number theory, we have from (8)

$$\begin{cases} \sum\limits_{i=1}^{m} e_i = s + \lambda_{-1} q, & \lambda_{-1} \geqslant 0, \\ \sum\limits_{i=1}^{m} c_{i,0} + \lambda_{-1} = s + 3 + \lambda_0 p, & \lambda_0 \geqslant 0, \\ \sum\limits_{i=1}^{m} c_{i,1} + \lambda_0 = s + 4 + \lambda_1 p, & \lambda_1 \geqslant 0, \\ \sum\limits_{i=1}^{m} c_{i,2} + \lambda_1 = s + 3 + \lambda_2 p, & \lambda_2 \geqslant 0, \\ \sum\limits_{i=1}^{m} c_{i,3} + \lambda_2 = s + 4 + \lambda_3 p, & \lambda_3 \geqslant 0. \end{cases} \tag{9}$$

By the facts that $\dim h_{i,j} = \dim a_i = 1$ and $\dim b_{i,j} = 2$, we can have

$$6 \le m \le s+6 < p-4+6 = p+2.$$

From $\dim h = \sum_{i=1}^{m} \dim x_i = s+6$. Notice that $e_i = 0$ or 1, $c_{i,j} = 0$ or 1, and $m < p+2$. From (9), we have

$$0 \le \sum_{i=1}^{m} e_i,\ \ \sum_{i=1}^{m} c_{i,j} \le m < p+2.$$

It follows that the number sequence $(\lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \lambda_3)$ must be equal to the sequence $(0,0,0,0,0)$. Then (9) can turn into

$$\begin{cases} \sum\limits_{i=1}^{m} e_i = s, \\ \sum\limits_{i=1}^{m} c_{i,0} = s+3, \\ \sum\limits_{i=1}^{m} c_{i,1} = s+4, \\ \sum\limits_{i=1}^{m} c_{i,2} = s+3, \\ \sum\limits_{i=1}^{m} c_{i,3} = s+4. \end{cases} \tag{10}$$

By $c_{i,3} = 0$ or 1, we can get $m \ge s+4$ from $\sum_{i=1}^{m} c_{i,1} = s+4$. We also have $m \le s+6$.

Thus m can be equal to $s+4, s+5$ or $s+6$.

Since $\sum_{i=1}^{m} e_i = s$, $\deg h_{i,j} \equiv 0 (\mathrm{mod}\, q)$, $i > 0$, $j \ge 0$. Then we can assume that $h = a_0^x a_1^y a_2^z a_3^k a_4^l h'$ with $h' = x_{s+1}x_{s+2}x_{s+3}\cdots x_m \in E_1^{6,t',*}$, where $0 \le x, y, z, k, l \le s, x+y+z+k+l = s$.

From (10) we can get

$$\begin{cases} \sum\limits_{i=s+1}^{m} e_i = 0, \\ \sum\limits_{i=s+1}^{m} c_{i,0} = s+3-y-z-k-l, \\ \sum\limits_{i=s+1}^{m} c_{i,1} = s+4-z-k-l, \\ \sum\limits_{i=s+1}^{m} c_{i,2} = s+3-k-l, \\ \sum\limits_{i=s+1}^{m} c_{i,3} = s+4-l, \end{cases} \tag{11}$$

where $t' = (s+4-l)p^3q + (s+3-k-l)p^2q + (s+4-z-k-l)pq + (s+3-y-z-k-l)q$.

Case 1 $m = s+4$.

(11) can turn into

$$\begin{cases} \sum\limits_{i=s+1}^{s+4} e_i = s, \\ \sum\limits_{i=s+1}^{s+4} c_{i,0} = s+3-y-z-k-l, \\ \sum\limits_{i=s+1}^{s+4} c_{i,1} = s+4-z-k-l, \\ \sum\limits_{i=s+1}^{s+4} c_{i,2} = s+3-k-l, \\ \sum\limits_{i=s+1}^{s+4} c_{i,3} = s+4-l. \end{cases} \tag{12}$$

Since $\sum_{i=s+1}^{s+4} c_{i,3} = s+4-l$ in (11), we have that $l = s+4-\sum_{i=s+1}^{s+4} c_{i,3} \geq s+4-4 = s$. Note that $0 \leq l \leq s$. Thus $l = s$ and $x = y = z = k = 0$. By (12), $h' = x_{s+1}x_{s+2}x_{s+3}x_{s+4} \in E_1^{6,4p^3+3p^2+4pq+3q,*} = 0$. Thus in this case it is impossible for h to exist.

Case 2 $m = s+5$.

(11) can turn into

$$\begin{cases} \sum\limits_{i=s+1}^{s+5} e_i = s, \\ \sum\limits_{i=s+1}^{s+5} c_{i,0} = s+3-y-z-k-l, \\ \sum\limits_{i=s+1}^{s+5} c_{i,1} = s+4-z-k-l, \\ \sum\limits_{i=s+1}^{s+5} c_{i,2} = s+3-k-l, \\ \sum\limits_{i=s+1}^{s+5} c_{i,3} = s+4-l. \end{cases} \tag{13}$$

Similarly, from $\sum_{i=s+1}^{s+5} c_{i,3} = s+4-l$, we have that $l = s+4-\sum_{i=s+1}^{s+5} c_{i,3} \geq s-1$. Thus there are five possibilities satisfying $0 \leq x, y, z, k, l \leq s$, and $x+y+z+k+l = s$. We list all the possibilities in the following table.

The possiblility	l	k	z	y	x	$E_1^{6,t',\star}$	$h' = x_{s+1}x_{s+2}x_{s+3}x_{s+4}x_{s+5}$
The 1st	$s-1$	1	0	0	0	$E_1^{6,q(5p^3+3p^2+4p+3),*} = 0$	Nonexistence
The 2nd	$s-1$	0	1	0	0	$E_1^{6,q(5p^3+4p^2+4p+3),*} = 0$	Nonexistence
The 3rd	$s-1$	0	0	1	0	$E_1^{6,q(5p^3+4p^2+5p+3),*} = 0$	Nonexistence
The 4th	$s-1$	0	0	0	1	$E_1^{6,q(5p^3+4p^2+5p+4),*} = 0$	Nonexistence
The 5th	s	0	0	0	0	$E_1^{6,q(4p^3+3p^2+4p+3),*} = 0$	Nonexistence

Table 1 $m = s+5$ h' all the possible

From the above table, it follows that in this case h cannot exist either.

Case 3 $m = s+6$.

As in Case 2, one has $l = s+6-\sum_{i=s+1}^{s+6} c_{i,3} \geq s-2$ from $\sum_{i=s+1}^{s+6} c_{i,3} = s+4-l$ in (11). Thus $x+y+z+k+l=s$. We list all the possibilities in the following table.

The possiblility	l	k	z	y	x	$E_1^{6,t',\star}$	$h' = x_{s+1}x_{s+2}x_{s+3}x_{s+4}x_{s+5}$
The 1st	$s-2$	2	0	0	0	$E_1^{6,q(6p^3+3p^2+4p+3),*}=0$	Nonexistence
The 2nd	$s-2$	0	2	0	0	$E_1^{6,q(6p^3+5p^2+4p+3),*}=0$	Nonexistence
The 3rd	$s-2$	0	0	2	0	$E_1^{6,q(6p^3+5p^2+6p+3),*}=0$	Nonexistence
The 4th	$s-2$	0	0	0	2	$E_1^{6,q(6p^3+5p^2+6p+5),*}=0$	Nonexistence
The 5th	$s-2$	1	1	0	0	$E_1^{6,q(6p^3+4p^2+4p+3),*}=0$	Nonexistence
The 6th	$s-2$	1	0	1	0	$E_1^{6,q(6p^3+4p^2+5p+3),*}=0$	Nonexistence
The 7th	$s-2$	1	0	0	1	$E_1^{6,q(6p^3+4p^2+5p+4),*}=0$	Nonexistence
The 8th	$s-2$	0	1	1	0	$E_1^{6,q(6p^3+5p^2+5p+3),*}=0$	Nonexistence
The 9th	$s-2$	0	1	0	1	$E_1^{6,q(6p^3+5p^2+5p+4),*}=0$	Nonexistence
The 10th	$s-2$	0	0	1	1	$E_1^{6,q(6p^3+5p^2+6p+4),*}=0$	Nonexistence
The 11th	$s-1$	1	0	0	0	$E_1^{6,q(5p^3+3p^2+4p+3),*}=0$	Nonexistence
The 12th	$s-1$	0	1	0	0	$E_1^{6,q(5p^3+4p^2+4p+3),*}=0$	Nonexistence
The 13th	$s-1$	0	0	1	0	$E_1^{6,q(5p^3+4p^2+5p+3),*}=0$	Nonexistence
The 14th	$s-1$	0	0	0	1	$E_1^{6,q(5p^3+4p^2+5p+4),*}=0$	Nonexistence
The 15th	s	0	0	0	0	$E_1^{6,q(4p^3+3p^2+4p+3),*}=0$	Nonexistence

Table 2 $m = s+6$ h' all the possible

From the above table, it follows that in this case h cannot exist either.

From Cases 1 and 2, the lemma follows. □

Now we give the proof of Theorem 1.2.

Proof of Theorem 1.2 Since $h_{2,0}h_{1,1}$, $h_{1,0}$ and $a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_1^{*,*,*}$ are permanent cycles in the MSS and converge nontrivially to $k_0, h_0, \tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{*,*}(Z_p, Z_p)$ for $n \geq 0$, we have

$$h_{1,0}h_{2,0}h_{1,1}a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_1^{q[(s+4)p^3+(s+3)p^2+(s+4)p+(s+3)]+s,*}$$

is an infinite cycle in the MSS and converges to $h_0 k_0 \tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+7,*}(Z_p, Z_p)$.

From Lemma 3.2, we see that

$$E_1^{s+6,q[(s+4)p^3+(s+3)p^2+(s+4)p+(s+3)]+s,*} = 0,$$

then for $r \geq 1$,

$$E_r^{s+6,q[(s+4)p^3+(s+3)p^2+(s+4)p+(s+3)]+s,*} = 0.$$

Thus the infinite cycle $h_{1,0}h_{2,0}h_{1,1}a_3^s h_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_r^{s+7,*,*}$ is not bounded. That is to say,

$$h_{1,0}h_{2,0}h_{1,1}a_3^s h_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_r^{s+7,*,*}$$

cannot be hit by any differential in the MSS. It follows that $h_{1,0}h_{2,0}h_{1,1}a_3^s h_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_r^{s+7,*,*}$ is an infinite cycle in the May spectral sequence and converges nontrivially to $h_0k_0\tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+7,*}(Z_p, Z_p)$. It follows that

$$h_0k_0\tilde{\delta}_{s+4} \neq 0 \in \mathrm{Ext}_A^{s+7,q[(s+4)p^3+(s+3)p^2+(s+4)p+(s+3)]+s,*}(Z_p, Z_p).$$

Theorem 1.2 is proved. □

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