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The Closeness of the τ-Standard Part of a Set

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Abstract In this paper, the closeness of the τ-standard part of a set is discussed. Some related propositions of the τ-neighborhood system of a set are given. And then some related conclusions of the τ-monad of a set and the τ-standard part of a set are presented. And based on it, the necessary and sufficient conditions of the enlarged model and the saturated model are showed. Finally, some sufficient conditions that the τ-standard part of a set is closed are proved in the enlarged model and the saturated model.

Keywords enlarged models; saturated models; τ-standard part; τ-closed set.

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1. Introduction and preliminaries

Nonstandard analysis is the mathematical theory which studies all kinds of mathematical problems by nonstandad models. Aside from theorems that tell us that nonstandard notions are equivalent to corresponding standard notions, all the results we obtain can be proved by standard methods. Therefore, nonstandard analysis can only be claimed to be of importance insofar as it leads to simpler, more accessible expositions, or to mathematical discoveries. Usually, we require these nonstandard models with better properties. There are two kinds of important models which are nonstandard enlarged models and saturated models in nonstandard analysis. We discuss many problems in nonstandard enlarged models. For example, Abraham Robinson proved many theorems in nonstandard enlarged models in [1]. But only the enlarged models are not enough, the saturated models are also important. For example, infinitesimal prolongation theorem was extended from sequences to nets based on κ-saturated model in [2]. The structure of ${}^*\tau_x$ and its properties were discussed in κ-saturated model in [3]. The extension of Robinson's sequential lemma and its applications were presented in κ-saturated model in [4]. The necessary and sufficient conditions and some applications of nonstandard enlarged models and saturated models were showed in [5] and [6]. In this paper, the closeness of the τ-standard part of a set will be discussed in the two kinds of models.

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Let S be a set of individuals. And assume that $\mathbf{V}(S)$ and $\mathbf{V}(^*S)$ are universes with individuals S and $^*S$ respectively, and $\mathbf{V}(^*S)$ is a nonstandard model of $\mathbf{V}(S)$.

Definition 1 *Let S be a set of individuals and let $\mathbf{V}(S)$ be a superstructure with individuals S. Then a relation r is called concurrent in $\mathbf{V}(S)$ if $r \in \mathbf{V}(S)$ and if whenever $x_1, x_2, \ldots, x_n \in \operatorname{dom}(r)$, there is an element $y \in \operatorname{ran}(r)$ such that $\langle x_i, y\rangle \in r$ for $i = 1, 2, \ldots, n$.*

Definition 2 *Let S be a set of individuals and let $\mathbf{V}(S)$ be a superstructure with individuals S, and let $\mathbf{V}(^*S)$ be a nonstandard model of $\mathbf{V}(S)$. Then $\mathbf{V}(^*S)$ is called a nonstandard enlargement of $\mathbf{V}(S)$ whenever for every concurrent relation r of $\mathbf{V}(S)$, there exists an element $y \in \mathbf{V}(^*S)$ such that $\langle ^*x, y\rangle \in {}^*r$ for all $x \in \operatorname{dom}(r)$.*

Definition 3 *Let κ be an infinite cardinal. A nonstandard model $\mathbf{V}(^*S)$ of $\mathbf{V}(S)$ is called κ-saturated whenever for every internal binary relation r in $\mathbf{V}(^*S)$, it is concurrent on a subset A of $\operatorname{dom}(r)$, that is, for every finite set $x_1, x_2, \ldots, x_n \in A$ there exists an element $y \in \operatorname{ran}(y)$ such that $\langle x, y\rangle \in r$ and if $\operatorname{card}(A) < \kappa$, then there exists an element y in $\mathbf{V}(^*S)$ in the range of r such that for all $x \in A$, $\langle x, y\rangle \in r$.*

Let (X, τ) be a topological space and let X be contained in the set of individuals of standard universe $\mathbf{V}(S)$, and let $\mathbf{V}(^*S)$ be an enlargement of $\mathbf{V}(S)$. Then we have $X \subseteq {}^*X$. Suppose $A \subseteq {}^*X$, then $\mathcal{N}_\tau(A) = \{H \subseteq X \mid \exists G \in \tau, A \subseteq {}^*G \subseteq {}^*H\}$ is called the τ-neighborhood system of A. In particular, $\mathcal{N}_\tau^\circ(A) = \{H \in \tau \mid A \subseteq {}^*H\}$ is called the τ-open neighborhood system of A.

Proposition 1 *Let $A \subseteq {}^*X$. Then $\mathcal{N}_\tau(A)$ is a filter on ${}^*X$.*

Proof Obviously, $X \in \mathcal{N}_\tau(A)$ and $\emptyset \notin \mathcal{N}_\tau(A)$. Let $H_1, H_2 \in \mathcal{N}_\tau(A)$. Then there exist $G_1, G_2 \in \tau$ such that $A \subseteq {}^*G_1 \subseteq {}^*H_1$, $A \subseteq {}^*G_2 \subseteq {}^*H_2$. Since $G_1 \cap G_2 \in \tau$ and $A \subseteq {}^*(G_1 \cap G_2) \subseteq {}^*(H_1 \cap H_2)$, $H_1 \cap H_2 \in \mathcal{N}_\tau(A)$. Suppose $H_1 \in \mathcal{N}_\tau(A)$ and $H_1 \subseteq H_2$. Then there exists $G \in \tau$ such that $A \subseteq {}^*G \subseteq {}^*H_1 \subseteq {}^*H_2$, and thus $H_2 \in \mathcal{N}_\tau(A)$. It is proved that $\mathcal{N}_\tau(A)$ is a filter on ${}^*X$. $\square$

Definition 4 *Let $\mathcal{A}$ be a nonempty family of sets of $\mathbf{V}(S)$. Then $\mathcal{A}$ has the finite intersection property if $\bigcap_{i=1}^n A_i \neq \emptyset$ for $A_1, A_2, \ldots, A_n \in \mathcal{A}$.*

Proposition 2 *Let $\mathcal{B}$ be a nonempty family of subsets on I. Then $\mathcal{B}$ has the finite intersection property if and only if there exists a filter $\mathcal{F}$ on I such that $\mathcal{B} \subseteq \mathcal{F}$.*

Proof If $\mathcal{B} \subseteq \mathcal{F}$, then $B_1 \in \mathcal{F}, B_2 \in \mathcal{F}, \ldots, B_n \in \mathcal{F}$ for all $\{B_1, B_2, \ldots, B_n\} \subseteq \mathcal{B}$. Since $\mathcal{F}$ is a filter on I, $\bigcap_{i=1}^n B_i \neq \emptyset$. That is, $\mathcal{B}$ has the finite intersection property. Conversely, let $\mathcal{F} = \{F \subseteq I \mid F$ contain the intersection of finite elements of $\mathcal{B}\}$. Then $\mathcal{B} \subseteq \mathcal{F}$ is obvious. Since $\bigcap_{i=1}^n B_i \neq \emptyset$ for every nonempty set $B_1, B_2, \ldots, B_n \in \mathcal{B}$, we have $\emptyset \notin \mathcal{F}$. Clearly, $I \in \mathcal{F}$. If $F_1 \in \mathcal{F}$, $F_2 \in \mathcal{F}$, then there exist $B_1, B_2, \ldots, B_m \in \mathcal{B}$, $B_{m+1}, \ldots, B_n \in \mathcal{B}$ such that $F_1 \supseteq \bigcap_{i=1}^m B_i$, $F_2 \supseteq \bigcap_{i=m+1}^n B_i$. Thus $F_1 \cap F_2 \supseteq \bigcap_{i=1}^n B_i$, and then $F_1 \cap F_2 \in \mathcal{F}$. Let $F_1 \in \mathcal{F}$ and $F_1 \subseteq F_2$. Then for some $B_1, B_2, \ldots, B_n \in \mathcal{B}$, we have $\bigcap_{i=1}^n B_i \subseteq F_1 \subseteq F_2$, and so $F_2 \in \mathcal{F}$. This shows that

$\mathcal{F}$ is a filter on I and $\mathcal{B} \subseteq \mathcal{F}$. $\square$

Lemma 1 (Transfer Principle) ([7]) *Let $\mathcal{L}$ be a language of $\mathbf{V}(S)$ and α be a sentence of $\mathcal{L}$. Then ${}^*\alpha$ is true in $\mathbf{V}({}^*S)$ if and only if α is true in $\mathbf{V}(S)$.*

Proposition 3 *Let $A \subseteq {}^*X$. Then $\mathcal{N}_\tau^\circ(A)$ is a filter subbasis of $\mathcal{N}_\tau(A)$.*

Proof For every $H_1, H_2, \ldots, H_n \in \mathcal{N}_\tau^\circ(A)$, we have $A \subseteq {}^*H_1, A \subseteq {}^*H_2, \ldots, A \subseteq {}^*H_n$. Thus $A \subseteq {}^*H_1 \cap {}^*H_2 \cap \cdots \cap {}^*H_n = {}^*(H_1 \cap H_2 \cap \cdots \cap H_n)$, and then ${}^*(H_1 \cap H_2 \cap \cdots \cap H_n) \neq \emptyset$. By Transfer Principle, we have $H_1 \cap H_2 \cap \cdots \cap H_n \neq \emptyset$, that is, $\mathcal{N}_\tau^\circ(A)$ has the finite intersection property. Then $\mathcal{N}_\tau^\circ(A)$ is a filter subbasis of $\mathcal{N}_\tau(A)$ by Proposition 2. $\square$

2. The τ-standard part of a set

In this section, some related conclusions of the τ-standard part of a set will be given.

Definition 5 *Let $A \subseteq {}^*X$. The monad of the filter $\mathcal{N}_\tau(A)$ is called τ-monad of A. It is denoted by $\mu_\tau(A)$.*

In particular, if $A = \{x\}$, we shall denote $\mu_\tau(A)$ by $\mu_\tau(x)$.

Proposition 4 *If $\mathcal{B}$ is a subbasis of a filter $\mathcal{F}$, then $\mu(\mathcal{F}) = \mu(\mathcal{B})$.*

Proof $\mathcal{B}$ is a subbasis of the filter $\mathcal{F}$, so $\mathcal{B} \subseteq \mathcal{F}$. Then $\mu(\mathcal{B}) \supseteq \mu(\mathcal{F})$. On the other hand, for every $F \in \mathcal{F}$ and $\mathcal{B}$ is a subbasis of the filter $\mathcal{F}$, so there exist $B_1, B_2, \ldots, B_n \in \mathcal{B}$ such that $F \supseteq \bigcap_{i=1}^n B_i$. By Transfer Principle, we have ${}^*F \supseteq {}^*\bigcap_{i=1}^n B_i = \bigcap_{i=1}^n {}^*B_i$. Since $\bigcap_{i=1}^n {}^*B_i \supseteq \mu(\mathcal{B})$, $\bigcap_{F \in \mathcal{F}} F = \mu(\mathcal{F}) \supseteq \bigcap_{i=1}^n {}^*B_i \supseteq \mu(\mathcal{B})$. So $\mu(\mathcal{F}) = \mu(\mathcal{B})$. $\square$

We know $\mu_\tau(A) = \mu(\mathcal{N}_\tau(A)) = \mu(\mathcal{N}_\tau^\circ(A))$ by Definitions 4 and 5. So for convenience we shall use $\mu_\tau(A) = \mu(\mathcal{N}_\tau^\circ(A))$ in the following discussion.

Definition 6 *A point $a \in {}^*X$ is called a τ-near-standard point whenever there exists a standard point $x \in X$ such that $a \in \mu_\tau(x)$. The set of all τ-near-standard points will be denoted by $ns_\tau({}^*(X, \tau))$ or simply $ns_\tau({}^*X)$.*

Definition 7 *Let $A \subseteq {}^*X$. Then $st_\tau(A) = \{x \mid x \in A \text{ and } \mu_\tau(x) \cap A \neq \emptyset\}$ is called τ-standard part of A.*

3. The closeness of the τ-standard part of a set in enlarged models

The definition of the enlarged model and some propositions have been known. In this section, the sufficient conditions that the τ-standard part of A is closed in the enlarged model will be showed.

Definition 8 *Let $A \subseteq \mathbf{V}({}^*S)$. The monad is called the discrete monad of A whenever there exists the smallest filter monad containing A. It is denoted by $\mu_d(A)$.*

Lemma 2 ([8]) *Let $\emptyset \neq A \subseteq {}^*X$. Then $\mathcal{D}_A = \{F \mid F \subseteq X$ and $A \subseteq {}^*F\}$ is a filter which satisfies $\mu_d(A) = \mu(\mathcal{D}_A)$.*

This Lemma was proved by Luxemburg [8]. He also proved that μ_d is a closure operator and defined the discrete S-topology (the topology induced by the closure poerator μ_d on ${}^*X$), or simply the S-topology. A is called S-closed, if $\mu_d(A) = A$. In the following lemma, $\vee\mathcal{F}$ and $\mu(\mathcal{F} \vee \mathcal{D}_A) = \mu(\mathcal{F}) \cap \mu_d(A)$ are also showed in [8].

Lemma 3 *Let $\mathbf{V}({}^*S)$ be an enlargement of $\mathbf{V}(S)$ and let $\mathcal{F}$ be a filter. If A is S-closed, that is, A is the monad of a filter, and ${}^*F \cap A \neq \emptyset$ for all $F \in \mathcal{F}$, then $A \cap \mu(\mathcal{F}) \neq \emptyset$.*

Proof Since A is S-closed, it follows that $A = \mu_d(A) = \mu(\mathcal{D}_A)$. For every $F \in \mathcal{F}$, ${}^*F \cap A \neq \emptyset$, so $\mu(\mathcal{D}_A) \cap {}^*F \neq \emptyset$. And then we have ${}^*E \cap {}^*F \neq \emptyset$ for all $E \in \mathcal{D}_A$. By Transfer Principle, $E \cap F \neq \emptyset$ implies that $F \vee \mathcal{D}_A$ exists. So we have $\emptyset \neq \mu(\mathcal{F} \vee \mathcal{D}_A) = \mu(\mathcal{F}) \cap \mu_d(A) = \mu(\mathcal{F}) \cap A$. □

Lemma 4 ([9]) *A is closed if and only if $\overline{A} = A$.*

Theorem 1 *Let $\mathbf{V}({}^*S)$ be an enlargement of $\mathbf{V}(S)$. If $A \subseteq {}^*X$ is S-closed, that is, A is the monad of a filter, then $st_\tau(A)$ is τ-closed.*

Proof Obviously, $st_\tau(A) \subseteq \overline{st_\tau(A)}$. On the other hand, for all $a \in \overline{st_\tau(A)}$, we have $V \cap st_\tau(A) \neq \emptyset$ for every $V \in \mathcal{N}_\tau^\circ(a)$, so there exists an element $y \in V$ and $y \in st_\tau(A)$. Then $\mu_\tau(y) \subseteq {}^*V$ and $A \cap \mu_\tau(y) \neq \emptyset$. Thus we have $A \cap {}^*V \neq \emptyset$ for every $V \in \mathcal{N}_\tau^\circ(a)$, $A \cap \mu_\tau(a) \neq \emptyset$ by Lemma 3. It follows that $a \in st_\tau(A)$, and then $\overline{st_\tau(A)} \subseteq st_\tau(A)$, so $st_\tau(A) = \overline{st_\tau(A)}$. It is proved that $st_\tau(A)$ is τ-closed. □

Proposition 5 *Let $\emptyset \neq A \subseteq {}^*X$. Then A is standard if and only if A is S-open and S-closed.*

Proof Assume that A is standard. Then there exists $E \subseteq X$ such that $A = {}^*E$, and so $\mu_d(A) = \mu_d({}^*E) = {}^*E = A$. That is, A is S-closed. Since ${}^*X \setminus A = {}^*(X \setminus E)$, $\mu_d({}^*X \setminus A) = \mu_d({}^*(X \setminus E)) = {}^*(X \setminus E) = {}^*X \setminus A$. So ${}^*X \setminus A$ is S-closed, and then A is S-open. Conversely, if A is S-open, then ${}^*X \setminus A$ is S-closed. Since A is S-closed, $A = {}^*X$ or $A = \emptyset$. This means that A is standard. □

Corollary 1 *Let $\mathbf{V}({}^*S)$ be an enlargement of $\mathbf{V}(S)$. If A is standard, then $st_\tau(A)$ is τ-closed.*

Proof If A is standard, then A is S-closed by proposition 5. So $st_\tau(A)$ is τ-closed by Theorem 1. □

Corollary 2 *Let $\mathbf{V}({}^*S)$ be an enlargement of $\mathbf{V}(S)$ and $A \subseteq X$. Then $st_\tau({}^*A) = \overline{A}$.*

Proof For all $x \in st_\tau({}^*a)$ if and only if $\mu_\tau(x) \cap {}^*A \neq \emptyset$ if and only if $x \in \overline{A}$. So $st_\tau({}^*A) = \overline{A}$. □

Lemma 5 *If $\mathcal{F}$ is a filter, then $st_\tau(\bigcap {}^*F) = \bigcap st_\tau({}^*F)$ for every $F \in \mathcal{F}$.*

Proof Since ${}^*F \supseteq \bigcap {}^*F$, we have $st_\tau({}^*F) \supseteq st_\tau(\bigcap {}^*F)$, and then $\bigcap st_\tau({}^*F) \supseteq st_\tau(\bigcap {}^*F)$. On the other hand, for every $x \in \bigcap st_\tau({}^*F)$, we have $x \in st_\tau({}^*F)$ for all $F \in \mathcal{F}$, and so

${}^*F \cap \mu_\tau(x) \neq \emptyset$. Since $\mu_\tau(x)$ is S-closed, it follows that $\mu_\tau(x) \cap \mu(\mathcal{F}) \neq \emptyset$, and then $x \in st_\tau(\mu(\mathcal{F}))$. Thus $\bigcap st_\tau({}^*F) \subseteq st_\tau(\bigcap {}^*F)$ and so $st_\tau(\bigcap {}^*F) = \bigcap st_\tau({}^*F)$. □

Theorem 2 *Let $\mathbf{V}({}^*S)$ be an enlargement of $\mathbf{V}(S)$. If $\mathcal{F}$ is a filter and $A = \mu(\mathcal{F})$, then $st_\tau(A) = \bigcap\{\overline{F} \mid F \in \mathcal{F}\}$.*

Proof $st_\tau(A) = st_\tau(\mu(\mathcal{F})) = st_\tau(\bigcap {}^*F) = \bigcap st_\tau({}^*F) = \bigcap \overline{F}$, So $st_\tau(A) = \bigcap\{\overline{F} \mid F \in \mathcal{F}\}$. □

Lemma 6 *Let $\mathbf{V}({}^*S)$ be an enlargement of $\mathbf{V}(S)$. If A is internal and $\mathcal{F}$ has a countable subbasis, then $A \cap {}^*F \neq \emptyset$ for all $F \in \mathcal{F}$ implies $A \cap \mu(\mathcal{F}) \neq \emptyset$.*

Proof Let $\Omega = \{F_n \mid n \in N\}$ be a countable subbasis of $\mathcal{F}$. If $\{F'_n \mid n \in N\} \in \Omega$, then $F_n = \bigcap_{i=1}^n F'_n$. Let $F : N \longrightarrow \mathcal{F}$. Then $F : {}^*N \longrightarrow {}^*\mathcal{F}$ and ${}^*\Omega = \{{}^*F_n \mid n \in {}^*N\}$ by Transfer Principle. Since A is internal, we obtain that $D = \{m \in {}^*N \mid F_m \cap A \neq \emptyset$ and $F_m \in {}^*\Omega\}$ is internal. $A \cap {}^*F \neq \emptyset$ for every $F \in \mathcal{F}$, so ${}^*F_n \cap A \neq \emptyset$ for all $n \in {}^*N$. Then $n \in D$, and so $N \subseteq D$. Since D is internal and N is external, $N \neq D$. Then $D \cap ({}^*N \setminus N) \neq \emptyset$, and there exists $\nu \in {}^*N \setminus N$ such that $\nu \in D$. It follows that ${}^*F_\nu \cap A \neq \emptyset$. Since Ω is a decreasing countable subbasis of $\mathcal{F}$ for all $n \in N$, we have ${}^*F_n \supseteq {}^*F_\nu$. Then $\bigcap_{n \in N} {}^*F = \mu(\mathcal{F}) \supseteq {}^*F_\nu$. We obtain $A \cap \mu(\mathcal{F}) \neq \emptyset$, since $A \cap {}^*F_\nu \neq \emptyset$. □

Theorem 3 *Let $\mathbf{V}({}^*S)$ be an enlargement of $\mathbf{V}(S)$. If $A \subseteq {}^*X$ is internal, and the open neighborhood system of a has countable subbasis for every $a \in \overline{st_\tau(A)}$, then $st_\tau(A)$ is τ-closed.*

Proof $st_\tau(A) \subseteq \overline{st_\tau(A)}$ is obvious. On the other hand, for all $a \in \overline{st_\tau(A)}$, we have $V \cap st_\tau(A) \neq \emptyset$ for every $V \in \mathcal{N}_\tau^\circ(a)$, so there exists an element $y \in V$ and $y \in st_\tau(A)$, and then $\mu_\tau(y) \subseteq {}^*V$ and $A \cap \mu_\tau(y) \neq \emptyset$. It follows that $A \cap {}^*V \neq \emptyset$ for all $V \in \mathcal{N}_\tau^\circ(a)$. Since A is internal and $\mathcal{N}_\tau^\circ(a)$ has countable subbasis, we obtain that $A \cap \mu_\tau(a) \neq \emptyset$ by Lemma 6. Then $a \in st_\tau(A)$, and so $\overline{st_\tau(A)} \subseteq st_\tau(A)$. It is proved that $st_\tau(A) = \overline{st_\tau(A)}$, that is, $st_\tau(A)$ is τ-closed. □

4. The closeness of the τ-standard part of a set in saturated models

The definition of κ-saturated models and some related conclusions have been known. In this section, a sufficient condition that τ-standard part of a set is closed in saturated models will be presented.

Proposition 6 *Let $\mathbf{V}({}^*S)$ be a nonstandard model of $\mathbf{V}(S)$. Then $\mathbf{V}({}^*S)$ is κ-saturated if and only if for every family $\{A_i\}_{i \in I}$ of internal set of $\mathbf{V}({}^*S)$ with the finite intersection property and* card$(I) < \kappa$ *there holds $\bigcap_{i \in I} A_i \neq \emptyset$.*

Proof Assume that $\mathbf{V}({}^*S)$ is κ-saturated. Let $\mathcal{A} = \{A_i\}_{i \in I}$ and $A \in \mathcal{A}$. We denote by $\mathcal{P}_{int}(A)$ a family of all internal subsets. Let $r = \{\langle x, y\rangle \mid x \in \mathcal{P}_{int}(A), y \in A, y \in x\}$. Then r is an internal binary relation in $\mathbf{V}({}^*S)$. Since $A \cap \mathcal{A} = \{A \cap B \mid B \in \mathcal{A}\} \subseteq \mathrm{dom}(r)$, there is an element $y \in (A \cap A_1) \cap (A \cap A_2) \cap \cdots \cap (A \cap A_n)$ for all $A \cap A_1, A \cap A_2, \ldots, A \cap A_n \in A \cap \mathcal{A}$. $(A \cap A_1) \cap (A \cap A_2) \cap \cdots \cap (A \cap A_n) \neq \emptyset$, so $y \in A$, and then $\langle A \cap A_i, y\rangle \in r$ for $i = 1, 2, \ldots, n$.

That is, r is a concurrent relation on $A \cap \mathcal{A}$. Since $\mathrm{card}(A \cap \mathcal{A}) < \mathrm{card}(\mathcal{A}) < \kappa$ and $\mathbf{V}({}^*S)$ is κ-saturated, there is an element $y \in \bigcap_{i \in I}(A \cap A_i) = \bigcap_{i \in I} A_i$ such that $\langle A \cap A_i, y\rangle \in r$ for all $A \cap A_i \in A \cap \mathcal{A}$ for $i \in I$. So $\bigcap_{i \in I} A_i \neq \emptyset$.

Conversely, assume that r is an internal binary relation and assume that r is concurrent on $A \cap \mathcal{A} \subseteq \mathrm{dom}(r)$ for $A \in \mathcal{A}$. Let $\mathrm{ran}(r) = \{y \mid y \in A \cap A_i, i \in I\}$. Since the family $\{A_i\}_{i \in I}$ of internal set has the finite intersection property, there exists an element $y \in \mathrm{ran}(r)$ such that $\langle A \cap \mathcal{A}, y\rangle \in r$. Now $\mathrm{card}(A \cap \mathcal{A}) < \mathrm{card}(\mathcal{A}) < \kappa$ and $\bigcap_{i \in I} A_i \neq \emptyset$, and so there exists an elememt $y \in \bigcap_{i \in I} A_i = \bigcap_{i \in I}(A \cap A_i)$ such that $\langle A \cap A_i, y\rangle \in r$ for all $A \cap A_i \in A \cap \mathcal{A}$. So $\mathbf{V}({}^*S)$ is κ-saturated. □

Lemma 7 *Let $\mathbf{V}({}^*S)$ be a nonstandard model of $\mathbf{V}(S)$ and let $\mathbf{V}({}^*S)$ be κ- saturated. If $\mathcal{F}$ is a filter and A is internal such that $A \cap {}^*F \neq \emptyset$ for all $F \in \mathcal{F}$, then $\mu(\mathcal{F}) \cap A \neq \emptyset$.*

Proof Let $\Omega_i = \{A \cap {}^*F_i \mid F_i \in \mathcal{F}\}$ for $i = 1, 2, \ldots, n$. Since $\mathcal{F}$ is a filter, $\bigcap_{i=1}^n F_i \in \mathcal{F}$. If A is internal such that $A \cap {}^*F \neq \emptyset$ for all $F \in \mathcal{F}$, then $\bigcap_{i=1}^n \Omega_i = \bigcap_{i=1}^n \{A \cap {}^*F_i \mid F_i \in \mathcal{F}\} \neq \emptyset$. Since $\mathbf{V}({}^*S)$ is κ-saturated, by Proposition 6, we have $\bigcap \Omega = \bigcap\{A \cap {}^*F \mid F \in \mathcal{F}\} \neq \emptyset$. That is, $\mu(\mathcal{F}) \cap A \neq \emptyset$. □

Theorem 4 *Let $\mathbf{V}({}^*S)$ be a nonstandard model of $\mathbf{V}(S)$ and let $\mathbf{V}({}^*S)$ be κ-saturated. If $A \subseteq {}^*X$ is internal, then $st_\tau(A)$ is τ-closed.*

Proof Clearly, $st_\tau(A) \subseteq \overline{st_\tau(A)}$. On the other hand, for every $x \in \overline{st_\tau(A)}$ and $V \in \mathcal{N}_\tau(x)$, we have ${}^*V \cap A \neq \emptyset$. Since $\mathbf{V}({}^*S)$ is κ-saturated and A is internal, we have $\mu_\tau(A) \cap A \neq \emptyset$ and then $x \in st_\tau(A)$. So $\overline{st_\tau(A)} \subseteq st_\tau(A)$, and then $st_\tau(A) = \overline{st_\tau(A)}$. That is, $st_\tau(A)$ is τ-closed. □

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