Journal of Mathematical Research & Exposition
Mar., 2011, Vol. 31, No. 2, pp. 377–380
DOI:10.3770/j.issn:1000-341X.2011.02.024
Http://jmre.dlut.edu.cn

Nonexistence of the Solution for a Class of Nonlinear Hyperbolic Equation

Yan Ping WANG
Department of Physics and Mathematics, Zhengzhou Institute of Aeronautical Industry Management, Henan 450015, P. R. China

Abstract In this paper, we prove the existence and uniqueness of the local generalized solution of the Cauchy problem for a class of nonlinear hyperbolic equation of higher order are proved. Moreover, we give the sufficient conditions for blow-up of the solution of the problem in finite time will be given.

Keywords nonlinear hyperbolic equation; Cauchy problem; local solution; nonexistence of solution.

Document code A
MR(2010) Subject Classification 35L35; 35G30
Chinese Library Classification O175.29

1. Introduction

In this paper, we study the following Cauchy problem for a class of nonlinear hyperbolic equation

$$u_{tt} + \alpha u_{xxxx} + \beta u_{xxxxtt} = f(u)_{xx},\ x \in R,\ t > 0, \tag{1.1}$$

$$u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x),\ x \in R, \tag{1.2}$$

where, $u(x,t)$ denotes the unknown function with respect to variables x and t, $\alpha > 0$ and $\beta > 0$ are physical constants, $f(x)$ is the given nonlinear function, and $\varphi(x)$ and $\phi(x)$ are known initial functions.

In the study of lattice dynamics and the study of water wave, the following model equation can be obtained [1]

$$u_{tt} + \alpha u_{xxxx} + \beta u_{xxxxtt} = \gamma(u^2)_{xx}, \tag{1.3}$$

where $\alpha > 0$, $\beta > 0$ and $\gamma \neq 0$ are constants. Obviously, Eq.(1.1) is the generalized type of Eq.(1.3). This kind of equation is also called the Boussinesq type equation (Bq equation). There are a lot of results on the solitary wave solution and traveling wave solution of Bq equations [2–6]. The paper [3] studied the initial boundary value problem for the Eq.(1.1), and proved the existence and uniqueness of the local generalized solution for the problem. Moreover, the

Received February 19, 2009; Accepted July 3, 2009
Supported by the National Natural Science Foundation of China (Grant No. 10671182) and the Excellent Youth Teachers Foundation of High College of Henan Province (Grant No. 2006110016).
E-mail address: wangyanping68@126.com

blow-up properties of the solution for the problem was discussed in [3]. The initial boundary problem or initial value problem for some Bq type equations were studied in [4–6].

In this paper, we first prove the existence and uniqueness of the local generalized solution of the Cauchy problem (1.1), (1.2), and then we discuss the blow-up of solution by means of concavity method.

Throughout the paper, we use the following notations: L^2 denotes the usual space of all L^2-functions on R with norm $\|u\| = \|u\|_{L^2}$; H^s denotes the usual Sobolev space on R with norm $\|u\|_{H^s} = \|(I-\partial_x^2)^{\frac{s}{2}}\| = \|(1+|\xi|^2)^{\frac{s}{2}}\hat{u}\|$, $\dot{H}^s$ denotes the corresponding homogeneous space on R with semi-norm $\|u\|_{\dot{H}^s} = \||\xi|^s\hat{u}\|$, where $s \in R$, I is a unitary operator, $\hat{u}(\xi,t)$ is the Fourier transformation of $u(x,\ t)$ with respect to x, i.e.,

$$\hat{u}(\xi,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ix\xi}u(x,t)\mathrm{d}x.$$

2. Local solution and the nonexistence of the global solution

By the standard method as in [3], we can prove the following conclusion about the problem (1.1), (1.2):

Theorem 2.1 *Assume that $s \geq 1$, $\varphi \in H^s$, $\psi \in H^s$ and $f \in C^{[s]+3}(R)$, then the problem (1.1), (1.2) has a unique local solution $u(x,t)$, defined on a maximal time interval $[0,T_0)$, $T_0 > 0$ with $u \in C([0,T_0);\ H^s) \cap C^1([0,T_0);\ H^s) \cap C^2([0,T_0);\ H^s)$.*

In order to investigate the nonexistence of the global solution for the problem (1.1), (1.2), let us introduce the following lemma:

Lemma 2.1 ([7]) *Suppose that for $t \geq 0$, a positive twice-differentiable $H(t)$ satisfies the inequality*

$$\ddot{H}H - (1+\delta)\dot{H}^2 \geq 0,$$

where $\delta > 0$ is a constant. If $H(0) > 0$ and $\dot{H}(0) > 0$, then there is a $t_1 \leq \dfrac{H(0)}{\delta\dot{H}(0)}$ such that $H(t) \to \infty$ as $t \to t_1$.

Next, we give the energy identity for the solution $u(x,t)$ of the problem (1.1), (1.2).

Lemma 2.2 *Suppose that $f \in C(R)$, $F(s) = \int_0^s f(\tau)\mathrm{d}\tau$, $\varphi \in H^1$, $\psi \in H^1 \cap \dot{H}^{-1}$, $F(\varphi) \in L^1$. Then the following energy identity holds*

$$E(t) = \|u_t(\cdot,t)\|^2_{\dot{H}^{-1}} + \alpha\|u_x(\cdot,t)\|^2 + \beta\|u_{xt}(\cdot,t)\|^2 + 2\int_{-\infty}^{+\infty} F(u(x,t))\mathrm{d}x = E(0). \tag{2.1}$$

Proof By the straightforward calculation, it follows from the equation (1.1) that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = 0.$$

Integrating the above identity with respect to t, we can obtain (2.1).

Theorem 2.2 *Under the assumptions of Lemma 2.2, if $sf(s) \leq 2(1+2\lambda)F(s)$, $\forall s \in R$ ($\lambda > 0$*

is a constant), then the solution $u(x,t)$ of the problem (1.1), (1.2) blows up in finite time if one of the following conditions holds:

(1) $E(0) < 0$*;*

(2) $E(0) = 0,\ (|\xi|^{-1}\hat{\varphi},\ |\xi|^{-1}\hat{\psi}) + \beta(|\xi|\hat{\varphi},\ |\xi|\hat{\psi}) > 0$*;*

(3) $E(0) > 0,\ (|\xi|^{-1}\hat{\varphi},\ |\xi|^{-1}\hat{\psi}) + \beta(|\xi|\hat{\varphi},\ |\xi|\hat{\psi}) > 2\sqrt{E(0)(\|\varphi\|^2_{\dot{H}^{-1}} + \beta\|\varphi\|^2_{\dot{H}^1})}$*.*

Proof Let

$$H(t) = \||\xi|^{-1}\hat{u}\|^2 + \beta\||\xi|\hat{u}\|^2 + \gamma_0(t+t_0)^2, \tag{2.2}$$

where γ_0 and t_0 are non-negative constants to be defined later. Then

$$\dot{H}(t) = 2(|\xi|^{-1}\hat{u},\ |\xi|^{-1}\hat{u}_t) + 2\beta(|\xi|\hat{u},\ |\xi|\hat{u}_t) + 2\gamma_0(t+t_0).$$

Using the Schwartz inequality, we can get

$$\dot{H}(t)^2 \le 4H(t)[\||\xi|^{-1}\hat{u}_t\|^2 + \beta\||\xi|\hat{u}_t\|^2 + \gamma_0]. \tag{2.3}$$

By the aid of the equation (1.1) and the energy identity (2.1), we have

$$\begin{aligned}\ddot{H}(t) =& 2\||\xi|^{-1}\hat{u}_t\|^2 + 2\beta\||\xi|\hat{u}_t\|^2 + 2\gamma_0 + 2(|\xi|^{-1}\hat{u},\ |\xi|^{-1}\hat{u}_{tt}) + 2\beta(|\xi|\hat{u},\ |\xi|\hat{u}_{tt})\\ =& 2\||\xi|^{-1}\hat{u}_t\|^2 + 2\beta\||\xi|\hat{u}_t\|^2 + 2\gamma_0 + 2(\hat{u},\ -\alpha|\xi|^2\hat{u} - \widehat{f(u)})\\ =& 2\||\xi|^{-1}\hat{u}_t\|^2 + 2\beta\||\xi|\hat{u}_t\|^2 + 2\gamma_0 - 2\alpha\||\xi|\hat{u}\|^2 - 2\int_{-\infty}^{+\infty} uf(u)\mathrm{d}x\\ =& -(2+4\lambda)(E(0)+\gamma_0) + 4(1+\lambda)(\||\xi|^{-1}\hat{u}_t\|^2 + \beta\||\xi|\hat{u}_t\|^2 + \gamma_0)\\ & 4\lambda\alpha\||\xi|\hat{u}\|^2 + 2\int_{-\infty}^{+\infty}[2(1+2\lambda)F(u) - uf(u)]\mathrm{d}x.\end{aligned} \tag{2.4}$$

From (2.2)–(2.4), it follows that

$$H(t)\ddot{H}(t) - (1+\lambda)\dot{H}(t)^2 \ge -2(1+2\lambda)(E(0)+\gamma_0)H(t). \tag{2.5}$$

If $E(0) < 0$, taking $\gamma_0 = -E(0) > 0$, from (2.5) we have

$$H(t)\ddot{H}(t) - (1+\lambda)\dot{H}(t)^2 \ge 0. \tag{2.6}$$

Obviously, if t_0 is sufficiently large, $\dot{H}(0) > 0$. From Lemma 2.1, we know that $H(t)$ becomes infinite at T_1 at most equal to $\frac{H(0)}{\lambda\dot{H}(0)} < +\infty$.

If $E(0) = 0$, taking $\gamma_0 = 0$, we see that (2.6) holds too. Noting the assumption (2) of Theorem 2.2, it follows that $\dot{H}(0) > 0$. By the aid of Lemma 2.1, we know that $H(t)$ becomes infinite at T_2 at most equal to $\frac{H(0)}{\lambda\dot{H}(0)} < +\infty$.

If $E(0) > 0$, taking $\gamma_0 = 0$, then $H(t)$ satisfies

$$H(t)\ddot{H}(t) - (1+\lambda)\dot{H}(t)^2 \ge -2(1+2\lambda)E(0)H(t).$$

Define $I(t) = H^{-\lambda}(t)$, we see

$$\begin{aligned}\dot{I}(t) =& -\lambda H^{-\lambda-1}(t)\dot{H}(t),\\ \ddot{I}(t) =& \lambda(\lambda+1)H^{-\lambda-2}(t)\dot{H}(t)^2 - \lambda H^{-\lambda-1}(t)\ddot{H}(t)\\ =& -\lambda H^{-\lambda-2}(t)[H(t)\ddot{H}(t) - (1+\lambda)\dot{H}(t)^2]\end{aligned}$$

$$\leq 2\lambda(1+2\lambda)E(0)H^{-\lambda-1}(t). \tag{2.7}$$

By the assumption (3) in Theorem 2.2 we have $\dot{I}(0) < 0$. Let

$$\tilde{t} = \sup\{t|\dot{I}(\tau) < 0,\ \tau \in [0,\ t)\}.$$

By the continuity of $\dot{I}(t)$, $\tilde{t}$ is positive.

Multiplying (2.7) by $2\dot{I}(t)$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{I}(t)^2) &\geq -4(1+2\lambda)\lambda^2 E(0)H^{-2\lambda-2}(t)\dot{H}(t)\\ &= 4\lambda^2 E(0)\frac{\mathrm{d}}{\mathrm{d}t}H^{-2\lambda-1}(t), \quad \forall t \in [0,\tilde{t}).\end{aligned} \tag{2.8}$$

Integrating (2.8) with respect to t in $[0,t)$ for $t \in [0,\tilde{t})$ yields

$$\dot{I}(t)^2 \geq \dot{I}(0)^2 + 4\lambda^2 E(0)[H^{-2\lambda-1}(t) - H^{-2\lambda-1}(0)].$$

By the assumption (3), it is true that

$$\dot{I}(0)^2 - 4\lambda^2 E(0)H^{-2\lambda-1}(0) > 0.$$

It follows from the continuity of $\dot{I}(t)$ that

$$\dot{I}(t) \leq -\sqrt{\dot{I}(0)^2 - 4\lambda^2 E(0)H^{-2\lambda-1}(0)},\ t \in [0,\ \tilde{t}).$$

By the definition of $\tilde{t}$, the above inequality holds for all $t \geq 0$. Hence,

$$I(t) \leq I(0) - t\sqrt{\dot{I}(0)^2 - 4\lambda^2 E(0)H^{-2\lambda-1}(0)},\ \forall t > 0.$$

Therefore, $I(T_1) = 0$ for some T_1 and $0 < T_1 \leq T_0$, where

$$T_0 = I(0)[\dot{I}(0)^2 - 4\lambda^2 E(0)H^{-2\lambda-1}(0)]^{-\frac{1}{2}}.$$

Thus, $H(t)$ becomes infinite at T_1.

This completes the proof of the theorem. □

References

[1] ROSENAU P. *Dynamics of dense lattices* [J]. Phys. Rev. B (3), 1987, **36**(11): 5868–5876.

[2] PORUBOV A V. *Strain solitary waves in an elastic rod with microstructure* [J]. Rend. Sem. Mat. Univ. Politec. Torino, 2000, **58**(2): 189–198.

[3] WANG Yanping, GUO Boling. *Blow-up of solution for a generalized Boussinesq equation* [J]. Appl. Math. Mech. (English Ed.), 2007, **28**(11): 1437–1443.

[4] WANG Yanping, GUO Boling. *The Cauchy problem for a generalized Boussinesq type equation* [J]. Chinese Ann. Math. Ser. A, 2008, **29**(2): 185–194. (in Chinese)

[5] CHEN Guowang, WANG Yanping, ZHAO Zhancai. *Blow-up of solution of an initial boundary value problem for a damped nonlinear hyperbolic equation* [J]. Appl. Math. Lett., 2004, **17**(5): 491–497.

[6] CHEN Guowang, WANG Yanping, WANG Shubin. *Initial boundary value problem of the generalized cubic double dispersion equation* [J]. J. Math. Anal. Appl., 2004, **299**(2): 563–577.

[7] LEVINE H A. *Instability and nonexistence of global solutions to nonlinear wave equations of the form* $Pu_{tt} = -Au + \mathcal{F}(u)$ [J]. Trans. Amer. Math. Soc., 1974, **192**: 1–21.