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3-Dimensional Cahn-Hilliard Equation with Concentration Dependent Mobility and Gradient Dependent Potential

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Abstract In this paper we investigate the initial boundary value problem of Cahn-Hilliard equation with concentration dependent mobility and gradient dependent potential. By the energy method and the theory of Campanato spaces, we prove the existence and the uniqueness of classical solutions in 3-dimensional space.

Keywords Cahn-Hilliard equation; concentration dependent mobility; gradient dependent potential.

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1. Introduction

In this paper, we consider the following initial boundary value problem of Cahn-Hilliard equation with concentration dependent mobility and gradient dependent potential

$$\frac{\partial u}{\partial t} + \operatorname{div}\left[m(u)\left(k\nabla\Delta u - \vec{\Phi}(\nabla u)\right)\right] = 0, \quad (x,t)\in Q_T, \tag{1.1}$$

$$\nabla u\cdot\nu\Big|_{\partial\Omega} = \mu\cdot\nu\Big|_{\partial\Omega} = 0, \quad t\in(0,T), \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x\in\Omega, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary, $Q_T = \Omega\times(0,T)$, ν denotes the unit exterior normal to the boundary $\partial\Omega$, $\mu = k\nabla\Delta u - \vec{\Phi}(\nabla u)$ is the flux, k is a positive constant, $m(u)$ denotes the mobility which is dependent on the concentration u, and $\vec{\Phi} = (\Phi_1,\Phi_2,\Phi_3)$ is a smooth vector function from $\mathbb{R}^3$ to $\mathbb{R}^3$.

The problem (1.1)–(1.3) models many interesting phenomena in mathematical biology, fluid mechanics, phase transition, etc. We refer the readers to [1] for the derivation of the equation (1.1) based on the continuum model for epitaxial thin film growth. Recently, such type of equations, especially in the case of one spatial dimension, have aroused the interests of many

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mathematicians, for example [2–5]. For the multi-dimensional case of the equation (1.1) with constant mobility, there are also some results which have been obtained recently. For example, Yin and Huang [1] proved the existence and uniqueness of global solutions. Li et al. [6] considered the time periodic solutions. For the equation (1.1) with concentration dependent mobility and gradient dependent potential, Huang et al. [7] considered the problem (1.1)–(1.3), and proved the existence and uniqueness of the classical solutions in 2-dimensional space. In this paper, we consider the problem (1.1)–(1.3) in 3-dimensional space, and obtain the existence and uniqueness of the classical solutions with small initial data by the energy method combined with the theory of Campanato spaces. This paper can be viewed as an extension to the previous work [7].

The main result of this paper is the following theorem.

Theorem 1.1 *Suppose $m(s)$ and $\vec{\Phi}(\xi)$ satisfy the following assumptions:*

(H1) $m(s) \in C^{1+\alpha}(\mathbb{R})$, $0 < \alpha < 1$, $M_1 \le m(s) \le M_2$, $|m'(s)| \le M_3$, $\forall s \in \mathbb{R}$;

(H2) $\Phi_i(\xi) \in C^{1+\alpha}(\mathbb{R}^3)$ *and*

$$|\vec{\Phi}(\xi)| \le C|\xi|,\ |\frac{\partial \Phi_i}{\partial \xi_j}| \le C,\ 1 \le i, j \le 3,\ \forall \xi \in \mathbb{R}^3;$$

(H3) $\vec{\Phi}(\xi) \cdot \nu = 0$, $\forall \xi \in \{\xi \in \mathbb{R}^3; \xi \cdot \nu = 0\}$,

where C, M_1, M_2 and M_3 are positive constants. If $u_0(x) \in C^{4+\alpha}(\bar{\Omega})$, and $\|u_0(x)\|_{H^1(\Omega)}$ is suitably small, then the problem (1.1)–(1.3) admits a unique classical solution on $\overline{Q_T}$.

Remark 1.1 Comparing to the previous work [7], in this paper we consider the problem in 3-dimensional space. Obviously, the main difficulty comes from the proof of the regularity of the solutions. In this paper, we use the energy method combined with the theory of Campanato spaces to overcome this difficulty provided that the initial value is suitably small.

2. Proof of main result

In this section we give the proof of the main result in this paper. We first have the following lemma.

Lemma 2.1 *If u is a solution of the problem (1.1)–(1.3), then*

$$|u(x_1, t_1) - u(x_2, t_2)| \le C \left(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\alpha/4} \right)$$

holds for any given $(x_1, t_1), (x_2, t_2) \in Q_T$ with $0 < \alpha < 1$.

Proof Multiplying both sides of the equation (1.1) by Δu and integrating the result with respect to x on Ω, we have

$$\int_\Omega u_t \Delta u \mathrm{d}x + \int_\Omega \operatorname{div} \left[m(u) \left(k \nabla \Delta u - \vec{\Phi}(\nabla u) \right) \right] \Delta u \mathrm{d}x = 0.$$

By the boundary value conditions (1.2) and using the assumptions (H1) and (H2), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla u|^2 \mathrm{d}x + 2k \int_\Omega m(u) |\nabla \Delta u|^2 \mathrm{d}x &= 2 \int_\Omega m(u) \vec{\Phi}(\nabla u) \nabla \Delta u \mathrm{d}x \\ &\le k \int_\Omega m(u) |\nabla \Delta u|^2 \mathrm{d}x + \frac{1}{k} \int_\Omega m(u) |\vec{\Phi}(\nabla u)|^2 \mathrm{d}x \end{aligned}$$

$$\leq k\int_\Omega m(u)|\nabla\Delta u|^2\mathrm{d}x + C\int_\Omega |\nabla u|^2\mathrm{d}x.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^2\mathrm{d}x + kM_1\int_\Omega |\nabla\Delta u|^2\mathrm{d}x \leq C\int_\Omega |\nabla u|^2\mathrm{d}x.$$

It follows from the Gronwall inequality that

$$\int_\Omega |\nabla u|^2\mathrm{d}x \leq C\int_\Omega |\nabla u_0|^2\mathrm{d}x, \quad \forall 0<t<T, \tag{2.1}$$

$$\iint_{Q_T} |\nabla\Delta u|^2\mathrm{d}x \leq C\int_\Omega |\nabla u_0|^2\mathrm{d}x. \tag{2.2}$$

By the assumption (H3), we know that the boundary value conditions (1.2) can be rewritten as

$$\nabla u\cdot\nu\Big|_{\partial\Omega} = \nabla\Delta u\cdot\nu\Big|_{\partial\Omega} = 0. \tag{2.3}$$

Multiplying both sides of the equation (1.1) by $\Delta^2 u$ and integrating the result with respect to x on Ω, we have

$$\int_\Omega u_t\Delta^2 u\mathrm{d}x + \int_\Omega \mathrm{div}\left[m(u)\left(k\nabla\Delta u - \vec{\Phi}(\nabla u)\right)\right]\Delta^2 u\mathrm{d}x = 0.$$

By (2.3) and integrating by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Delta u|^2\mathrm{d}x + k\int_\Omega m(u)|\Delta^2 u|^2\mathrm{d}x + k\int_\Omega m'(u)\nabla u\nabla\Delta u\Delta^2 u\mathrm{d}x - \\ &\int_\Omega m(u)\mathrm{div}\vec{\Phi}(\nabla u)\Delta^2 u\mathrm{d}x - \int_\Omega m'(u)\nabla u\vec{\Phi}(\nabla u)\Delta^2 u\mathrm{d}x = 0.\end{aligned}$$

It follows from (H1), (H2) and the Hölder inequality that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Delta u|^2\mathrm{d}x + kM_1\int_\Omega |\Delta^2 u|^2\mathrm{d}x \\ &\leq \frac{kM_1}{4}\int_\Omega |\Delta^2 u|^2\mathrm{d}x + C\int_\Omega |\nabla u|^2|\nabla\Delta u|^2\mathrm{d}x + \\ &\quad C\int_\Omega |\Delta u|^2\mathrm{d}x + C\int_\Omega |\nabla u|^4\mathrm{d}x \\ &\leq \frac{kM_1}{4}\int_\Omega |\Delta^2 u|^2\mathrm{d}x + C\left(\int_\Omega |\nabla u|^8\mathrm{d}x\right)^{1/4}\left(\int_\Omega |\nabla\Delta u|^{8/3}\mathrm{d}x\right)^{3/4} + \\ &\quad C\int_\Omega |\Delta u|^2\mathrm{d}x + C\int_\Omega |\nabla u|^4\mathrm{d}x.\end{aligned} \tag{2.4}$$

Now we estimate the terms of the right hand side of (2.4). By the Cagliardo-Nirenberg inequality, we have

$$\left(\int_\Omega |\nabla u|^8\mathrm{d}x\right)^{1/8} \leq C\left(\int_\Omega |\nabla u|^2\mathrm{d}x\right)^{3/8}\left(\left(\int_\Omega |\Delta^2 u|^2\mathrm{d}x\right)^{1/8} + \left(\int_\Omega |\nabla u|^2\mathrm{d}x\right)^{1/8}\right),$$

and

$$\left(\int_\Omega |\nabla\Delta u|^{8/3}\mathrm{d}x\right)^{3/8} \leq C\left(\int_\Omega |\nabla u|^2\mathrm{d}x\right)^{1/8}\left(\left(\int_\Omega |\Delta^2 u|^2\mathrm{d}x\right)^{3/8} + \left(\int_\Omega |\nabla u|^2\mathrm{d}x\right)^{3/8}\right).$$

Combining (2.1) with the Young inequality, we have the following estimation on the second term of the right hand side of (2.4)

$$
\begin{aligned}
&\left(\int_\Omega |\nabla u|^8 \mathrm{d}x\right)^{1/4} \left(\int_\Omega |\nabla \Delta u|^{8/3} \mathrm{d}x\right)^{3/4} \\
&\le C \int_\Omega |\nabla u|^2 \mathrm{d}x \left(\left(\int_\Omega |\Delta^2 u|^2 \mathrm{d}x\right)^{1/4} + \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{1/4}\right) \cdot \\
&\quad \left(\left(\int_\Omega |\Delta^2 u|^2 \mathrm{d}x\right)^{3/4} + \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{3/4}\right) \\
&\le C \int_\Omega |\nabla u|^2 \mathrm{d}x \left(\int_\Omega |\Delta^2 u|^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 \mathrm{d}x + \right. \\
&\quad \left. \left(\int_\Omega |\Delta^2 u|^2 \mathrm{d}x\right)^{1/4} \cdot \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{3/4} + \left(\int_\Omega |\Delta^2 u|^2 \mathrm{d}x\right)^{3/4} \cdot \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{1/4}\right) \\
&\le C \int_\Omega |\nabla u|^2 \mathrm{d}x \left(\int_\Omega |\Delta^2 u|^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 \mathrm{d}x\right) \\
&\le C \int_\Omega |\nabla u_0|^2 \mathrm{d}x \left(\int_\Omega |\Delta^2 u|^2 \mathrm{d}x + \int_\Omega |\nabla u_0|^2 \mathrm{d}x\right).
\end{aligned}
$$

For the third term of the right hand side of (2.4), we first notice that

$$
\begin{aligned}
\int_\Omega |\Delta u|^2 \mathrm{d}x &= -\int_\Omega \nabla u \nabla \Delta u \mathrm{d}x \\
&\le \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{1/2} \left(\int_\Omega |\nabla \Delta u|^2 \mathrm{d}x\right)^{1/2} \\
&= \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{1/2} \left(-\int_\Omega \Delta u \Delta^2 u \mathrm{d}x\right)^{1/2} \\
&\le \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{1/2} \left(\int_\Omega |\Delta u|^2 \mathrm{d}x\right)^{1/4} \left(\int_\Omega |\Delta^2 u|^2 \mathrm{d}x\right)^{1/4} \\
&\le C \left(\int_\Omega |\nabla u_0|^2 \mathrm{d}x\right)^{1/2} \left(\int_\Omega |\Delta u|^2 \mathrm{d}x\right)^{1/4} \left(\int_\Omega |\Delta^2 u|^2 \mathrm{d}x\right)^{1/4}.
\end{aligned}
$$

Combining the above inequality with the Young inequality, we have

$$
\begin{aligned}
\int_\Omega |\Delta u|^2 \mathrm{d}x &\le C \left(\int_\Omega |\nabla u_0|^2 \mathrm{d}x\right)^{2/3} \left(\int_\Omega |\Delta^2 u|^2 \mathrm{d}x\right)^{1/3} \\
&\le \varepsilon \int_\Omega |\Delta^2 u|^2 \mathrm{d}x + C_\varepsilon \int_\Omega |\nabla u_0|^2 \mathrm{d}x,
\end{aligned}
$$

where ε is a positive constant which can be small enough, and C_ε is a positive constant depending on ε.

For the fourth term of the right hand side of (2.4), it follows from the Cagliardo-Nirenberg inequality that

$$
\left(\int_\Omega |\nabla u|^4 \mathrm{d}x\right)^{1/4} \le C \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{5/12} \left(\int_\Omega |\Delta^2 u|^2 \mathrm{d}x\right)^{1/12} + C \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{1/2}.
$$

By (2.1) and the Young inequality, we have

$$\begin{aligned}\int_\Omega |\nabla u|^4 \mathrm{d}x &\le C\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{5/3}\left(\int_\Omega |\Delta^2 u|^2 \mathrm{d}x\right)^{1/3} + C\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^2 \\ &\le C\left(\int_\Omega |\nabla u_0|^2 \mathrm{d}x\right)^{5/3}\left(\int_\Omega |\Delta^2 u|^2 \mathrm{d}x\right)^{1/3} + C\left(\int_\Omega |\nabla u_0|^2 \mathrm{d}x\right)^2 \\ &\le C\int_\Omega |\nabla u_0|^2 \mathrm{d}x \int_\Omega |\Delta^2 u|^2 \mathrm{d}x + C\left(\int_\Omega |\nabla u_0|^2 \mathrm{d}x\right)^2.\end{aligned}$$

For the inequality (2.4), we know from the assumptions of the Theorem 1.1 that for suitably small $\|u_0(x)\|_{H^1(\Omega)}$, there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Delta u|^2 \mathrm{d}x + kM_1 \int_\Omega |\Delta^2 u|^2 \mathrm{d}x \le C.$$

Then

$$\sup_{0<t<T}\int_\Omega |\Delta u(x,t)|^2 \mathrm{d}x \le C, \tag{2.5}$$

$$\iint_{Q_T} |\Delta^2 u|^2 \mathrm{d}x\mathrm{d}t \le C.$$

By the inequality (2.5) and Sobolev embedding theorem, we conclude that there exists a constant $0 < \alpha < 1$ such that

$$|u(x_1,t) - u(x_2,t)| \le C|x_1 - x_2|^\alpha$$

holds for any given $(x_1,t),(x_2,t) \in Q_T$. Then by using the inequality (2.2) and the equation (1.1) itself, we are informed by a completely similar argument of the proof of the Lemma 2.3 of [8] that

$$|u(x,t_1) - u(x,t_2)| \le C|t_1 - t_2|^{\alpha/4}$$

holds for any given $(x,t_1),(x,t_2) \in Q_T$. The proof of this lemma is completed. □

Before giving the proof of the main theorem, we first give the following technical lemma which is required to estimate the Hölder norm of ∇u. One can find its proof in Giaquinta [9].

Lemma 2.2 *Let $\varphi(\rho)$ be a nonnegative and nondecreasing function satisfying*

$$\varphi(\rho) \le A\left(\frac{\rho}{R}\right)^\alpha \varphi(R) + BR^\beta, \quad \forall 0 < \rho \le R \le R_0,$$

where A, B, α, β are positive constants with $\beta < \alpha$. Then there exists a positive constant C depending only on α, β and A, such that

$$\varphi(\rho) \le C\left(\frac{\rho}{R}\right)^\beta \left[\varphi(R) + BR^\beta\right], \quad \forall 0 < \rho \le R \le R_0.$$

Now we are in the position to give the proof of the main result of this paper.

The Proof of Theorem 1.1 We first rewrite the equation (1.1) into the following form

$$\frac{\partial u}{\partial t} + \mathrm{div}(a(x,t)\nabla\Delta u) = \mathrm{div}\vec{F},$$

where $a(x,t) = km(u(x,t))$, $\vec{F} = m(u)\vec{\Phi}(\nabla u)$. The key step of the proof is the Hölder norm estimation of ∇u. In the following text, we employ the theory of Campanato spaces to obtain this key estimation. For any given point (x_0, t_0) in $\Omega \times (0, T)$, we define

$$\varphi(u,\rho) = \iint_{S_\rho} \left(|\nabla u - (\nabla u)_\rho|^2 + \rho^4 |\nabla \Delta u|^2 \right) \mathrm{d}x\mathrm{d}t,$$

where

$$S_\rho = (t_0 - \rho^4, t_0 + \rho^4) \times B_\rho(x_0), \quad (\nabla u)_\rho = \frac{1}{|S_\rho|} \iint_{S_\rho} \nabla u \mathrm{d}x\mathrm{d}t,$$

where $B_\rho(x_0)$ is the ball with radius ρ and center at the point x_0 in the Euclidean space.

We split the solution u of the problem (1.1)–(1.3) on S_R as $u = u_1 + u_2$, where u_1 is the solution of the following problem

$$\frac{\partial u_1}{\partial t} + a(x_0,t_0)\Delta^2 u_1 = 0, \quad (x,t) \in S_R, \tag{2.6}$$

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u}{\partial \nu}, \quad \frac{\partial \Delta u_1}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu}, \quad (x,t) \in (t_0 - R^4, t_0 + R^4) \times \partial B_R(x_0), \tag{2.7}$$

$$u_1 = u, \quad t = t_0 - R^4, \quad x \in B_R(x_0), \tag{2.8}$$

and u_2 solves the problem

$$\frac{\partial u_2}{\partial t} + a(x_0,t_0)\Delta^2 u_2 = \mathrm{div}\,[(a(x_0,t_0) - a(x,t))\nabla \Delta u] + \mathrm{div}\vec{F}, \quad (x,t) \in S_R, \tag{2.9}$$

$$\frac{\partial u_2}{\partial \nu} = 0, \quad \frac{\partial \Delta u_2}{\partial \nu} = 0, \quad (x,t) \in (t_0 - R^4, t_0 + R^4) \times \partial B_R(x_0), \tag{2.10}$$

$$u_2 = 0, \quad t = t_0 - R^4, \quad x \in B_R(x_0). \tag{2.11}$$

By the classical linear theory, the above decomposition is uniquely determined by u.

By the Lemma 2.1 and the assumption (H1), we know that the function $a(x,t)$ is also Hölder continuous. Namely, there exists a $\sigma \in (0,1)$ such that for any $(x,t) \in B_R(x_0) \times (t_0 - R^4, t_0 + R^4)$ we have

$$|a(x,t) - a(x_0,t_0)| \le C \left(|x - x_0|^\sigma + |t - t_0|^{\sigma/4} \right). \tag{2.12}$$

Multiplying both sides of the equation (2.9) by Δu_2 and integrating the resulting with respect to (t,x) on $(t_0 - R^4, t) \times B_R(x_0)$, we know from the inequality (2.12) and the assumptions (H1) and (H2) that

$$\begin{aligned}
&\frac{1}{2} \int_{B_R(x_0)} |\nabla u_2(x,t)|^2 \mathrm{d}x + a(x_0,t_0) \int_{t_0-R^4}^t \int_{B_R(x_0)} |\nabla \Delta u_2|^2 \mathrm{d}x\mathrm{d}t \\
&= \int_{t_0-R^4}^t \int_{B_R(x_0)} \left(a(x_0,t_0) - a(x,t) \right) \nabla \Delta u \nabla \Delta u_2 \mathrm{d}x\mathrm{d}t + \\
&\quad \int_{t_0-R^4}^t \int_{B_R(x_0)} \vec{F} \nabla \Delta u_2 \mathrm{d}x\mathrm{d}t \\
&\le \frac{a(x_0,t_0)}{2} \int_{t_0-R^4}^t \int_{B_R(x_0)} |\nabla \Delta u_2|^2 \mathrm{d}x\mathrm{d}t + C \int_{t_0-R^4}^t \int_{B_R(x_0)} \left| \vec{F} \right|^2 \mathrm{d}x\mathrm{d}t + \\
&\quad C \int_{t_0-R^4}^t \int_{B_R(x_0)} |(a(x,t) - a(x_0,t_0))\nabla \Delta u|^2 \mathrm{d}x\mathrm{d}t \\
&\le \frac{a(x_0,t_0)}{2} \int_{t_0-R^4}^t \int_{B_R(x_0)} |\nabla \Delta u_2|^2 \mathrm{d}x\mathrm{d}t + CR^7 \sup_{S_R} |\nabla u|^2 +
\end{aligned}$$

$$CR^{2\sigma}\int_{t_0-R^4}^{t}\int_{B_R(x_0)}|\nabla\Delta u|^2\,\mathrm{d}x\mathrm{d}t.$$

Then

$$\sup_{(t_0-R^4,t_0+R^4)}\int_{B_R(x_0)}|\nabla u_2(x,t)|^2\mathrm{d}x+\iint_{S_R}|\nabla\Delta u_2|^2\mathrm{d}x\mathrm{d}t$$
$$\leq CR^{2\sigma}\iint_{S_R}|\nabla\Delta u|^2\,\mathrm{d}x\mathrm{d}t+CR^7\sup_{S_R}|\nabla u|^2. \tag{2.13}$$

For u_1, we first know from the Sobolev embedding theorem that, for any $(x_1,t_1),(x_2,t_2)\in S_\rho$, there holds

$$\frac{|\nabla u_1(x_1,t_1)-\nabla u_1(x_2,t_2)|^2}{|x_1-x_2|}$$
$$\leq C\sup_{(t_0-\rho^4,t_0+\rho^4)}\int_{B_\rho(x_0)}\left(\rho^{-4}\left|\nabla u_1-(\nabla u_1)_\rho\right|^2+\rho|\nabla\Delta^k u_1|^2\right)\mathrm{d}x,$$

where k is a positive constant which is not less than $5/4$. Then by the equation (2.6) itself, we have

$$\frac{|\nabla u_1(x_1,t_1)-\nabla u_1(x_2,t_2)|^2}{|x_1-x_2|+|t_1-t_2|^{1/4}}$$
$$\leq C\sup_{(t_0-\rho^4,t_0+\rho^4)}\int_{B_\rho(x_0)}\left(\rho^{-4}\left|\nabla u_1-(\nabla u_1)_\rho\right|^2+\rho|\nabla\Delta^k u_1|^2\right)\mathrm{d}x+$$
$$C\iint_{S_\rho}\left(\rho^{-4}|\nabla\Delta u_1|^2+\rho|\nabla\Delta^{k+1}u_1|\right)\mathrm{d}x\mathrm{d}t, \tag{2.14}$$

where k is a positive constant which is not less than $5/4$. Similarly to the proof of the Lemma 4.4 in [1], we know that the following Caccioppoli type inequalities hold

$$\sup_{(t_0-\rho^4,t_0+\rho^4)}\int_{B_\rho(x^0)}|\nabla u_1-(\nabla u_1)_R|^2\mathrm{d}x+\iint_{S_\rho}|\nabla\Delta u_1|^2\mathrm{d}x\mathrm{d}t$$
$$\leq\frac{C}{(R-\rho)^4}\iint_{S_R}|\nabla u_1-(\nabla u_1)_R|^2\mathrm{d}x\mathrm{d}t, \tag{2.15}$$

$$\sup_{(t_0-\rho^4,t_0+\rho^4)}\int_{B_\rho(x^0)}|\Delta u_1|^2\mathrm{d}x+\iint_{S_\rho}|\Delta^2 u_1|^2\mathrm{d}x\mathrm{d}t$$
$$\leq\frac{C}{(R-\rho)^4}\iint_{S_R}|\Delta u_1|^2\mathrm{d}x\mathrm{d}t\leq\frac{C}{(R-\rho)^6}\iint_{S_{2R}}|\nabla u_1-(\nabla u_1)_R|^2\mathrm{d}x\mathrm{d}t, \tag{2.16}$$

$$\sup_{(t_0-\rho^4,t_0+\rho^4)}\int_{B_\rho(x^0)}|\nabla\Delta u_1|^2\mathrm{d}x+\iint_{S_\rho}|\nabla\Delta^2 u_1|^2\mathrm{d}x\mathrm{d}t$$
$$\leq\frac{C}{(R-\rho)^4}\iint_{S_R}|\nabla\Delta u_1|^2\mathrm{d}x\mathrm{d}t. \tag{2.17}$$

Then, by (2.15) and (2.17), we have

$$\sup_{(t_0-(R/2)^4,t_0+(R/2)^4)}\int_{B_{R/2}}|\nabla\Delta^n u_1|^2\mathrm{d}x\leq\frac{C}{R^{4n+4}}\iint_{S_R}|\nabla u_1-(\nabla u_1)_R|^2\mathrm{d}x, \tag{2.18}$$

and

$$\iint_{S_{R/2}} |\nabla \Delta^{n+1} u_1|^2 \mathrm{d}x \le \frac{C}{R^{4n+4}} \iint_{S_R} |\nabla u_1 - (\nabla u_1)_R|^2 \mathrm{d}x \tag{2.19}$$

hold for all positive integer n.

Note that for any $R/2 < \rho < R$, we only need to take $C = 2^8$ to obtain

$$\varphi(u_1, \rho) \le C \left(\frac{\rho}{R}\right)^8 \varphi(u_1, R).$$

If $0 < \rho \le R/2$, by the mean value theorem we know that there exists a point $(x^*, t_*) \in S_\rho$ such that

$$(\nabla u_1)_\rho = \nabla u_1(x^*, t_*).$$

Then, by the inequalities (2.14)–(2.19) and noticing that $k \ge 5/4$, we have

$$\begin{aligned}
&\iint_{S_\rho} |\nabla u_1 - (\nabla u_1)_\rho|^2 \mathrm{d}x\mathrm{d}t = \iint_{S_\rho} |\nabla u_1 - \nabla u_1(x^*, t_*)|^2 \mathrm{d}x\mathrm{d}t \le C\rho^7 \sup_{(x,t)\in S_\rho} |\nabla u_1 - \nabla u_1(x^*, t_*)|^2 \\
&\le C\rho^8 \sup_{t\in(t_0-(R/2)^4, t_0+(R/2)^4)} \int_{B_{R/2}(x^0)} \left(R^{-4}|\nabla u_1 - (\nabla u_1)_R|^2 + R|\nabla \Delta^k u_1|^2\right) \mathrm{d}x + \\
&\quad C\rho^8 \iint_{S_{R/2}} (R^{-4}|\nabla \Delta u_1|^2 + R|\nabla \Delta^{k+1} u_1|^2) \mathrm{d}x\mathrm{d}t \\
&\le C\rho^8 \iint_{S_R} \left(\frac{1}{R^8} + \frac{1}{R^{4k+3}}\right) |\nabla u_1 - (\nabla u_1)_R|^2 \mathrm{d}x\mathrm{d}t \\
&\le C \left(\frac{\rho}{R}\right)^8 \iint_{S_R} |\nabla u_1 - (\nabla u_1)_R|^2 \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Thus

$$\varphi(u_1, \rho) \le C \left(\frac{\rho}{R}\right)^8 \varphi(u_1, R), \quad \forall \rho \in (0, R). \tag{2.20}$$

By the inequalities (2.13) and (2.20) and using the Lemma 2.2, we conclude that for any constant $\lambda \in (7, 8)$ there holds

$$\varphi(u, \rho) \le C \left(1 + \sup_{S_R} |\nabla u|^2\right) \rho^\lambda, \quad \forall 0 < \rho \le R \le R_0, \tag{2.21}$$

where $R_0 \triangleq \min\{\mathrm{dist}(x_0, \partial\Omega), t_0^{1/4}\}$. The proof of the above inequality is completely similar to the proof of the Lemma 4.7 in [1], and we omit the details here.

By the inequality (2.21), and noticing the integral character of the Hölder continuous functions, we have

$$|\nabla u(x_1, t_1) - \nabla(x_2, t_2)| \le C \left(1 + \sup_{S_R} |\nabla u|\right) \left(|x_1 - x_2|^{(\lambda-7)/2} + |t_1 - t_2|^{(\lambda-7)/8}\right),$$

where $(x_1, t_1), (x_2, t_2)$ are any given two points in S_R. It follows from the interpolation inequality that

$$|\nabla u(x_1, t_1) - \nabla u(x_2, t_2)| \le C \left(|x_1 - x_2|^{(\lambda-7)/2} + |t_1 - t_2|^{(\lambda-7)/8}\right).$$

For the estimations near the boundary of Q_T, we can obtain them by the same method. Let $(x^0, t_0) \in \partial\Omega \times (0, T)$ be fixed and assume that $\partial\Omega$ can be explicitly expressed by a function

$x_2 = \phi(x_1)$ in some neighborhood of x^0. We split u as $u_1 + u_2$ in $\hat{S_R} = (t_0 - R^4, t_0 + R^4) \times \Omega_R(x_0)$ with $\Omega_R(x_0) = B_R(x^0) \cap \Omega$. u_1 solves the following problem

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} + a(x_0, t_0)\Delta^2 u_1 = 0, \quad (x,t) \in \hat{S_R}, \\
&\frac{\partial u_1}{\partial \nu} = \frac{\partial u}{\partial \nu}, \quad \frac{\partial \Delta u_1}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu}, \quad (x,t) \in \partial\Omega_R(x_0) \times (t_0 - R^4, t_0 + R^4), \\
&u_1 = u, \quad t = t_0 - R^4, \quad x \in \Omega_R(x_0),
\end{aligned}$$

and u_2 solves the problem

$$\begin{aligned}
&\frac{\partial u_2}{\partial t} + a(x_0, t_0)\Delta^2 u_2 = \operatorname{div}\left[(a(x_0, t_0) - a(x,t))\nabla\Delta u\right] + \operatorname{div}\vec{F}, \quad (x,t) \in \hat{S_R}, \\
&\frac{\partial u_2}{\partial \nu} = 0, \quad \frac{\partial \Delta u_2}{\partial \nu} = 0, \quad (x,t) \in \partial\Omega_R(x_0) \times (t_0 - R^4, t_0 + R^4), \\
&u_2 = 0, \quad t = t_0 - R^4, \quad x \in \Omega_R(x_0).
\end{aligned}$$

We can modify the function $\varphi(u, \rho)$ as

$$\varphi(u, \rho) = \iint_{S_\rho} \left(|\partial_n u|^2 + |\partial_\tau u - (\partial_\tau u)_\rho|^2 + \rho^4 |\nabla\Delta u|^2\right) \mathrm{d}x\mathrm{d}t,$$

where

$$\partial_n = \phi'(x)\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}, \quad \partial_\tau = \frac{\partial}{\partial x_1} + \phi'(x)\frac{\partial}{\partial x_2}$$

denote the normal and tangential derivatives, respectively. The remaining part of the proof is completely similar to that of Theorem 1.1 of [8]. Namely, we can employ the Leray-Schauder fixed point theorem to prove the existence of the solutions and use the Holmgren's approach to prove the uniqueness of the solutions. We omit the details here. The proof of the theorem is completed. □

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