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Characteristic Conditions for the Generation of α-Times Resolvent Families on a Hilbert Space

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Abstract In 2000, Shi and Feng gave the characteristic conditions for the generation of C_0-semigroups on a Hilbert space. In this paper, we will extend them to the generation of α-times resolvent operator families. Such characteristic conditions can be applied to show rank-1 perturbation theorem and relatively-bounded perturbation theorem for α-times resolvent operator families.

Keywords α-times resolvent family; resolvent; rank-1 perturbation; relatively-bounded perturbation.

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1. Introduction

The Hille-Yosida theorem told us how to characterize an operator which is generator of some strongly continuous semigroup. However, it is hard to use the Hille-Yosida theorem to check whether an operator generates a C_0-semigroup. In fact, the difficulty is in finding the expression of $R^k(\lambda, A)$ and their estimates for all $k \geq 2$. Shi and Feng in 2000 gave a new necessary and sufficient condition in terms of $R(\lambda, A)$ and $R(\lambda, A^*)$ which makes sure that A generates a C_0-semigroup on a Hilbert space. Such condition is easy to verify and convenient to use.

The notion of α-times resolvent families was introduced by Bajlecova [2] to study the Cauchy problem of fractional order:

$$D_t^{\alpha} u(t) = Au(t).$$

It is known that the class of α-times resolvent operator families interpolates C_0-semigroups and cosine functions. So it is also interesting to consider the characterization of the generators of these families on Hilbert spaces.

Let us first recall the definitions of α-times resolvent operator families. Let A be a closed densely defined linear operator on a Banach space X and $\alpha \in (0, 2]$.

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Definition 1.1 *A family $S_\alpha(t) \subset B(X)$ is called an α-times resolvent operator family for A if the following conditions are satisfied:*

1) *$S_\alpha(t)$ is strongly continuous for $t \geq 0$ and $S_\alpha(0) = I$;*
2) *$S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for $x \in D(A)$ and $t \geq 0$;*
3) *For $x \in D(A)$, $S_\alpha(t)x$ satisfies*

$$S_\alpha(t)x = x + \int_0^t g_\alpha(t-s)S_\alpha(s)Ax\mathrm{d}s, \quad t \geq 0,$$

where $g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$.
If $\|S_\alpha(t)\| \leq M_A e^{\omega_A t}$ where $M_A \geq 1$, $\omega_A \geq 0$, we write as $A \in C^\alpha(M_A, \omega_A)$ (or shortly $A \in C^\alpha$).

Lemma 1.2 ([2]) *Let $0 \leq \alpha \leq 2$. Then $A \in C^\alpha(M_A, \omega_A)$ if and only if $(\omega_A^\alpha, \infty) \subset \rho(A)$ and there is a strongly continuous operator-valued function $S(t)$ satisfying $\|S(t)\| \leq M_A e^{\omega_A t}$, $t \geq 0$, and such that*

$$\lambda^{\alpha-1}R(\lambda^\alpha, A)x = \int_0^\infty e^{-\lambda t}S(t)x\mathrm{d}t, \quad \lambda > \omega_A,\ x \in X.$$

In Section 2, we will give characteristic conditions for α-times resolvent operator families on a Hilbert space, which extends the result for C_0-semigroups [1]. As an application, we will show some perturbation theorems in Section 3.

2. Characteristic conditions of the generation

Theorem 2.1 *Let A be a closed densely defined linear operator on a Hilbert space H and $\alpha \in [1,2]$. Then the following statements are equivalent:*

1) *$A \in C^\alpha$;*
2) *There is a constant $\omega_A \in \mathbb{R}$ such that $\{\lambda^\alpha : \mathrm{Re}\,\lambda > \omega_A\} \subseteq \rho(A)$,*

$$\sup_{\omega > \omega_A} (\omega - \omega_A) \int_{\mathbb{R}} \|(\omega + i\tau)^{\alpha-1}R((\omega+i\tau)^\alpha, A)x\|^2 \mathrm{d}\tau < +\infty, \quad \forall x \in H, \tag{1}$$

and

$$\sup_{\omega > \omega_A} (\omega - \omega_A) \int_{\mathbb{R}} \|(\omega - i\tau)^{\alpha-1}R((\omega-i\tau)^\alpha, A^*)y\|^2 \mathrm{d}\tau < +\infty, \quad \forall y \in H. \tag{2}$$

In order to prove the theorem, we need the following lemmas.

Lemma 2.2 *If (1) and (2) hold, then for every $x \in H$, $\omega > \omega_A$,*

$$\|\lambda^{\alpha-1}R(\lambda^\alpha, A)x\| \to 0, \text{ when } \mathrm{Re}\lambda > \omega \text{ and } |\lambda| \to +\infty.$$

Proof Without loss of generality, we assume that $\omega_A \geq 0$. Under the conditions (1) and (2), we have

$$\int_{\mathbb{R}} \|(\omega + i\tau)^{\alpha-1}R((\omega+i\tau)^\alpha, A)x\|^2 \mathrm{d}\tau \leq \frac{M_1^2}{\omega - \omega_A}\|x\|^2, \quad \forall x \in H, \tag{3}$$

$$\int_{\mathbb{R}} \|(\omega - i\tau)^{\alpha-1}R((\omega-i\tau)^\alpha, A^*)y\|^2 \mathrm{d}\tau \leq \frac{M_2^2}{\omega - \omega_A}\|y\|^2, \quad \forall y \in H, \tag{4}$$

for some constants M_1, $M_2>0$. By the Schwartz inequality and Cauchy inequality,

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\|(\omega+i\tau)^{2\alpha-2}R^2((\omega+i\tau)^\alpha,A)x\|\mathrm{d}\tau\\
&=\sup_{y\in H,\|y\|=1}\int_{\tau_1}^{\tau_2}((\omega+i\tau)^{2\alpha-2}R^2((\omega+i\tau)^\alpha,A)x,y)_H\mathrm{d}\tau\\
&=\sup_{y\in H,\|y\|=1}\int_{\tau_1}^{\tau_2}((\omega+i\tau)^{\alpha-1}R((\omega+i\tau)^\alpha,A)x,(\omega-i\tau)^{\alpha-1}R((\omega-i\tau)^\alpha,A^*)y)_H\mathrm{d}\tau\\
&\le\sup_{y\in H,\|y\|=1}\Big(\int_{\tau_1}^{\tau_2}\|(\omega+i\tau)^{\alpha-1}R((\omega+i\tau)^\alpha,A)x\|^2\mathrm{d}\tau\Big)^{1/2}\cdot\\
&\quad\Big(\int_{\tau_1}^{\tau_2}\|(\omega-i\tau)^{\alpha-1}R((\omega-i\tau)^\alpha,A^*)y\|^2\mathrm{d}\tau\Big)^{1/2}\\
&\le\frac{M_1M_2}{\omega-\omega_A}\|x\|,\quad\forall\tau_1,\ \tau_2\in\mathbb{R}.
\end{aligned}$$

Therefore, the integral $\int_{\mathbb{R}}(\omega+i\tau)^{2\alpha-2}R^2((\omega+i\tau)^\alpha,A)x\mathrm{d}\tau$ exists. Similarly,

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\|(\omega+i\tau)^{\alpha-2}R((\omega+i\tau)^\alpha,A)x\|\mathrm{d}\tau\\
&=\sup_{y\in H,\|y\|=1}\int_{\tau_1}^{\tau_2}((\omega+i\tau)^{\alpha-2}R((\omega+i\tau)^\alpha,A)x,y)_H\mathrm{d}\tau\\
&=\sup_{y\in H,\|y\|=1}\int_{\tau_1}^{\tau_2}((\omega+i\tau)^{\alpha-1}R((\omega+i\tau)^\alpha,A)x,(\omega-i\tau)^{-1}y)_H\mathrm{d}\tau\\
&\le\sup_{y\in H,\|y\|=1}\Big(\int_{\tau_1}^{\tau_2}\|(\omega+i\tau)^{\alpha-1}R((\omega+i\tau)^\alpha,A)x\|^2\mathrm{d}\tau\Big)^{1/2}\cdot\\
&\quad\Big(\int_{\tau_1}^{\tau_2}\|(\omega-i\tau)^{-1}y\|^2\mathrm{d}\tau\Big)^{1/2}\\
&\le\frac{M_1\|x\|}{(\omega-\omega_A)^{1/2}}\Big(\frac{\arctan\frac{\tau_2}{\omega}-\arctan\frac{\tau_1}{\omega}}{\omega}\Big)^{1/2}\le\frac{M_1M_3}{\omega-\omega_A}\|x\|,\quad\forall\tau_1,\ \tau_2\in\mathbb{R},
\end{aligned}$$

where $M_3=(\arctan\frac{\tau_2}{\omega}-\arctan\frac{\tau_1}{\omega})^{1/2}$. Thus the integral $\int_{\mathbb{R}}(\omega+i\tau)^{\alpha-2}R((\omega+i\tau)^\alpha,A)x\mathrm{d}\tau$ is also convergent. Since

$$\begin{aligned}
&(\omega+i\tau_1)^{\alpha-1}R((\omega+i\tau_1)^\alpha,A)x\\
&=(\omega+i\tau_0)^{\alpha-1}R((\omega+i\tau_0)^\alpha,A)x+i(\alpha-1)\int_{\tau_0}^{\tau_1}(\omega+i\tau)^{\alpha-2}R((\omega+i\tau)^\alpha,A)x\mathrm{d}\tau-\\
&\quad i\alpha\int_{\tau_0}^{\tau_1}(\omega+i\tau)^{2\alpha-2}R^2((\omega+i\tau)^\alpha,A)x\mathrm{d}\tau,
\end{aligned}$$

the limit $\lim_{|\tau|\to\infty}(\omega+i\tau)^{\alpha-1}R((\omega+i\tau)^\alpha,A)x$ exists. Together with (3), we have

$$\lim_{|\tau|\to\infty}(\omega+i\tau)^{\alpha-1}R((\omega+i\tau)^\alpha,A)x=0,\quad\forall x\in H,\ \omega>\omega_A.$$

Moreover,

$$\begin{aligned}
&\|(\omega+i\tau_1)^{\alpha-1}R((\omega+i\tau_1)^\alpha,A)x\|\\
&\le\|(\omega+i\tau_0)^{\alpha-1}R((\omega+i\tau_0)^\alpha,A)x\|+\frac{\alpha M_1M_2}{\omega-\omega_A}\|x\|+\frac{(\alpha-1)M_1M_3}{\omega-\omega_A}\|x\|
\end{aligned}$$

$$\leq \|(\omega+i\tau_0)^{\alpha-1}R((\omega+i\tau_0)^\alpha, A)x\| + \frac{M_4}{\omega-\omega_A}\|x\|,$$

where $M_4=\alpha M_1M_2+(\alpha-1)M_1M_3$. Letting $\tau_0\to\infty$ gives

$$\|(\omega+i\tau)^{\alpha-1}R((\omega+i\tau)^\alpha, A)\| \leq \frac{M_4}{\omega-\omega_A}.$$

Let $\omega_1>\omega_A$. If $\omega\geq\max\{\omega_1,\tau\}$, then $\omega\to\infty$ if and only if $|\lambda|\to\infty$,

$$\|\lambda^{\alpha-1}R(\lambda^\alpha, A)x\| \leq \frac{M_4}{\omega-\omega_A}\|x\|\to 0, \quad \text{as} \quad |\lambda|\to\infty.$$

Otherwise if $\omega_1\leq\omega\leq|\tau|$, without loss of generality, we assume ω is bounded, then $|\tau|\to\infty$ if and only if $|\lambda|\to\infty$. Since

$$\begin{aligned}&(\omega+i\tau)^{\alpha-1}R((\omega+i\tau)^\alpha, A)x-(\omega_1+i\tau)^{\alpha-1}R((\omega_1+i\tau)^\alpha, A)x\\ &=(\omega+i\tau)^{\alpha-1}R((\omega+i\tau)^\alpha, A)\frac{(\omega_1+i\tau)^\alpha-(\omega+i\tau)^\alpha}{(\omega_1+i\tau)^{\alpha-1}}(\omega_1+i\tau)^{\alpha-1}R((\omega_1+i\tau)^\alpha, A)x+\\ &\quad[(\frac{\omega+i\tau}{\omega_1+i\tau})^{\alpha-1}-1](\omega_1+i\tau)^{\alpha-1}R((\omega_1+i\tau)^\alpha, A)x,\end{aligned}$$

the limit $\|(\frac{\omega+i\tau}{\omega_1+i\tau})^{\alpha-1}-1\|\to 0$ as $|\tau|\to\infty$ implies

$$\|[(\frac{\omega+i\tau}{\omega_1+i\tau})^{\alpha-1}-1](\omega+i\tau)^{\alpha-1}R((\omega_1+i\tau)^\alpha, A)x\|\to 0 \quad \text{as} \quad |\tau|\to\infty. \tag{5}$$

It is easy to know that $\|(\omega+i\tau)^{\alpha-1}R((\omega+i\tau)^\alpha, A)\|$ is bounded because of the boundedness of ω. Moreover, $\|\frac{(\omega_1+i\tau)^\alpha-(\omega+i\tau)^\alpha}{(\omega+i\tau)^{\alpha-1}}\|\to\alpha|\omega_1-\omega|$ as $|\tau|\to\infty$, we get

$$\|(\omega+i\tau)^{\alpha-1}R((\omega+i\tau)^\alpha, A)\frac{(\omega_1+i\tau)^\alpha-(\omega+i\tau)^\alpha}{(\omega+i\tau)^{\alpha-1}}(\omega+i\tau)^{\alpha-1}R((\omega_1+i\tau)^\alpha, A)x\|\to 0$$

$$\text{as} \quad |\tau|\to\infty. \tag{6}$$

By (5) and (6), we can obtain that $\|\lambda^{\alpha-1}R(\lambda^\alpha, A)x\|\to 0$ as $|\lambda|\to\infty$ following by $\|(\omega_1+i\tau)^{\alpha-1}R((\omega_1+i\tau)^\alpha, A)x\|\to 0$ as $|\tau|\to\infty$. From the above discussion, the desired is obtained. □

Lemma 2.3 *Let A be a closed densely defined linear operator on a Hilbert space H. If $A\in C^\alpha(M_A,\omega_A)$, then $A^*\in C^\alpha(M_A,\omega_A)$.*

Proof We will show that $S_\alpha^*(t):=(S_\alpha(t))^*$ is the α-times resolvent operator family generated by A^*. If $y\in D(A^*)$, then for $T\geq t>s\geq 0$ and any $x\in H$,

$$\begin{aligned}|(x,S_\alpha^*(t)y-S_\alpha^*(s)y)| &=|(S_\alpha(t)x-S_\alpha(s)x,y)|=\Big|(A\int_s^t g_\alpha(t-\tau)S_\alpha(\tau)x\mathrm{d}\tau,y)\Big|\\ &=\Big|(\int_s^t g_\alpha(t-\tau)S_\alpha(\tau)x\mathrm{d}\tau,A^*y)\Big|\leq M_T(t-s)\|x\|\|A^*y\|,\end{aligned}$$

where M_T is a constant depending on T. This shows that $t\mapsto S_\alpha^*(t)y$ is continuous. Since $D(A^*)$ is dense [4], we show that $S_\alpha^*(t)x$ is continuous for all $x\in H$. Moreover $\|S_\alpha^*(t)\|=\|S_\alpha(t)\|$, by Lemma 1.2 we obtain that $S_\alpha^*(t)$ is the α-times resolvent family generated by A^*. □

The Proof of Theorem 2.1 1)$\Longrightarrow$2). For every $\omega>\omega_A$,

$$(\omega+i\tau)^{\alpha-1}R((\omega+i\tau)^\alpha,A)x=\int_0^{+\infty}e^{-(\omega+i\tau)t}S_\alpha(t)x\mathrm{d}t=\int_0^{+\infty}e^{-i\tau t}(e^{-\omega t}S_\alpha(t)x)\mathrm{d}t$$
$$=\widehat{e^{-\omega t}S_\alpha(t)}x(\tau).$$

Since $\|e^{-\omega t}S_\alpha(t)\|\le M_Ae^{(\omega_A-\omega)t}$, $e^{-\omega t}S_\alpha(t)x\in L^2(\mathbb{R}_+)$, by using the Plancherel's theorem [6], we obtain

$$\begin{aligned}&\int_{\mathbb{R}}\|(\omega+i\tau)^{\alpha-1}R((\omega+i\tau)^\alpha,A)x\|^2\mathrm{d}\tau\\&=\int_{\mathbb{R}}\|\widehat{e^{-\omega t}S_\alpha(t)}x(\tau)\|^2\mathrm{d}\tau=2\pi\int_0^{+\infty}e^{-2\omega t}\|S_\alpha(t)x\|^2\mathrm{d}t\\&\le 2\pi\|x\|^2\int_0^{+\infty}M_A^2e^{-2(\omega-\omega_A)t}\mathrm{d}t=\frac{\pi M_A^2}{\omega-\omega_A}\|x\|^2.\end{aligned}$$

This means that (1) holds. And (2) follows by Lemma 2.3.

2)$\Longrightarrow$1). Fix $\omega>\omega_A$ and define the linear operator $S_\omega(t)$ by

$$S_\omega(t)x=\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha,A)x\mathrm{d}\lambda=\frac{e^{\omega t}}{2\pi}\int_{-\infty}^{+\infty}e^{it\tau}(\omega+i\tau)^{\alpha-1}R((\omega+i\tau)^\alpha,A)x\mathrm{d}\tau.$$

Since

$$\begin{aligned}&\int_{\tau_1}^{\tau_2}e^{it\tau}(\omega+i\tau)^{\alpha-1}R((\omega+i\tau)^\alpha,A)x\mathrm{d}\tau\\&=\frac{e^{it\tau}}{it}(\omega+i\tau)^{\alpha-1}R((\omega+i\tau)^\alpha,A)x|_{\tau_1}^{\tau_2}-\frac{\alpha-1}{t}\int_{\tau_1}^{\tau_2}e^{it\tau}(\omega+i\tau)^{\alpha-2}R((\omega+i\tau)^\alpha,A)x\mathrm{d}\tau+\\&\quad\frac{\alpha}{t}\int_{\tau_1}^{\tau_2}e^{it\tau}(\omega+i\tau)^{2\alpha-2}R^2((\omega+i\tau)^\alpha,A)x\mathrm{d}\tau,\end{aligned}$$

similarly to the proof of Lemma 2.2, we can obtain that the integral

$$\frac{e^{\omega t}}{2\pi}\int_{\mathbb{R}}e^{it\tau}(\omega+i\tau)^{\alpha-1}R((\omega+i\tau)^\alpha,A)x\mathrm{d}\tau$$

converges, and

$$\left\|\frac{e^{\omega t}}{2\pi}\int_{\mathbb{R}}e^{it\tau}(\omega+i\tau)^{\alpha-1}R((\omega+i\tau)^\alpha,A)x\mathrm{d}\tau\right\|\le\frac{M_4e^{\omega t}}{2\pi t(\omega-\omega_A)}\|x\|.$$

This means that $S_\omega(t)$ is a linear bounded operator. Now we verify that $S_\omega(t)$ is an α-times resolvent family generated by A.

1) We show that $S_\omega(t)$ is independent of ω. Choose $\omega_1>\omega_A$. Without loss of generality, assume $\omega_1>\omega$.

For every $\beta>0$, let $\Gamma_\beta:=\{\omega_1+i\tau,-\beta\le\tau\le\beta\}\cup\{s+i\beta,\omega\le s\le\omega_1\}\cup\{\omega+i\tau,-\beta\le\tau\le\beta\}\cup\{s-i\beta,\omega\le s\le\omega_1\}$ be oriented counterclockwise, and denote them by Γ_1, Γ_2, Γ_3, Γ_4, respectively. By Cauchy's theorem,

$$\int_{\Gamma_\beta}e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha,A)x\mathrm{d}\lambda=0.$$

That is

$$\Big(\int_{\Gamma_1}-\int_{\Gamma_2}-\int_{\Gamma_3}+\int_{\Gamma 4}\Big)e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha,A)x\mathrm{d}\lambda=0. \tag{7}$$

By Lemma 2.2,

$$\begin{aligned}\Big\|\int_{\Gamma_2}e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha,A)x\mathrm{d}\lambda\Big\| &= \Big\|\int_{\omega}^{\omega_1}e^{(s+i\beta)t}(s+i\beta)^{\alpha-1}R((s+i\beta)^\alpha,A)x\mathrm{d}s\Big\| \\ &\le e^{\omega_1 t}\int_{\omega}^{\omega_1}\|(s+i\beta)^{\alpha-1}R((s+i\beta)^\alpha,A)x\|\mathrm{d}s\to 0,\quad \text{as}\quad \beta\to\infty,\end{aligned}$$

and similarly,

$$\Big\|\int_{\Gamma_4}e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha,A)x\mathrm{d}\lambda\Big\|\to 0,\quad \text{as}\quad \beta\to\infty.$$

Letting $\beta\to\infty$ in (7) yields

$$\lim_{\beta\to\infty}\Big(\int_{\Gamma_1}-\int_{\Gamma_3}\Big)e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha,A)x\mathrm{d}\lambda=0.$$

Therefore

$$\int_{\omega-i\infty}^{\omega+i\infty}e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha,A)x\mathrm{d}\lambda=\int_{\omega_1-i\infty}^{\omega_1+i\infty}e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha,A)x\mathrm{d}\lambda,$$

that is to say $S_\omega(t)=S_{\omega_1}(t)$. So we can denote $S_\omega(t)$ as $S(t)$.

2) We show that $S(t)$ is exponential bounded. We have

$$\|S(t)\|\le\inf_{\omega>\omega_A}\frac{M_4e^{\omega t}}{2\pi t(\omega-\omega_A)}=\frac{M_4e^{\omega_A t}}{2\pi}\inf_{\omega>\omega_A}\frac{e^{(\omega-\omega_A)t}}{(\omega-\omega_A)t}.$$

Let $f(x)=\frac{e^x}{x}$. Then $f'(x)=\frac{e^x(x-1)}{x^2}$. Thus $\frac{e^x}{x}$ gets its minimum at $x=1$, so we have

$$\|S(t)\|\le\frac{M_4e}{2\pi}e^{\omega_A t}=M_Ae^{\omega_A t}.$$

3) We show that $S(t)$ is strongly continuous. Let $x\in D(A)$. Then

$$\begin{aligned}S(t)x-x&=\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha,A)x\mathrm{d}\lambda-x\\ &=\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{e^{\lambda t}R(\lambda^\alpha,A)Ax}{\lambda}\mathrm{d}\lambda+\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{e^{\lambda t}}{\lambda}x\mathrm{d}\lambda-x\\ &=\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{e^{\lambda t}R(\lambda^\alpha,A)Ax}{\lambda}\mathrm{d}\lambda.\end{aligned}$$

Since

$$\int_{\omega-i\infty}^{\omega+i\infty}\frac{\|e^{\lambda t}R(\lambda^\alpha,A)Ax\|}{|\lambda|}\mathrm{d}\lambda\le e^{\omega t}\Big(\int_{\omega-i\infty}^{\omega+i\infty}\|\lambda^{\alpha-1}R(\lambda^\alpha,A)Ax\|^2\mathrm{d}\lambda\Big)^{1/2}\Big(\int_{\omega-i\infty}^{\omega+i\infty}\frac{1}{|\lambda^\alpha|^2}\mathrm{d}\lambda\Big)^{1/2},$$

it follows from (1) and $\alpha\ge 1$ that the integral $\int_{\omega-i\infty}^{\omega+i\infty}\frac{\|e^{\lambda t}R(\lambda^\alpha,A)Ax\|}{|\lambda|}\mathrm{d}\lambda$ is convergent. Hence, we have by the Lebesgue's dominated convergence theorem and Cauchy theorem that

$$\lim_{t\to 0^+}\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{e^{\lambda t}R(\lambda^\alpha,A)Ax}{\lambda}\mathrm{d}\lambda=\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{R(\lambda^\alpha,A)Ax}{\lambda}\mathrm{d}\lambda=0.$$

The strong continuity follows from the fact that $D(A)$ is dense and $S(t)$ is bounded near zero by 2).

4) For every $x \in H$, $\omega_1 > \omega > \omega_A$,

$$\begin{aligned}\int_0^{+\infty} e^{-\omega_1 t} S(t)x\mathrm{d}t &= \int_0^{+\infty} e^{-\omega_1 t}\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha, A)x\mathrm{d}\lambda\mathrm{d}t\\ &=\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\int_0^{+\infty} e^{(\lambda-\omega_1)t}\mathrm{d}t\lambda^{\alpha-1}R(\lambda^\alpha, A)x\mathrm{d}\lambda\\ &=\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{\lambda^{\alpha-1}R(\lambda^\alpha, A)x}{\omega_1-\lambda}\mathrm{d}\lambda\\ &=\omega_1^{\alpha-1}R(\omega_1^\alpha, A)x.\end{aligned}$$

From all the verifications above and by Lemma 1.2, we obtain $A \in C^\alpha$. □

3. Perturbations of generators of α-times resolvent families

In this section, we will use the characteristic conditions given in the previous section to study the perturbations of α-times resolvent families. First we consider rank-1 perturbation.

Definition 3.1 *Let A be a closed densely defined linear operator on a Banach space X, given $a \in X$, $b^* \in X^*$. We call the operator B a rank-1 perturbation of A, if*

$$Bx = b^*(A_\beta x)a, \text{ where } x \in D(A),\ \beta > 0,\ A_\beta = (\sigma - A)^\beta,\ \sigma > \omega_A^\alpha.$$

We denote this operator B by ab^*A_β.

Theorem 3.2 *Let A be a closed densely defined linear operator on a Hilbert space H and $\alpha \in [1, 2]$. If $A \in C^\alpha$ and A satisfies $\|R(\sigma + is, A)\| = O(|s|^{-\beta})$ as $|s| \to \infty$ for $\beta > 0$, B is a rank-1 perturbation operator, then*

1) There exists an $r \geq 0$, such that $\{\mathrm{Re}\mu \geq \sigma, |\mu| \geq r\} \subset \rho(A+B)$ and $\|R(\sigma+is, A+B)\| = O(|s|^{-\beta})$ as $|s| \to \infty$;

2) $A + B \in C^\alpha$.

Proof 1) The proof is similar to Proposition 4.2(1) in [3], so we omit it.

2) By Lemma 1.1 in [3], we obtain that if $\lambda^\alpha \in \rho(A)$ and $b^*A_\beta R(\lambda^\alpha, A)a \neq 1$, then $\lambda^\alpha \in \rho(A + ab^*A_\beta)$ and

$$R(\lambda^\alpha, A + ab^*A_\beta)x = R(\lambda^\alpha, A)x + \frac{b^*A_\beta R(\lambda^\alpha, A)x}{1 - b^*A_\beta R(\lambda^\alpha, A)a}R(\lambda^\alpha, A)a.$$

Set

$$Q(\lambda^\alpha) = \frac{b^*A_\beta R(\lambda^\alpha, A)}{1 - b^*A_\beta R(\lambda^\alpha, A)a}R(\lambda^\alpha, A)a.$$

From the proof of Proposition 4.2(1) in [3], we have that if $|\lambda^\alpha|$ is sufficiently large, then $\lambda^\alpha \in \rho(A + B)$ and

$$\lambda^{\alpha-1}R(\lambda^\alpha, A + B) = \lambda^{\alpha-1}R(\lambda^\alpha, A) + \lambda^{\alpha-1}Q(\lambda^\alpha),$$

where $\|\lambda^{\alpha-1}Q(\lambda^\alpha)\| \leq c\|\lambda^{\alpha-1}R(\lambda^\alpha, A)a\|$. Thus

$$\|\lambda^{\alpha-1}R(\lambda^\alpha, A + B)\| \leq (1 + c\|a\|)\|\lambda^{\alpha-1}R(\lambda^\alpha, A)\|.$$

Since $A \in C^\alpha$ for $x \in H$ and $\omega > \omega_A$, by the proof of Theorem 2.1 (1)$\Longrightarrow$ 2))

$$\int_{\mathbb{R}} \|(\omega + i\tau)^{\alpha-1} R((\omega + i\tau)^\alpha, A)x\|^2 \mathrm{d}\tau \leq \frac{\pi M_A^2}{\omega - \omega_A} \|x\|^2.$$

If ω is sufficiently large,

$$\int_{\mathbb{R}} \|\lambda^{\alpha-1} Q(\lambda^\alpha)\|^2 \mathrm{d}\tau \leq c^2 \int_{-\infty}^{+\infty} \|(\omega + i\tau)^{\alpha-1} R((\omega + i\tau)^\alpha, A)ax\|^2 \mathrm{d}\tau \leq \frac{\pi c^2 \|a\|^2 M_A^2}{\omega - \omega_A} \|x\|^2.$$

Then

$$\int_{\mathbb{R}} \|(\omega + i\tau)^{\alpha-1} R((\omega + i\tau)^\alpha, A + B)x\|^2 \mathrm{d}\tau \leq \frac{\pi(1 + c\|a\|)^2 M_A^2}{\omega - \omega_A} \|x\|^2.$$

Similarly, we have

$$\int_{\mathbb{R}} \|(\omega - i\tau)^{\alpha-1} R((\omega - i\tau)^\alpha, A^* + B^*)x\|^2 \mathrm{d}\tau \leq \frac{\pi(1 + c\|a\|)^2 M_A^2}{\omega - \omega_A} \|x\|^2.$$

Then by Theorem 2.1, we have $A + B \in C^\alpha$. □

Next we consider relatively-bounded perturbation.

Theorem 3.3 *Let A be a closed densely defined linear operator on a Hilbert space H and $\alpha \in [1, 2]$. $(A, D(A)) \in C^\alpha(M_A, \omega_A)$ and $(B, D(B))$ is a closed operator on H such that $D(B) \supseteq D(A)$. Assume that there exists a constant $M \in [0, 1)$ such that*

$$\|BR(\lambda, A)x\| \leq M\|x\| \text{ and } \|R(\lambda, A)By\| \leq M\|y\|$$

for $\{\lambda \in \mathbb{C}; \mathrm{Re}\lambda > \omega_A^\alpha\}$ and $\forall x \in H$, $\forall y \in D(B)$, then $(A + B, D(A)) \in C^\alpha$.

Proof For $\forall x \in D(A)$, when $\omega > \omega_A$, by the proof of Theorem 2.1 1)$\Longrightarrow$ 2), we have

$$\int_{\mathbb{R}} \|(\omega + i\tau)^{\alpha-1} R((\omega + i\tau)^\alpha, A)x\|^2 \mathrm{d}\tau \leq \frac{\pi M_A^2}{\omega - \omega_A} \|x\|^2.$$

By the proof of Lemma 5.1 in [8], we have that $(A+B, D(A))$ is a closed densely defined operator and

$$(\omega + i\tau)^{\alpha-1} R((\omega + i\tau)^\alpha, A + B)x = (1 - R((\omega + i\tau)^\alpha, A)B)^{-1} (\omega + i\tau)^{\alpha-1} R((\omega + i\tau)^\alpha, A)x.$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}} \|(\omega + i\tau)^{\alpha-1} R((\omega + i\tau)^\alpha, A + B)x\|^2 \mathrm{d}\tau \\
&= \int_{\mathbb{R}} \|(1 - R((\omega + i\tau)^\alpha, A)B)^{-1} (\omega + i\tau)^{\alpha-1} R((\omega + i\tau)^\alpha, A)x\|^2 \mathrm{d}\tau \\
&\leq \frac{1}{(1 - M)^2} \int_{\mathbb{R}} \|(\omega + i\tau)^{\alpha-1} R((\omega + i\tau)^\alpha, A)x\|^2 \mathrm{d}\tau \\
&\leq \frac{\pi M_A^2}{(1 - M)^2 (\omega - \omega_A)} \|x\|^2.
\end{aligned}$$

Similarly,

$$\int_{\mathbb{R}} \|(\omega - i\tau)^{\alpha-1} R((\omega - i\tau)^\alpha, A^* + B^*)x\|^2 \mathrm{d}\tau \leq \frac{\pi M_A^2}{(1 - M)^2 (\omega - \omega_A)} \|x\|^2.$$

Thus by Theorem 2.1, we have $((A + B, D(A)) \in C^\alpha$. □

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