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Gap Theorem on Complete Noncompact Riemannian Manifold

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Abstract A gap theorem on complete noncompact n-dimensional locally conformally flat Riemannian manifold with nonnegative and bounded Ricci curvature is proved. If there holds the following condition:

$$\int_0^r sk(x_0, s)\mathrm{d}s = o(\log r)$$

then the manifold is flat.

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1. Introduction

Let M be an n-dimensional complete noncompact Riemannian manifold with nonnegative Ricci curvature. Mok, Siu and Yau [7] proved that if a complete noncompact Kähler-Stein manifold of nonnegative and bounded holomorphic bisectional curvature of complex dimension $n \geq 2$ has maximal volume growth and the scalar curvature decays faster than quadratic, in the sense that, for some $C > 0$ and $\varepsilon > 0$, $R(x) \leq Cd(x_0, x)^{-(2+\varepsilon)}$, then M is isometrically biholomorphic to C^n. This can be interpreted as a gap phenomenon of the bisectional curvature on Kähler manifolds (A more general theorem in Riemannian category was proved by Greene and Wu in [10]).

Later the similar result was extended to the Riemannian manifold with maximal volume growth and nonnegative Ricci curvature by Bando, Kasue and Nakajima in [11]. Recently, Chen and Zhu obtained the gap theorem on the locally conformally flat manifolds [2]. They showed that:

Let M be an n-dimensional ($n \geq 3$) complete noncompact locally conformally flat Riemannian manifolds with nonnegative Ricci curvature. If the scalar curvature is bounded and there exists

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a positive function $\varepsilon: R \to R$ with $\lim_{r\to\infty}\varepsilon(r)=0$ such that

$$\frac{1}{\mathrm{vol}(B(x_0,\,r))}\int_{B(x_0,\,r)} R(x)\mathrm{d}v \le \frac{\varepsilon(r)}{r^2}, \text{ for } \; x_0\in M, \quad r>0, \tag{1}$$

then M is flat.

Later in [3], they used the theory of the Ricci flow to obtain the analogous gap theorem on Kähler manifold as follows.

Suppose M is a complete noncompact Kähler manifold of complex dimension $n \ge 2$ with bounded and nonnegative holomorphic bisectional curvature, and the condition (1) is satisfied, then M is flat.

In [9], Ni and Tam changed condition (1) into the following condition $(*)$

$$\int_0^r sk(x_0,s)\mathrm{d}s = o(\log r) \tag{*}$$

where $k(x_0,s)=\frac{1}{\mathrm{vol}(B(x_0,s))}\int_{B(x_0,s)} R(x)\mathrm{d}v$. They got the same result on Kähler manifold.

Stimulated by Ni and Tam's result, we consider condition $(*)$ on the locally conformally flat Riemannian manifold, and also get the analogous gap theorem.

Theorem *Let M be an n-dimensional $(n\ge 3)$ complete noncompact locally conformally flat manifolds with nonnegative Ricci curvature. If the scalar curvature is bounded and satisfies condition $(*)$, then M is flat.*

We remark that the condition (1) is stronger than condition $(*)$, because from $(*)$, we can only know the average curvature decay in infinity. In fact, if the scalar curvature satisfies the condition (1), then

$$\lim_{r\to\infty}\frac{\int_0^r sk(x_0,s)\mathrm{d}s}{\log r} = \lim_{r\to\infty}\frac{\int_0^r \frac{\varepsilon(s)}{s}\mathrm{d}s}{\log r} = \lim_{r\to\infty}\varepsilon(r)=0.$$

2. Yamabe flow and estimate for its solution

Suppose g_{ij} and R_{ij} are the metric tensor and the Ricci tensor on M, respectively. The Yamabe flow is the following evolution equation for the metric:

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -R(x,\,t)g_{ij}(x,\,t), & x\in M,\ t>0; \\ g_{ij}(x,\,0)=g_{ij}(x), & x\in M. \end{cases} \tag{2.1}$$

Write $g_{ij}(x,\,t)=(u(x,\,t))^{\frac{4}{n-2}}g_{ij}(x)$ for some positive function $u(x,\,t)$. Then (2.1) can be written in the equivalent form:

$$\begin{cases} \dfrac{\partial u^N(x,\,t)}{\partial t} = (n-1)N\Big[\triangle u(x,\,t) - \frac{n-2}{4(n-1)}R(x)u(x,\,t)\Big]; \\ u(x,\,t)>0, \quad x\in M, \quad t>0; \\ u(x,\,0)=1, \quad x\in M, \end{cases} \tag{2.2}$$

where $N=\frac{n+2}{n-2}$, $\triangle$ is the Laplace operator with respect to the initial metric $g_{ij}(x)$.

We know from [2] that the Yamabe flow (2.1) has a smooth solution, namely,

Proposition 2.1 *Let M be an n-dimensional $(n\ge 3)$ complete noncompact Riemannian manifolds with nonnegative Ricci curvature. If the scalar curvature is bounded, then the Yamabe* *flow (2.1) has a smooth solution on a maximal time interval* $[0, t_{\max})$ *with* $t_{\max} > 0$ *such that either* $t_{\max} = +\infty$*, or the evolving metric contracts to a point at a finite time* $t_{\max}$ *in the sense that for any curve* γ *on* M*, the length of* γ *with respect to the evolving metric* $g_{ij}(x, t)$ *tends to zero as* $t \to t_{\max}$.

Now we want to drive some estimate for the solution of the Yamabe flow (2.2). From above proposition we have a smooth solution $g_{ij}(x, t)$ of (2.1) on a maximal time interval $[0, t_{\max})$. Write the solution as $g_{ij}(x, t) = (u(x, t))^{\frac{4}{n-2}} g_{ij}(x)$, where $u(x, t)$ is a positive solution of (2.2) on $[0, t_{\max})$.

Lemma 2.1 *For* $r \to +\infty$, $t > 0$, *we have*

$$-\int_{B(x_0, r)} \log u(x, t)\mathrm{d}v \le C\,[1 - (\log u)_{\min}(t)]\, tr^{-2}\mathrm{vol}(B(x_0, r)), \tag{2.3}$$

where $(\log u)_{\min}(t)$ *is the infimum of* $\log u(x, t)$ *for* $x \in M$, $\epsilon > 0$ *and* C *is some positive constant depending only on* n.

Proof Since the Ricci curvature of the initial metric $g_{ij}(x)$ is nonnegative, we know that there exists a constant $C > 0$ depending only on the dimension such that for any fixed point $x_0 \in M$ and $r > 0$ there exists a smooth function $\varphi(x) \in C^\infty(M)$ such that

$$\begin{cases} \exp\left[-C(1 + \frac{d(x, x_0)}{r})\right] \le \varphi(x) \le \exp\left[-(1 + \frac{d(x, x_0)}{r})\right], \\ \mid \nabla\varphi(x) \mid \le \frac{C}{r}\varphi(x), \\ \mid \triangle\varphi(x) \mid \le \frac{C}{r^2}\varphi(x). \end{cases} \tag{2.4}$$

Suppose $u(x, t)$ is solution of (2.2). Since $\frac{\partial u}{\partial t} < 0$ and $u(x, o) = 1$, we have $u \le 1$, and

$$\begin{aligned} \frac{\partial}{\partial t}\int_M \varphi u^N \mathrm{d}v &= \int_M \varphi\left[(n-1)N\triangle u - \frac{n-2}{4}Ru\right]\mathrm{d}v \\ &= (n-1)N\int_M \triangle\varphi u \mathrm{d}v - \frac{n-2}{4}\int_M R\varphi u \mathrm{d}v. \end{aligned} \tag{2.5}$$

So by integrating (2.5) from 0 to t, we obtain:

$$\int_M \varphi(1 - u^N)\mathrm{d}v \le \frac{C}{r^2}\int_0^t \mathrm{d}t\int_M u\varphi \mathrm{d}v\; C\int_0^t \mathrm{d}t\int_M R\varphi u\mathrm{d}v. \tag{2.6}$$

From $(*)$ we know

$$\lim_{r\to\infty} k(x_0, r) \le o(\frac{1}{r^2}) \le Ct(\frac{1}{r^2}\int_M \varphi \mathrm{d}v + \int_M R\varphi \mathrm{d}v),$$

$$\begin{aligned} \int_M R\varphi \mathrm{d}v &\le \int_M Re^{-(1+\frac{d(x, x_0)}{r})}\mathrm{d}v \\ &= \lim_{r\to\infty}\int_{B(x_0, r)} R(x)e^{-(1+\frac{d(x, x_0)}{r})}\mathrm{d}v + \lim_{r\to\infty}\int_{M\setminus B(x_0, r)} R(x)e^{-(1+\frac{d(x, x_0)}{r})}\mathrm{d}v \\ &\le \lim_{r\to\infty}\int_{B(x_0, r)} R(x)\mathrm{d}v \le \lim_{r\to\infty} o(r^{-2})(\mathrm{vol}(B(x_0, r))). \end{aligned}$$

Thus we get

$$\int_M R\varphi \mathrm{d}v \le \lim_{r\to\infty} o(r^{-2})\mathrm{vol}(B(x_0, r)). \tag{2.7}$$

Similarly, we have

$$\int_M \varphi \mathrm{d}v \le \lim_{r\to\infty} C\mathrm{vol}(B(x_0, r)). \tag{2.8}$$

Substituting (2.7) and (2.8) into (2.6), we deduce

$$\int_M \varphi(1-u^N)\mathrm{d}v \le \lim_{r\to\infty} Ct(\frac{1}{r^2})\mathrm{vol}(B(x_0, r)) \tag{2.9}$$

for some positive constant C depending only on n.

On the other hand, since $u \le 1$ and $1-e^x \ge -\frac{x}{2}$ for $-1 \le x \le 0$, we get

$$\begin{aligned}\int_M \varphi(1-u^N)\mathrm{d}v &\ge \int_M \varphi(1-u)\mathrm{d}v\\ &= \int_{\{\log u\ge -1\}} \varphi(1-e^{\log u})\mathrm{d}v + \int_{\{\log u \le -1\}} \varphi(1-e^{\log u})\mathrm{d}v\\ &\ge -\frac{1}{2}\int_{\{\log u \ge -1\}} \varphi \log u \mathrm{d}v + \frac{1}{2}\int_{\{\log u\le -1\}} \varphi \mathrm{d}v.\end{aligned}$$

Noting that the last two terms are positive, we deduce

$$-\int_{\{\log u\ge -1\}} \varphi \log u\mathrm{d}v \le 2\int_M \varphi(1-u^N)\mathrm{d}v$$

and

$$\begin{aligned}&-\int_{\{\log u \le -1\}} \varphi \log u \mathrm{d}v \le -(\log u)_{\min}(t)\int_{\{\log u\le -1\}} \varphi \mathrm{d}v\\ &\le -2(\log u)_{\min}(t)\int_M \varphi(1-u^N)\mathrm{d}v.\end{aligned}$$

So

$$-\int_M \varphi \log u \mathrm{d}v \le 2(1-(\log u)_{\min}(t))\int_M \varphi(1-u^N)\mathrm{d}v. \tag{2.10}$$

By (2.4), we get

$$-\int_M \varphi \log u\mathrm{d}v \ge -\int_M \log u\, e^{-C(1+\frac{d(x,\,x_0)}{r})}\mathrm{d}v \ge -e^{-2c(n)}\int_{B(x_0,\,r)} \log u \mathrm{d}v. \tag{2.11}$$

Combining (2.9) and (2.10) with (2.11), we obtain the estimate (2.3). □

Lemma 2.2 *There exists a positive constant C such that*

$$\log u(x, t) \ge -C\Big[\int_0^r sk(x_0, s)\mathrm{d}s - \frac{1}{\mathrm{vol}(B(x_0, r))}\int_{B(x_0,\,r)} \log u\mathrm{d}v\Big] \tag{2.12}$$

for $\forall\, t > 0$ and $\forall\, r \ge 1$.

Proof Since in general M must not be nonparabolic, we use a trick of Shi in [6] by considering a new manifold $\widetilde{M} = M \times R^3$, where R^3 is equipped with the flat Euclidean metric and $\widetilde{M}$ is equipped with the product metric. Let $x \in M$, $\tilde{x} \in \widetilde{M}$. Denote by $R(x)$ and $\widetilde{R}(\tilde{x})$ the scalar curvature of M at x, and the scalar curvature of $\widetilde{M}$ at $\tilde{x}$, respectively. Obviously $\widetilde{R}(\tilde{x}) = R(x)$ for $\forall\, x \in M$, $\tilde{x} = (x, y) \in \widetilde{M}, \forall y \in R^3$. Thus we define $\tilde{u}(\tilde{x}, t) = u(x, t)$ for $\forall\, \tilde{x} \in \widetilde{M}$, $\forall\, t > 0$, then $\tilde{u}$ is a solution of the following evolution equation.

$$\begin{cases} \frac{\partial \tilde{u}}{\partial t} = (n-1)N[\widetilde{\triangle}\tilde{u} - \frac{n-2}{4(n-1)}\widetilde{R}(\tilde{x})\tilde{u}], \ \tilde{x} \in \widetilde{M}, \forall\, t > 0; \\ \tilde{u}(\tilde{x}, t) > 0, \ \tilde{x} \in \widetilde{M}, \forall\, t > 0; \\ \tilde{u}(\tilde{x}, 0) = 1, \ \tilde{x} \in \widetilde{M}, \end{cases}$$

where $\widetilde{\triangle}$ is the Laplacian operator of $\widetilde{M}$.

As we know, $\widetilde{M}$ still has nonnegative Ricci curvature and bounded scalar curvature, moreover

$$B_M(x_0, \frac{r}{2}) \times B_{R^3}(y_0, \frac{r}{2}) \subset \widetilde{B}(\tilde{x}_0, r) \subset B_M(x_0, r) \times B_{R^3}(y_0, r) \tag{2.13}$$

where $\tilde{x}_0 = (x_0, y_0)$, $x_0 \in M$, $y_0 \in R^3$. So from (2.13) and the volume comparison theorem, we get

$$C\,(\frac{r_2}{r_1})^3 \leq \frac{\mathrm{vol}[\widetilde{B}(\tilde{x}_0, r_2)]}{\mathrm{vol}[\widetilde{B}(\tilde{x}_0, r_1)]} \leq (\frac{r_2}{r_1})^{n+3}, \text{ for } r_1 \leq r_2. \tag{2.14}$$

Then

$$\begin{aligned} \int_0^r sk(\tilde{x_0}, s)\mathrm{d}s &= \int_0^r \frac{s}{\mathrm{vol}(B(\tilde{x_0}, s))} \int_{B(\tilde{x_0}, s)} R(\tilde{x})\mathrm{d}\tilde{v}\mathrm{d}s \\ &\leq C \int_0^r \frac{s}{(\frac{s}{2})^3 \mathrm{vol}(B_M(x_0, \frac{s}{2}))} \int_{B_M(x_0,s)\times B_{R^3}(y_0,s)} \tilde{R}(\tilde{x})\mathrm{d}\tilde{v}\mathrm{d}s \\ &\leq C \int_0^r \frac{s}{\mathrm{vol}(B_M(x_0, s))} \int_{B_M(x_0,s)} R(x)\mathrm{d}v\mathrm{d}s \leq C \int_0^r sk(x_0, s)\mathrm{d}s. \end{aligned}$$

Thus from $(*)$, we have $\int_0^r sk(\tilde{x_0}, s)\mathrm{d}s = o(\log r)$. Since $\widetilde{R}(\tilde{x}) = R(x)$, we only prove that $\widetilde{R}(\tilde{x}) \equiv 0$. So we can discuss the problem on $\tilde{M}$. For convenience, we drop off the symbol $\tilde{}$ in the following.

From the fact that $g_{ij}(x, t) = (u(x, t))^{\frac{4}{n-2}} g_{ij}(x)$, we can compute directly:

$$u^{\frac{4}{n-2}} R(x, t) = R(x) - \frac{4(n-1)}{n-2} \triangle \log u. \tag{2.15}$$

From Li-Yau-Hamilton inequality on locally conformally flat manifolds [12], we obtain

$$\frac{\partial R}{\partial t} + \frac{R}{t} \geq 0, \quad \frac{\partial (R(x, t)t)}{\partial t} \geq 0.$$

Thus $R(x, t)t$ is nondecreasing in time. Since $R(x, 0) = R(x)$ is bounded, $R(x, t)t \geq 0$, for $t \geq 0$. Thus $R(x, t) \geq 0$.

From (2.15), we have

$$\triangle \log u(x, t) \leq \frac{n-2}{4(n-1)} R(x), \quad x \in M, \ t \geq 0. \tag{2.16}$$

Since the Ricci curvature of M is nonnegative and (2.14) holds, we know from [6] that there exists a positive Green function $G(x_0, x)$ satisfying

$$\begin{cases} \dfrac{d^2(x_0, x)}{C\,\mathrm{vol}[B(x_0, d(x_0, x))]} \leq G(x_0, x) \leq C \dfrac{d^2(x_0, x)}{\mathrm{vol}[B(x_0, d(x_0, x))]}, \\ \mid \nabla G(x_0, x) \mid \leq C \dfrac{d(x_0, x)}{\mathrm{vol}[B(x_0, d(x_0, x))]}. \end{cases} \tag{2.17}$$

For $\forall\, \alpha > 0$, we define $\Omega_\alpha = \{x \in M \mid G(x_0, x) \geq \alpha\}$.

From (2.14) and (2.17), we have

$$\frac{1}{C\,(d(x_0,\,x))^{n+1}\mathrm{vol}[B(x_0,\,1)]}\leq G(x_0,\,x)\leq\frac{C}{d(x_0,\,x)\mathrm{vol}[B(x_0,\,1)]}$$

for $d(x_0,\,x)\geq 1$. Thus $\overline{\Omega}_\alpha$ is compact subset of M, and $\partial\Omega_\alpha=\{x\in M\mid G(x_0,\,x)=\alpha\}$.

Moreover we assume that α satisfies $\alpha>0$, then there exists a number $d(\alpha)\geq 1$ such that

$$\frac{d^2(\alpha)}{\mathrm{vol}[B(x_0,\,d(\alpha))]}=\alpha. \tag{2.18}$$

Thus combining (2.17) and (2.18), we have: for $\forall x\in\partial\Omega_\alpha$

$$\frac{d^2(\alpha)}{\mathrm{vol}[B(x_0,\,d(\alpha))]}\leq\frac{C\,d^2(x_0,\,x)}{\mathrm{vol}[B(x_0,\,d(x_0,\,x))]},\quad\frac{d^2(\alpha)}{d^2(x_0,\,x)}\leq\frac{C\,\mathrm{vol}[B(x_0,\,d(\alpha))]}{\mathrm{vol}[B(x_0,\,d(x_0,\,x))]}$$

which together with (2.14) implies $d(x_0,\,x)\leq C\,d(\alpha)$, $\forall\,x\in\partial\Omega_\alpha$. Thus

$$\Omega_\alpha\subset B(x_0,\,C\,d(\alpha)). \tag{2.19}$$

By the Green formula, it follows

$$\log u(x_0,\,t)=\int_{\Omega_\alpha}(\alpha-G(x_0,\,x))\triangle\,\log u(x,\,t)\mathrm{d}v-\int_{\partial\Omega_\alpha}\log u\frac{\partial G}{\partial\vec{r}}\mathrm{d}\sigma,$$

where $\vec{r}$ denotes the outer unit normal vectors of $\partial\Omega_\alpha$, $\mathrm{d}\sigma$ denotes the volume element of $\partial\Omega_\alpha$. From (2.16), we get

$$\begin{aligned}\log u(x_0,\,t)&\geq\int_{\Omega_\alpha}(\alpha-G(x_0,\,x))\frac{n-2}{4(n-1)}R(x)\mathrm{d}v-\int_{\partial\Omega_\alpha}\log u\frac{\partial G}{\partial\vec{r}}\mathrm{d}\sigma\\&\geq -C\int_{\Omega_\alpha}G(x_0,\,x)R(x)\mathrm{d}v+\int_{\partial\Omega_\alpha}\log u\,\mid\nabla G(x_0,\,x)\mid\mathrm{d}\sigma.\end{aligned} \tag{2.20}$$

Now we estimate the last two terms respectively. Since Ricci curvature is nonnegative, from Ni-Shi-Tam's result [9], we know that

$$\int_{B(x_0,r)}G(x_0,x)R(x)\mathrm{d}v\leq C\int_0^r sk(x_0,s)\mathrm{d}s.$$

Thus from $(*)$ and (2.19), we obtain that

$$\begin{aligned}\int_{\Omega_\alpha}G(x_0,x)R(x)\mathrm{d}v&\leq\int_{B(x_0,Cd(\alpha))}G(x_0,x)R(x)\mathrm{d}v\\&\leq\int_0^{Cd(\alpha)}sk(x_0,s)\mathrm{d}s.\end{aligned} \tag{2.21}$$

By (2.17), we have

$$\int_{\partial\Omega_\alpha}\log u\,\mid\nabla G(x_0,\,x)\mathrm{d}\sigma\geq\frac{Cd(\alpha)}{\mathrm{vol}(B(x_0,\,d(\alpha)))}\int_{\partial\Omega_\alpha}\log u(x,\,t)\mathrm{d}\sigma.$$

By integrating two sides from α to 2α, we get

$$\int_\alpha^{2\alpha}\int_{\partial\Omega_\beta}\log u\,\mid\nabla G(x_0,\,x)\mid\mathrm{d}\sigma\,\mathrm{d}\beta\geq\frac{Cd(\alpha)}{\mathrm{vol}(B(x_0,\,d(\alpha)))}\int_\alpha^{2\alpha}\int_{\partial\Omega_\beta}\log u\mathrm{d}\sigma\,\mathrm{d}\beta. \tag{2.22}$$

We know from the definition of $d(\alpha)$ that for $\alpha\leq\beta\leq 2\alpha$

$$\frac{d^2(\alpha)}{\mathrm{vol}(B(x_0,\,d(\alpha)))}\leq\frac{d^2(\beta)}{\mathrm{vol}(B(x_0,\,d(\beta)))}\leq\frac{2\,d^2(\alpha)}{\mathrm{vol}(B(x_0,\,d(\alpha)))},$$

$$\frac{\operatorname{vol}(B(x_0,\, d(\beta)))}{\operatorname{vol}(B(x_0,\, d(\alpha)))} \le \frac{d^2(\beta)}{d^2(\alpha)} \le \frac{2\operatorname{vol}(B(x_0,\, d(\beta)))}{\operatorname{vol}(B(x_0,\, d(\alpha)))}$$

which together with (2.14) implies

$$C^{-1}d(\alpha) \le d(\beta) \le Cd(\alpha), \quad \alpha \le \beta \le 2\alpha.$$

Since $\mathrm{d}\beta = \frac{\partial G}{\partial \vec{r}}\mathrm{d}r$, from (2.14) and (2.17), we have

$$\begin{aligned} \mathrm{d}\sigma\mathrm{d}\beta &= \frac{\partial G}{\partial \vec{r}}\mathrm{d}\sigma\mathrm{d}r = \mid \frac{\partial G(x_0,\, x)}{\partial \vec{r}} \mid \mathrm{d}\sigma \mid \mathrm{d}r \mid \le \mid \nabla G(x_0,\, x) \mid \mathrm{d}v \\ &\le \frac{C\, d(\beta)}{\operatorname{vol}(B(x_0,\, d(\beta)))}\mathrm{d}v \le \frac{C\, d(\alpha)}{\operatorname{vol}(B(x_0,\, d(\alpha)))}\mathrm{d}v. \end{aligned}$$

Thus from (2.18), (2.19) and (2.22), we have

$$\begin{aligned} &\int_\alpha^{2\alpha}\int_{\partial\Omega_\beta} \log u \mid \nabla G(x_0,\, x) \mid \mathrm{d}\sigma\, \mathrm{d}\beta \ge C\Big(\frac{d(\alpha)}{\operatorname{vol}(B(x_0,\, d(\alpha)))}\Big)^2 \int_{\Omega_\alpha\setminus\Omega_{2\alpha}} \log u \mathrm{d}v \\ &\ge C\frac{\alpha}{\operatorname{vol}(B(x_0,\, d(\alpha)))}\int_{\Omega_\alpha} \log u(x,\, t)\mathrm{d}v \\ &\ge C\frac{\alpha}{\operatorname{vol}(B(x_0,\, Cd(\alpha)))}\int_{B(x_0, Cd(\alpha))} \log u(x,\, t)\mathrm{d}v. \end{aligned} \tag{2.23}$$

By integrating (2.20) from α to 2α, and combining (2.21) and (2.23), we have

$$\log u(x_0,\, t) \ge C\Big[\frac{1}{\operatorname{vol}(B(x_0,\, Cd(\alpha)))}\int_{B(x_0,\, Cd(\alpha))} \log u(x,\, t)\mathrm{d}v - \int_0^{Cd(\alpha)} sk(x_0, s)\mathrm{d}s\Big].$$

Let $r = Cd(\alpha)$. We get the desired estimate (2.12). □

3. Proof of Theorem

From Lemmas 2.1 and 2.2, we get the lower bound estimate for the solution of (2.2):

$$\begin{aligned} (\log u)_{\min}(t) &\ge \lim_{r\to+\infty} -C\Big[\int_0^r sk(x_0, s)\mathrm{d}s - \frac{1}{\operatorname{vol}(B(x_0,\, r))}\int_{B(x_0,\, r)} \log u \mathrm{d}v\Big] \\ &\ge \lim_{r\to+\infty} -C\Big[\int_0^r sk(x_0, s)\mathrm{d}s + t(r^{-2})(1 - (\log u)_{\min}(t))\Big]. \end{aligned}$$

From Proposition 2.1, we know the solution $u(x,\, t)$ of (2.2) exists for all times and satisfies

$$(\log u)_{\min}(t) \ge \lim_{r\to+\infty} -C\,\big[o(\log r) + t(r^{-2})(1 - (\log u)_{\min}(t))\big]$$

for all $t > 0$, where C is positive constant depending only on n.

Now let $t = r$. According to condition $(*)$, we have

$$\lim_{t\,\to\,+\infty} -\frac{(\log u)_{\min}(t)}{\log t} = 0. \tag{3.1}$$

By (2.1) and $g_{ij}(x,\, t) = (u(x,\, t))^{\frac{4}{n-2}}g_{ij}(x)$, we have

$$\begin{aligned} \frac{\partial g_{ij}(x,\, t)}{\partial t} &= \frac{4}{n-2}(u(x,\, t))^{\frac{4}{n-2}-1}\frac{\partial u}{\partial t}g_{ij}(x), \\ -R(x,\, t)g_{ij}(x,\, t) &= \frac{4}{n-2}(u(x,\, t))^{\frac{4}{n-2}-1}\frac{\partial u}{\partial t}g_{ij}(x), \end{aligned}$$

$$\begin{aligned}
-R(x,\,t)g_{ij}(x,\,t) &= \frac{4}{n-2}(u(x,\,t))^{\frac{4}{n-2}}g_{ij}(x)\frac{\partial \log u}{\partial t},\\
\frac{\partial}{\partial t}\log u(x,\,t) &= -\frac{n-2}{4}R(x,\,t),\\
\int_0^t R(x,\,\tau)\mathrm{d}\tau &= -\frac{4}{n-2}\log u(x,\,t) \le -\frac{4}{n-2}(\log u)_{\min}(t).
\end{aligned}$$

Since $R(x,\,t)t$ is nondecreasing in time, we have

$$R(x,\,\tau) > R(x,\,\sqrt{t})\sqrt{t}\frac{1}{\tau},\ \text{for}\ \tau \ge \sqrt{t}.$$

Thus

$$\int_0^t R(x,\,\tau)\mathrm{d}\tau \ge \int_{\sqrt{t}}^t R(x,\,\tau)\mathrm{d}\tau \ge \frac{1}{2}R(x,\,\sqrt{t})\sqrt{t}\log t.$$

So we have

$$\frac{1}{2}R(x,\,\sqrt{t})\sqrt{t}\log t \le -\frac{4}{n-2}(\log u)_{\min}(t)$$

which together with (3.1) implies $\lim_{t\to+\infty} R(x,\,\sqrt{t})\sqrt{t} = 0$. Thus $R(x,\,\sqrt{t})\sqrt{t} \equiv 0$, namely, $R(x,\,t) \equiv 0$, for $\forall\, x \in M$, $t \ge 0$. Therefore the manifold M with the initial metric must be flat. □

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