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Some Convergence Properties for ψ-Mixing Sequences

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Abstract In this paper, we extend the Kolmogorov-type inequality to the case of ψ-mixing sequences. Moreover, we study the strong limit theorems for partial sums of ψ-mixing random variables. As a result, we extend the Khintchine-Kolmogorov-type convergence theorem, the three series theorem, Marcinkiewicz strong law of large number to the case of ψ-mixing sequences.

Keywords Kolmogorov-type inequality; Khintchine-Kolmogorov-type convergence theorem; strong law of large numbers.

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1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. $S_n \doteq \sum_{i=1}^n X_i,\ n \geq 1$. Let n and m be positive integers. Write $\mathcal{F}_n^m = \sigma(X_i, n \leq i \leq m)$. Given σ-algebras $\mathcal{B}, \mathcal{R}$ in $\mathcal{F}$, let

$$\psi(\mathcal{B}, \mathcal{R}) = \sup_{A\in\mathcal{B}, B\in\mathcal{R}, P(A)P(B)>0} \frac{|P(AB) - P(A)P(B)|}{P(A)P(B)},$$

$$\varphi(\mathcal{B}, \mathcal{R}) = \sup_{A\in\mathcal{B}, B\in\mathcal{R}, P(A)>0} |P(B|A) - P(B)|.$$

Define the mixing coefficients by

$$\psi(n) = \sup_{k\geq 1} \psi(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty), \quad \varphi(n) = \sup_{k\geq 1} \varphi(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty),\ n \geq 0.$$

Definition 1.1 *A sequence $\{X_n, n \geq 1\}$ of random variables is said to be a ψ-mixing (φ-mixing) sequence if $\psi(n) \downarrow 0$ ($\varphi(n) \downarrow 0$) as $n \to \infty$.*

It is easily seen that $\varphi(n) \leq \psi(n)$. Therefore, the family of φ-mixing contains ψ-mixing as a special case. ψ-mixing random variables were introduced by Blum, et al. [1] and some applications have been found. See for example, Blum, et al. [1] for strong law of large numbers,

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Yang [2] for almost sure convergence of weighted sums, and so forth. The main purpose of this paper is to study the strong limit theorems for partial sums of ψ-mixing random variables and try to get some new results.

Lemma 1.1 ([3, Lemma 1.2.11]) *Let $\{X_n, n \geq 1\}$ be a sequence of ψ-mixing random variables. Let $X \in \mathcal{F}_1^k$, $Y \in \mathcal{F}_{k+n}^{\infty}$, $E|X| < \infty$, $E|Y| < \infty$. Then $E|XY| < \infty$ and $|EXY - EXEY| \leq \psi(n)E|X|E|Y|$.*

Lemma 1.2 ([4, Lemma 2.2]) *Let $\{X_n, n \geq 1\}$ be a φ-mixing sequence. Put $T_a(n) = \sum_{i=a+1}^{a+n} X_i$. Suppose that there exists an array $\{C_{a,n}\}$ of positive numbers such that*

$$ET_a^2(n) \leq C_{a,n} \quad \text{for every} \quad a \geq 0 \quad \text{and} \quad n \geq 1.$$

Then for every $q \geq 2$, there exists a constant C depending only on q and $\varphi(\cdot)$ such that

$$E\Big(\max_{1\leq j\leq n} |T_a(j)|^q\Big) \leq C\Big[C_{a,n}^{q/2} + E(\max_{a+1\leq i\leq a+n} |X_i|^q)\Big], \text{ for every } a \geq 0 \text{ and } n \geq 1.$$

By Lemmas 1.1 and 1.2, we can easily get the following maximal inequality for ψ-mixing random variables. The details are omitted.

Lemma 1.3 *Let $\{X_n, n \geq 1\}$ be a sequence of ψ-mixing random variables satisfying $\sum_{n=1}^{\infty} \psi(n) < \infty$. Assume that $EX_n = 0$ and $E|X_n|^q < \infty$ for each $n \geq 1$ and $q \geq 2$. Then there exists a constant C depending only on q and $\psi(\cdot)$ such that for every $a \geq 0$ and $n \geq 1$*

$$E\Big(\max_{1\leq j\leq n} |\sum_{i=a+1}^{a+j} X_i|^q\Big) \leq C\Big[\sum_{i=a+1}^{a+n} E|X_i|^q + (\sum_{i=a+1}^{a+n} EX_i^2)^{q/2}\Big].$$

Throughout the paper, let $X^a = XI(|X| \leq a)$ for some $a > 0$ and $I(A)$ be the indicator function of the set A. C denotes a positive constant which may be different.

2. Main results and their proofs

Theorem 2.1 (Khintchine-Kolmogorov-type Convergence Theorem) *Let $\{X_n, n \geq 1\}$ be a sequence of ψ-mixing random variables satisfying $\sum_{n=1}^{\infty} \psi(n) < \infty$. Assume that*

$$\sum_{n=1}^{\infty} \mathrm{Var}(X_n) < \infty. \tag{1}$$

Then $\sum_{n=1}^{\infty}(X_n - EX_n)$ converges a.s..

Proof Without loss of generality, we assume that $EX_n = 0$ for all $n \geq 1$. Let $m < n$ be positive integers. By Lemma 1.3 and (1), we have

$$E(S_n - S_m)^2 = E(\sum_{i=m+1}^{n} X_i)^2 \leq C \sum_{i=m+1}^{n} EX_i^2 \to 0 \quad \text{as} \quad m, n \to \infty, \tag{2}$$

which implies that $\{S_n, n \geq 1\}$ is a Cauchy sequence in L_2. Then there exists a random variable $S \in L_2$ such that $E(S_n - S)^2 \to 0$ as $n \to \infty$, i.e., $S_n \xrightarrow{L_2} S$, which implies that $S_n \xrightarrow{P} S$. So there exist positive integers $n_k \to \infty$ such that

$$S_{n_k} \to S, \text{ a.s. as } k \to \infty. \tag{3}$$

By the method of subsequence, we only need to show

$$\max_{n_{k-1}<j\le n_k} |S_j - S_{n_{k-1}}| \to 0 \text{ a.s.}. \tag{4}$$

For any $\varepsilon > 0$, by Markov's inequality, Lemma 1.3 and (1), we can see that (letting $n_0 = 0$)

$$\begin{aligned}\sum_{k=1}^{\infty} P\Big(\max_{n_{k-1}<j\le n_k} |S_j - S_{n_{k-1}}| \ge \varepsilon\Big) &\le C\sum_{k=1}^{\infty} E\Big(\max_{n_{k-1}<j\le n_k} |S_j - S_{n_{k-1}}|\Big)^2 \\ &\le C\sum_{k=1}^{\infty}\sum_{j=n_{k-1}+1}^{n_k} EX_j^2 = C\sum_{j=1}^{\infty} EX_j^2 < \infty,\end{aligned}$$

which implies (4) by Borel-Cantelli Lemma. The proof is completed. □

Theorem 2.2 (Three Series Theorem) *Let $\{X_n, n \ge 1\}$ be a sequence of ψ-mixing random variables satisfying $\sum_{n=1}^{\infty} \psi(n) < \infty$. Assume that for some $c > 0$*

$$\sum_{n=1}^{\infty} P(|X_n| > c) < \infty, \tag{5}$$

$$\sum_{n=1}^{\infty} E(X_n^c) \text{ converges}, \tag{6}$$

$$\sum_{n=1}^{\infty} \mathrm{Var}(X_n^c) < \infty. \tag{7}$$

Then $\sum_{n=1}^{\infty} X_n$ converges a.s..

Proof According to (7) and Theorem 2.1, we can get that $\sum_{n=1}^{\infty}(X_n^c - EX_n^c)$ converges a.s.. Therefore, $\sum_{n=1}^{\infty} X_n^c$ converges a.s. following from (6). By (5),

$$\sum_{n=1}^{\infty} P(X_n \ne X_n^c) = \sum_{n=1}^{\infty} P(|X_n| > c) < \infty, \tag{8}$$

which implies that $P(X_n \ne X_n^c, \text{ i.o.}) = 0$. Hence, $\sum_{n=1}^{\infty} X_n$ converges a.s. □

Theorem 2.3 *Let $\{X_n, n \ge 1\}$ be a sequence of ψ-mixing random variables satisfying $\sum_{n=1}^{\infty} \psi(n) < \infty$ and $\{a_n, n \ge 1\}$ be a sequence of positive numbers. Let $\{g_n(x), n \ge 1\}$ be a sequence of even functions defined on $\mathbb{R}$, positive and nondecreasing on the half-line $x > 0$. One or the other of the following conditions is satisfied for every $n \ge 1$,*

(i) In the interval $(0, 1]$, there exists a $\delta > 0$ such that $g_n(x) \ge \delta x$;

(ii) In the interval $(0, 1]$, there exist $\beta \in (1, 2]$ and $\delta > 0$ such that $g_n(x) \ge \delta x^\beta$ and in the interval $(1, +\infty)$, there exists a $\delta > 0$ such that $g_n(x) \ge \delta x$. $EX_n = 0$ for all $n \ge 1$.

For some $M > 0$, we assume that

$$\sum_{n=1}^{\infty} Eg_n\Big(\frac{X_n}{Ma_n}\Big) < \infty. \tag{9}$$

Then $\sum_{n=1}^{\infty} X_n/a_n$ *converges a.s..*

Proof Let $X_n^{Ma_n} \doteq X_n I(|X_n| \le Ma_n)$. Thus

$$\frac{X_n^{Ma_n}}{Ma_n} = \frac{X_n}{Ma_n} I(|\frac{X_n}{Ma_n}| \le 1) \doteq (\frac{X_n}{Ma_n})^1.$$

By the definition of ψ-mixing sequence, we can see that $\{X_n/Ma_n, n \ge 1\}$ is also a ψ-mixing sequence. Therefore, we only need to test (5), (6) and (7), where $c = 1$.

Firstly, if the function $g_n(x)$ satisfies condition (i), then for $|X_n| > Ma_n > 0$, we have $\frac{1}{\delta} g_n(\frac{X_n}{Ma_n}) \ge 1$. Therefore,

$$P(|X_n| > Ma_n) = E(I(|X_n| > Ma_n)) \le \frac{1}{\delta} E g_n(\frac{X_n}{Ma_n}). \tag{10}$$

By (9) and (10),

$$\sum_{n=1}^{\infty} P(|X_n| > Ma_n) \le \frac{1}{\delta} \sum_{n=1}^{\infty} E g_n(\frac{X_n}{Ma_n}) < \infty. \tag{11}$$

If the function $g_n(x)$ satisfies condition (ii), then we also have (11).

Secondly, if the function $g_n(x)$ satisfies condition (i), we have

$$|EX_n^{Ma_n}| \le E(|X_n| I(|X_n| \le Ma_n)) = Ma_n E(\frac{|X_n|}{Ma_n} I(|X_n| \le Ma_n)) \le \frac{1}{\delta} Ma_n E g_n(\frac{X_n}{Ma_n}).$$

If the function $g_n(x)$ satisfies condition (ii), we can get

$$|EX_n^{Ma_n}| \le E(|X_n| I(|X_n| > Ma_n)) \le \frac{1}{\delta} Ma_n E g_n(\frac{X_n}{Ma_n}). \tag{12}$$

Therefore, whether $g_n(x)$ satisfies condition (i) or (ii), we can obtain

$$\sum_{n=1}^{\infty} \frac{|EX_n^{Ma_n}|}{Ma_n} \le \frac{1}{\delta} \sum_{n=1}^{\infty} E g_n(\frac{X_n}{Ma_n}) < \infty. \tag{13}$$

Finally, if the function $g_n(x)$ satisfies condition (i), we have

$$\sum_{n=1}^{\infty} \frac{E(X_n^{Ma_n})^2}{a_n^2} \le M^2 \sum_{n=1}^{\infty} E(\frac{|X_n|}{Ma_n} I(|X_n| \le Ma_n)) \le \frac{M^2}{\delta} \sum_{n=1}^{\infty} E g_n(\frac{X_n}{Ma_n}) < \infty.$$

If the function $g_n(x)$ satisfies condition (ii), we can get

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{E(X_n^{Ma_n})^2}{a_n^2} &\le M^2 \sum_{n=1}^{\infty} E(|\frac{X_n}{Ma_n}|^{\beta} I(|X_n| \le Ma_n)) \\ &\le \frac{M^2}{\delta} \sum_{n=1}^{\infty} E g_n(\frac{X_n}{Ma_n} I(|X_n| \le Ma_n)) < \infty. \end{aligned}$$

Therefore, whether $g_n(x)$ satisfies condition (i) or (ii), we can obtain

$$\sum_{n=1}^{\infty} \frac{\mathrm{Var}(X_n^{Ma_n})}{(Ma_n)^2} \le \sum_{n=1}^{\infty} \frac{E(X_n^{Ma_n})^2}{(Ma_n)^2} \le \frac{1}{\delta} \sum_{n=1}^{\infty} E g_n(\frac{X_n}{Ma_n}) < \infty. \tag{14}$$

Thus, $\sum_{n=1}^{\infty} X_n/a_n$ converges a.s. from Theorem 2.2, (11), (13) and (14). □

Corollary 2.1 *Let* $\{X_n, n \ge 1\}$ *be a sequence of* ψ*-mixing random variables satisfying* $\sum_{n=1}^{\infty} \psi(n)$

$< \infty$ and $\{a_n, n \geq 1\}$ be a positive number sequence satisfying $0 < a_n \uparrow \infty$. For some $M > 0$, one or the other of the following conditions is satisfied:

(i) $\sum_{n=1}^{\infty} E(\frac{|X_n|^\beta}{|Ma_n|^\beta + |X_n|^\beta}) < \infty$, $\exists \beta \in (0, 1]$;

(ii) $\sum_{n=1}^{\infty} E(\frac{|X_n|^\beta}{Ma_n|X_n|^{\beta-1} + |Ma_n|^\beta}) < \infty$, $\exists \beta \in (1, 2]$ *and* $EX_n = 0$ *for all* $n \geq 1$.

Then $\lim_{n\to\infty} \sum_{i=1}^{n} X_i/a_n = 0$ *a.s..*

By Corollary 2.1, we can get the following result.

Corollary 2.2 *Let $\{X_n, n \geq 1\}$ be a sequence of mean zero ψ-mixing random variables satisfying $\sum_{n=1}^{\infty} \psi(n) < \infty$ and $\{a_n, n \geq 1\}$ be a positive number sequence satisfying $0 < a_n \uparrow \infty$. If there exists some $\beta \in (0, 2]$ such that $\sum_{n=1}^{\infty} \frac{E|X_n|^\beta}{a_n^\beta} < \infty$, then $\lim_{n\to\infty} \sum_{i=1}^{n} X_i/a_n = 0$ a.s..*

Theorem 2.4 *Let $\{X_n, n \geq 1\}$ be a sequence of ψ-mixing random variables satisfying $\sum_{n=1}^{\infty} \psi(n) < \infty$ and $\{a_n, n \geq 1\}$ be a sequence of positive numbers. Let $\{g_n(x), n \geq 1\}$ be a sequence of even functions defined on $\mathbb{R}$, positive and nondecreasing on the half-line $x > 0$. There exist $\beta \in [2, \infty)$ and $\delta > 0$ such that $g_n(x) \geq \delta x^\beta$, $x > 0$ for all $n \geq 1$. If*

$$\sum_{n=1}^{\infty} (Eg_n(\frac{X_n}{Ma_n}))^{1/\beta} < \infty, \tag{15}$$

then $\sum_{n=1}^{\infty} X_n/a_n$ converges a.s..

Proof Since $\beta \geq 2$, by (15), we have

$$\sum_{n=1}^{\infty} (Eg_n(\frac{X_n}{Ma_n}))^{2/\beta} < \infty; \quad \sum_{n=1}^{\infty} Eg_n(\frac{X_n}{Ma_n}) < \infty. \tag{16}$$

By (16) and similarly to the proof of (11), we can get

$$\sum_{n=1}^{\infty} P(|X_n| > Ma_n) = \sum_{n=1}^{\infty} E(I(|X_n| > Ma_n)) \leq \frac{1}{\delta} \sum_{n=1}^{\infty} Eg_n(\frac{X_n}{Ma_n}) < \infty. \tag{17}$$

By Hölder's inequality and the assumption of the function $g_n(x)$,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{|EX_n^{Ma_n}|}{Ma_n} &\leq \sum_{n=1}^{\infty} (E(\frac{|X_n|}{Ma_n})^\beta I(|X_n| \leq Ma_n))^{1/\beta} \\ &\leq (\frac{1}{\delta})^{1/\beta} \sum_{n=1}^{\infty} (Eg_n(\frac{X_n}{Ma_n}))^{1/\beta} < \infty. \end{aligned} \tag{18}$$

Since $(E(|X|^r))^{1/r}$ is increasing for $r > 0$, by $\beta \geq 2$ and (16),

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\mathrm{Var}(X_n^{Ma_n})}{(Ma_n)^2} \leq \sum_{n=1}^{\infty} \frac{E(X_n^{Ma_n})^2}{(Ma_n)^2} &\leq \sum_{n=1}^{\infty} (E(\frac{|X_n|}{Ma_n})^\beta I(|X_n| \leq Ma_n))^{2/\beta} \\ &\leq (\frac{1}{\delta})^{2/\beta} \sum_{n=1}^{\infty} (Eg_n(\frac{X_n}{Ma_n}))^{2/\beta} < \infty. \end{aligned} \tag{19}$$

Thus, $\sum_{n=1}^{\infty} X_n/a_n$ converges a.s. by (17), (18), (19) and Theorem 2.2. □

Theorem 2.5 (Marcinkiewicz Strong Law of Large Numbers) *Let $\{X_n, n \geq 1\}$ be a sequence* *of identically distributed ψ-mixing random variables with $E|X_1|^p < \infty$ for $0 < p < 2$ and $\sum_{n=1}^{\infty} \psi(n) < \infty$. Assume that $EX_1 = 0$ if $1 \le p < 2$. Then $\lim_{n\to\infty} \frac{1}{n^{1/p}} \sum_{k=1}^{n} X_k \to 0$ a.s..*

Proof Denote $Y_n = X_n I(|X_n| < n^{1/p})$. Then

$$\sum_{n=1}^{\infty} P(X_n \neq Y_n) = \sum_{n=1}^{\infty} P(|X_n| \ge n^{1/p}) \le CE|X_1|^p < \infty, \tag{20}$$

which implies that $P(X_n \neq Y_n,\ \text{i.o.}) = 0$ by Borel-Cantelli Lemma. Thus $\frac{1}{n^{1/p}} \sum_{k=1}^{n} X_k \to 0$ a.s. if and only if $\frac{1}{n^{1/p}} \sum_{k=1}^{n} Y_k \to 0$ a.s.. So we only need to show that

$$\frac{1}{n^{1/p}} \sum_{k=1}^{n} (Y_k - EY_k) \to 0 \ \text{ a.s.}, \quad n \to \infty, \tag{21}$$

$$\frac{1}{n^{1/p}} \sum_{k=1}^{n} EY_k \to 0, \quad n \to \infty. \tag{22}$$

By Theorem 2.1 and Kronecker's Lemma, to prove (21), it suffices to show that

$$\sum_{n=1}^{\infty} \mathrm{Var}(\frac{Y_n}{n^{1/p}}) < \infty. \tag{23}$$

In fact,

$$\begin{aligned} \sum_{n=1}^{\infty} \mathrm{Var}(\frac{Y_n}{n^{1/p}}) &\le \sum_{n=1}^{\infty} \frac{1}{n^{2/p}} \sum_{k=1}^{n} EX_1^2 I(k-1 \le |X_1|^p < k) \\ &\le C \sum_{k=1}^{\infty} k^{1-2/p} E|X_1|^p k^{(2-p)/p} I(k-1 \le |X_1|^p < k) < \infty. \end{aligned}$$

Hence (21) holds. Next, we will prove (22). It will be divided into two cases:

(i) If $p = 1$, by Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} EY_n = \lim_{n\to\infty} \int_{\Omega} X_1(\omega) I(|X_1(\omega)| < n^{1/p}) P(\mathrm{d}\omega) = EX_1 = 0.$$

Thus, $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} EY_k = 0$.

(ii) If $p \neq 1$, by the Kronecker's Lemma, to prove (22), it suffices to show that

$$\sum_{n=1}^{\infty} \frac{|EY_n|}{n^{1/p}} < \infty. \tag{24}$$

For $0 < p < 1$,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{|EY_n|}{n^{1/p}} &\le \sum_{n=1}^{\infty} \sum_{j=1}^{n} n^{-1/p} E|X_1| I(j-1 \le |X_1|^p < j) \\ &\le C \sum_{j=1}^{\infty} j^{1-1/p} E|X_1|^p j^{(1-p)/p} I(j-1 \le |X_1|^p < j) < \infty. \end{aligned}$$

For $1 \le p < 2$, by $EX_n = 0$, we can see that

$$\sum_{n=1}^{\infty} \frac{|EY_n|}{n^{1/p}} \le \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} n^{-1/p} E|X_1| I(j \le |X_1|^p < j+1)$$

$$\leq C\sum_{j=1}^{\infty} j^{1-1/p}E|X_1|^p j^{(1-p)/p}I(j \leq |X_1|^p < j+1) < \infty.$$

Thus (24) holds. We get the desired result. □

Theorem 2.6 *Let $\{X_n, n \geq 1\}$ be a sequence of ψ-mixing identically distributed random variables satisfying $\sum_{n=1}^{\infty}\psi(n) < \infty$ and $\{b_n, n \geq 1\}$ be a sequence of positive numbers satisfying $0 < b_n \uparrow \infty$. Assume that*

$$b_n^2\sum_{j=n}^{\infty} b_j^{-2} \ll n, \tag{25}$$

$$\sum_{n=1}^{\infty} P(|X_1| > b_n) < \infty. \tag{26}$$

Then

$$\sum_{n=1}^{\infty}\frac{X_n - EX_1I(|X_1| \leq b_n)}{b_n} \quad \text{converges a.s.}. \tag{27}$$

Furthermore, if $EX_n = 0$ for each n and

$$b_n\sum_{j=1}^{n} b_j^{-1} \ll n, \tag{28}$$

then $\sum_{n=1}^{\infty}\frac{X_n}{b_n}$ converges a.s. and $b_n^{-1}\sum_{k=1}^{n}X_k \to 0$ a.s..

Proof Let $b_0 = 0$ and $Y_n = X_nI(|X_n| \leq b_n)$ for each $n \geq 1$. By (26),

$$\sum_{n=1}^{\infty} nP(b_{n-1} < |X_1| \leq b_n) = \sum_{n=1}^{\infty}\sum_{j=0}^{n-1} P(b_{n-1} < |X_1| \leq b_n) = \sum_{j=0}^{\infty} P(|X_1| > b_j) < \infty. \tag{29}$$

Together with (29) and (25), it follows that

$$\begin{aligned}\sum_{j=1}^{\infty}\mathrm{Var}(\frac{Y_j}{b_j}) &\leq \sum_{j=1}^{\infty} b_j^{-2}\sum_{n=1}^{j} EX_1^2I(b_{n-1} < |X_1| \leq b_n)\\ &\ll \sum_{n=1}^{\infty} nb_n^{-2}EX_1^2I(b_{n-1} < |X_1| \leq b_n) \leq \sum_{n=1}^{\infty} nP(b_{n-1} < |X_1| \leq b_n) < \infty.\end{aligned}$$

Thus, $\sum_{n=1}^{\infty} b_n^{-1}(Y_n - EY_n)$ converges a.s. following from the above inequality and Theorem 2.1. (26) implies that $\sum_{n=1}^{\infty} P(X_n \neq Y_n) = \sum_{n=1}^{\infty} P(|X_n| > b_n) = \sum_{n=1}^{\infty} P(|X_1| > b_n) < \infty$. Therefore, $\sum_{n=1}^{\infty} b_n^{-1}(X_n - EY_n)$ converges a.s., which implies (27).

Furthermore, if $EX_1 = 0$, then by (28)

$$\begin{aligned}\sum_{n=1}^{\infty} b_n^{-1}|EY_n| &\leq \sum_{n=1}^{\infty} b_n^{-1}E|X_1|I(|X_1| > b_n) = \sum_{n=1}^{\infty} b_n^{-1}\sum_{j=n}^{\infty} E|X_1|I(b_j < |X_1| \leq b_{j+1})\\ &\ll \sum_{j=1}^{\infty}(j+1)b_{j+1}^{-1}E|X_1|I(b_j < |X_1| \leq b_{j+1})\\ &\leq \sum_{j=1}^{\infty}(j+1)P(b_j < |X_1| \leq b_{j+1}) < \infty.\end{aligned}$$

Thus, $\sum_{n=1}^{\infty}\frac{X_n}{b_n}$ converges a.s. from the above inequality and (27). The proof is completed. □

Remark 2.1 Khintchine-Kolmogorov-type convergence theorem, three series theorem and Marcinkiewicz strong law of large numbers for ψ-mixing sequence in the paper reach to the results of independent sequence under the condition $\sum_{n=1}^{\infty}\psi(n)<\infty$.

Remark 2.2 Strong law of large numbers for ψ-mixing sequence has been studied by some authors, for example, Blum, et al. [1]. They obtained the following result: Let $\{X_n, n\geq 1\}$ be a ψ-mixing process such that $EX_n=0$, $EX_n^2<\infty$ for every n. Suppose

(i) The random variables of the process are uniformly integrable and

(ii) $\sum_{n=1}^{\infty}EX_n^2/n^2<\infty$.

Then $P\{\lim_{n\to\infty}S_n/n=0\}=1$.

Corollary 2.2 in the paper generalizes the result above for $\beta=2$ to the case of $\beta\in(0,2]$ and $a_n=n$ to the case of arbitrary positive sequence $\{a_n, n\geq 1\}$. In addition, the condition (i) above is not needed in Corollary 2.2. Therefore, our Theorem 2.3, Corollaries 2.1 and 2.2 generalize and improve the result of Blum, et al. [1].

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