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The Structure of Quantum Group $\mathcal{U}_q(\mathrm{osp}(1,2,f))$

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Abstract In this paper we construct a new quantum group $\mathcal{U}_q(\mathrm{osp}(1,2,f))$, which can be seen as a generalization of $\mathcal{U}_q(\mathrm{osp}(1,2))$. A necessary and sufficient condition for the algebra $\mathcal{U}_q(\mathrm{osp}(1,2,f))$ to be a super Hopf algebra is obtained and the center $Z(\mathcal{U}_q(\mathrm{osp}(1,2,f)))$ is given.

Keywords super Hopf algebra; quantum Casimir element; Verma module.

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1. Introduction

The Lie superalgebra $G=\mathrm{osp}(1,2n)$ is a special one among the contragredient classical Lie superalgebras. Kac [1, 2] studied the representations of the Lie superalgebra G and indicated that its finite dimensional representations are completely reducible, hence its representation theory is rather similar to that of a semisimple Lie algebra. The deformation theory of Lie superalgebras has a close relation to the quantum Yang–Baxter equation and finds applications in areas of supersymmetry, integrable system and knot theory. The canonical example is the quantized enveloping algebra $\mathcal{U}_q(\mathrm{osp}(1,2n))$ of the Lie superalgebra G. Zou and Musson [3, 4] studied the integrable representations and crystal bases of $\mathcal{U}_q(\mathrm{osp}(1,2n))$.

In this paper we construct and study a new quantum group $\mathcal{U}_q(\mathrm{osp}(1,2,f))$, which can be seen as a natural generalization of $\mathcal{U}_q(\mathrm{osp}(1,2))$. This paper is organized as follows: In Section 2 we indicate that the algebra $\mathcal{U}_q(\mathrm{osp}(1,2,f))$ is Noetherian and has no zero divisors, furthermore, the set $\{E^iF^jK^l\}_{i,j\in\mathbb{N},l\in\mathbb{Z}}$ is its PBW basis. In Section 3 we give a necessary and sufficient condition for the algebra $\mathcal{U}_q(\mathrm{osp}(1,2,f))$ to be a super Hopf algebra. In Section 4 we construct the quantum Casimir element C_q of $\mathcal{U}_q(\mathrm{osp}(1,2,f))$ and prove that it generates the center of $\mathcal{U}_q(\mathrm{osp}(1,2,f))$ as a polynomial algebra.

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2. Quantum group $\mathcal{U}_q(\mathrm{osp}(1,2,f))$

Throughout the paper we suppose that k is the complex field and $q \in k^* = k \setminus \{0\}$ is not a root of the unit.

Definition 2.1 *Define $\mathcal{U}_q(\mathrm{osp}(1,2,f))$ as the algebra generated by the four variables E, F, K, K^{-1} with the relations*

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = qE, \quad KFK^{-1} = q^{-1}F, \quad EF + FE = f(K),$$

where $f(K) = \sum_{j=-N}^{N} a_j K^j \in k[K, K^{-1}]$ and $N \in \mathbb{Z}^+$.

Lemma 2.2 *Let $m \in \mathbb{N}$, and $n \in \mathbb{Z}$. The following relations hold in $\mathcal{U}_q(\mathrm{osp}(1,2,f))$:*

$$E^m K^n = q^{-mn} K^n E^m, \quad F^m K^n = q^{mn} K^n F^m.$$

For any Laurent polynomial $g(K) = \sum_{j=-N}^{N} a_j K^j \in k[K, K^{-1}]$, we first introduce the following polynomials defined by the change of coefficients. For any $s, m \in \mathbb{N}$, let

$$g^{+(s)}(K) = \sum_{j=-N}^{N} q^{js} a_j K^j, \quad g^{-(s)}(K) = \sum_{j=-N}^{N} q^{-js} a_j K^j,$$

$$g_{+(m)}(K) = \sum_{j=-N}^{N} (m)_{-q^j} a_j K^j, \quad g_{-(m)}(K) = \sum_{j=-N}^{N} (m)_{-q^{-j}} a_j K^j,$$

where $(n)_q = 1 + q^2 + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1}$.

Lemma 2.3 *1) For any $s \in \mathbb{N}$,*

$$g(K)F^s = F^s g^{-(s)}(K), \quad F^s g(K) = g^{+(s)}(K) F^s.$$

2) For any $m \in \mathbb{N}$,

$$g_{+(m)}(K) = \sum_{s=0}^{m-1} (-1)^s g^{+(s)}(K), \quad g_{-(m)}(K) = \sum_{s=0}^{m-1} (-1)^s g^{-(s)}(K).$$

Proof 1) For any $s \in \mathbb{N}$, we have

$$g(K)F^s = \sum_{j=-N}^{N} a_j K^j F^s = \sum_{j=-N}^{N} q^{-js} a_j F^s K^j = F^s g^{-(s)}(K).$$

Similarly, we can get $F^s g(K) = g^{+(s)}(K) F^s$.

2) For any $m \in \mathbb{N}$, we have

$$\sum_{s=0}^{m-1} (-1)^s g^{+(s)}(K) = \sum_{s=0}^{m-1} \sum_{j=-N}^{N} (-1)^s q^{js} a_j K^j = \sum_{j=-N}^{N} (\sum_{s=0}^{m-1} (-q^j)^s) a_j K^j$$
$$= \sum_{j=-N}^{N} (m)_{-q^j} a_j K^j.$$

Similarly, we can get $\sum_{s=0}^{m-1} (-1)^s g^{-(s)}(K) = g_{-(m)}(K)$. □

Lemma 2.4 *Let $m>0$. The following relations hold in $\mathcal{U}_q(\mathrm{osp}(1,2,f))$:*

$$EF^m-(-1)^mF^mE=(-1)^{m-1}F^{m-1}f_{-(m)}(K)=(-1)^{m-1}f_{+(m)}(K)F^{m-1}, \tag{2.1}$$

$$E^mF-(-1)^mFE^m=(-1)^{m-1}E^{m-1}f_{+(m)}(K)=(-1)^{m-1}f_{-(m)}(K)E^{m-1}. \tag{2.2}$$

Proof We only prove the equation (2.1) holds. Suppose $m>0$ is odd, then we have

$$\begin{aligned}EF^m+F^mE=&EF^m+FEF^{m-1}+F^2EF^{m-2}+\cdots+F^{m-1}EF+F^mE-\\&FEF^{m-1}-F^2EF^{m-2}-\cdots-F^{m-1}EF\\=&(EF+FE)F^{m-1}+F^2(EF+FE)F^{m-3}+\cdots+F^{m-1}(EF+FE)-\\&F(EF+FE)F^{m-2}-\cdots-F^{m-2}(EF+FE)F\\=&\sum_{i=0}^{m-1}(-1)^iF^{m-1-i}(EF+FE)F^i=\sum_{i=0}^{m-1}(-1)^iF^{m-1-i}f(K)F^i\\=&\sum_{i=0}^{m-1}(-1)^iF^{m-1-i}F^if^{-(i)}(K)=F^{m-1}\sum_{i=0}^{m-1}(-1)^if^{-(i)}(K)\\=&F^{m-1}f_{-(m)}(K).\end{aligned}$$

In a similar way, we have

$$\begin{aligned}EF^m+F^mE&=\sum_{i=0}^{m-1}(-1)^iF^i(EF+FE)F^{m-1-i}=\sum_{i=0}^{m-1}(-1)^if^{+(i)}(K)F^{m-1}\\&=f_{+(m)}(K)F^{m-1}.\end{aligned}$$

Suppose $m>0$ is even, then we have

$$\begin{aligned}EF^m-F^mE=&EF^m+FEF^{m-1}+F^2EF^{m-2}+\cdots+F^{m-1}EF-\\&FEF^{m-1}-F^2EF^{m-2}-\cdots-F^{m-1}EF-F^mE\\=&(EF+FE)F^{m-1}+F^2(EF+FE)F^{m-3}+\cdots+F^{m-2}(EF+FE)F-\\&F(EF+FE)F^{m-2}-\cdots-F^{m-1}(EF+FE)\\=&\sum_{i=0}^{m-1}(-1)^{i+1}F^{m-1-i}(EF+FE)F^i=\sum_{i=0}^{m-1}(-1)^{i+1}F^{m-1-i}f(K)F^i\\=&\sum_{i=0}^{m-1}(-1)^{i+1}F^{m-1-i}F^if^{-(i)}(K)=F^{m-1}\sum_{i=0}^{m-1}(-1)^{i+1}f^{-(i)}(K)\\=&-F^{m-1}f_{-(m)}(K).\end{aligned}$$

In a similar way, we have

$$\begin{aligned}EF^m-F^mE&=\sum_{i=0}^{m-1}(-1)^{i+1}F^i(EF+FE)F^{m-1-i}=\sum_{i=0}^{m-1}(-1)^{i+1}f^{+(i)}(K)F^{m-1}\\&=-f_{+(m)}(K)F^{m-1}.\ \square\end{aligned}$$

Proposition 2.5 *The algebra $\mathcal{U}_q(\mathrm{osp}(1,2,f))$ is Noetherian and has no zero divisors. The set $\{E^iF^jK^l\}_{i,j\in\mathbb{N},\ l\in\mathbb{Z}}$ is a basis of $\mathcal{U}_q(\mathrm{osp}(1,2,f))$.*

Proof Define $A_0 = k[K, K^{-1}]$. We shall construct two Ore extensions $A_0 \subset A_1 \subset A_2$ such that A_2 is isomorphic to $\mathcal{U}_q(\mathrm{osp}(1,2,f))$. Firstly, it is clear that A_0 is Noetherian and has no zero divisors. The set $\{K^l\}_{l\in\mathbb{Z}}$ is a basis of A_0.

Secondly, consider the automorphism α_1 of A_0 determined by $\alpha_1(K) = qK$ and corresponding Ore extension $A_1 = A_0[F, \alpha_1, 0]$. Then A_1 is the algebra generated by F, K, K^{-1} and satisfies the relation $FK = \alpha_1(K)F = qKF$. Moreover, A_1 is Noetherian and has no zero divisors. The set $\{F^jK^l\}_{j\in\mathbb{N},l\in\mathbb{Z}}$ is a basis of A_1.

Finally, we establish an Ore extension $A_2 = A_1[E, \alpha_2, \delta]$ by an automorphism of A_1 and an α_2–derivation of A_1 δ. The automorphism α_2 is determined by

$$\alpha_2(F^jK^l) = (-1)^j q^{-l} F^jK^l.$$

Let us take it as given for a moment (in Lemma 2.6) that there exists an α_2-derivation δ such that $\delta(F) = f(K)$ and $\delta(K) = 0$.

Then the following relations hold in A_2:

$$EK = \alpha_2(K)E + \delta(K) = qKE,$$

$$EF = \alpha_2(F)E + \delta(F) = -FE + f(K).$$

Hence, one easily concludes that A_2 is isomorphic to $\mathcal{U}_q(\mathrm{osp}(1,2,f))$ and $\mathcal{U}_q(\mathrm{osp}(1,2,f))$ has the required properties. □

To complete the proof of Proposition 2.5, it remains to prove the following technical lemma.

Lemma 2.6 *Set* $\delta(K^l) = 0$ *and* $\delta(F^jK^l) = (-1)^{j-1} f_{+(j)}(K)F^{j-1}K^l = (EF^j - (-1)^jF^jE)K^l$ *when* $j > 0$*. Then* δ *extends to an* α_2*-derivation of* A_1*.*

Proof We must check that for all $j, m \in \mathbb{N}$ and all $l, n \in \mathbb{Z}$, we have

$$\delta(F^jK^l \cdot F^mK^n) = \alpha_2(F^jK^l)\delta(F^mK^n) + \delta(F^jK^l)F^mK^n.$$

In fact, by Lemma 2.4 we have

$$\begin{aligned}
&\alpha_2(F^jK^l)\delta(F^mK^n) + \delta(F^jK^l)F^mK^n \\
&\quad = (-1)^j q^{-l} F^jK^l(EF^m - (-1)^mF^mE)K^n + (EF^j - (-1)^jF^jE)K^lF^mK^n \\
&\quad = (-1)^{j+m-1} q^{-l} F^jK^l f_{+(m)}(K)F^{m-1}K^n + (EF^j - (-1)^jF^jE)K^lF^mK^n \\
&\quad = (-1)^{j+m-1} q^{-ml} F^j f_{+(m)}(K)F^{m-1}K^{l+n} + q^{-ml}(EF^j - (-1)^jF^jE)F^mK^{l+n} \\
&\quad = q^{-ml}((-1)^jF^j(EF^m - (-1)^mF^mE) + (EF^j - (-1)^jF^jE)F^m)K^{l+n} \\
&\quad = q^{-ml}(EF^{j+m} - (-1)^{j+m}F^{j+m}E)K^{l+n} \\
&\quad = q^{-ml}\delta(F^{j+m}K^{l+n}) = \delta(F^jK^l \cdot F^mK^n). \quad \square
\end{aligned}$$

Definition 2.7 *Let* V *be a* $\mathcal{U}_q(\mathrm{osp}(1,2,f))$*-module and* $\lambda \neq 0$ *a scalar. An element* $v \neq 0$ *of* V *is a highest weight vector of weight* λ *if* $K \cdot v = \lambda v$ *and* $E \cdot v = 0$*. A* $\mathcal{U}_q(\mathrm{osp}(1,2,f))$*-module is the highest weight module of weight* λ *if it is generated by the highest weight vector.*

Let us fix a scalar $\lambda \neq 0$ and consider an infinite dimensional vector space $V(\lambda)$ with denumerable basis $\{v_i\}_{i\in\mathbb{N}}$. For $n \geq 0$, set

$$K \cdot v_n = \lambda q^{-n} v_n, \quad K^{-1} \cdot v_n = \lambda^{-1} q^n v_n,$$

$$E \cdot v_{n+1} = (-1)^n f_{-(n+1)}(\lambda) v_n, \quad E \cdot v_0 = 0, \quad F \cdot v_n = v_{n+1}.$$

Lemma 2.8 *The above relations define a $\mathcal{U}_q(\mathrm{osp}(1,2,f))$-module structure on $V(\lambda)$. Moreover, $V(\lambda)$ is the highest weight module of weight λ with its generator v_0. We shall call $V(\lambda)$ a Verma module of highest weight λ.*

Proof We only prove the relation $(EF + FE) \cdot v_n = f(K) \cdot v_n$ holds

$$\begin{aligned}(EF + FE) \cdot v_n &= E \cdot v_{n+1} + (-1)^{n-1} f_{-(n)}(\lambda) F \cdot v_{n-1} \\ &= (-1)^n f_{-(n+1)}(\lambda) v_n + (-1)^{n-1} f_{-(n)}(\lambda) v_n \\ &= (-1)^n (f_{-(n+1)}(\lambda) - f_{-(n)}(\lambda)) v_n = (-1)^n (-1)^n f^{-(n)}(\lambda) v_n \\ &= (\sum_j a_j \lambda^j q^{-jn}) v_n = f(K) \cdot v_n.\end{aligned}$$

Clearly, v_0 is the highest weight vector of weight λ and generates $V(\lambda)$. □

Remark 2.9 If λ satisfies the equation $f_{-(n)}(\lambda) = 0$ for some $n > 0$, then we have $E \cdot v_n = 0$ and $K \cdot v_n = \lambda q^{-n} v_n$. Thus the set $\{v_i\}_{i\geq n}$ spans a non-trivial submodule $L(\lambda)$ of $V(\lambda)$ which is isomorphic to $V(\lambda q^{-n})$.

3. The super Hopf algebra structure of $\mathcal{U}_q(\mathrm{osp}(1,2,f))$

In this section, we construct a super Hopf algebra structure in $\mathcal{U}_q(\mathrm{osp}(1,2,f))$. We first give the definition of super Hopf algebras.

Definition 3.1 ([7]) *A $\mathbb{Z}_2$-graded super Hopf algebra is a direct sum $H = H_0 \oplus H_1$ of vector subspaces such that the following conditions hold:*

1) H is an algebra such that $H_n H_m \subseteq H_{n+m}$ for all $m, n \in \mathbb{Z}_2$ and $1 \in H_0$;

2) H is a coalgebra such that $\Delta(H_n) \subseteq \oplus_{i+j=n} H_i \otimes H_j$, and $\varepsilon(H_1) = 0$;

3) $\Delta : H \to H \otimes H$, $\varepsilon : H \to k$ are algebra homomorphisms, where the product $\bullet$ of $H \otimes H$ is defined by

$$(x \otimes y_m) \bullet (x'_n \otimes y') = (-1)^{mn} x x'_n \otimes y_m y', \tag{3.1}$$

for $x, y' \in H, x'_n \in H_n, y_m \in H_m$;

4) There exists a linear mapping S such that $\mathrm{id} * S = S * \mathrm{id} = \eta\varepsilon$, and $S(H_n) \subseteq H_n$. Here, $*$ is the convolution and η is the unit map of H.

Clearly, $\mathcal{U}_q(\mathrm{osp}(1,2,f))$ is $\mathbb{Z}_2$-graded with the grading given by

$$\deg E = \deg F = 1, \quad \deg K = \deg K^{-1} = 0.$$

Lemma 3.2 *Assume that Δ is a morphism of algebras from $\mathcal{U}_q(\mathrm{osp}(1,2,f))$ to $\mathcal{U}_q(\mathrm{osp}(1,2,f)) \otimes \mathcal{U}_q(\mathrm{osp}(1,2,f))$ such that K and K^{-1} are group-like elements. If the Laurent polynomial $g(K) \in$*

$K[K,K^{-1}]$ is a group-like element, then $g(K)=K^m$ for some $m\in\mathbb{Z}$.

Proof Suppose that $g(K)=\sum_{i=-N}^{N} b_iK_i$. By the assumption, we have

$$\Delta g(K)=\sum_{i=-N}^{N}\sum_{j=-N}^{N} b_ib_jK_i\otimes K^j.$$

If there exist $i\neq j$ such that $b_ib_j\neq 0$, then $\Delta g(K)$ is not a group-like element by Proposition 2.5. Therefore $g(K)=bK^m$ for some $b\in k$ and $m\in\mathbb{Z}$. It is clear that $b^2=b$, and so $b=1$ or $b=0$, but the last is impossible. □

Proposition 3.3 *Assume that Δ is a morphism of algebras from $\mathcal{U}_q(\mathrm{osp}(1,2,f))$ to $\mathcal{U}_q(\mathrm{osp}(1,2,f))\otimes\mathcal{U}_q(\mathrm{osp}(1,2,f))$ such that*

$$\Delta(K)=K\otimes K,\quad \Delta(K^{-1})=K^{-1}\otimes K^{-1},$$
$$\Delta(E)=K^s\otimes E+E\otimes K^t,\quad \Delta(F)=K^n\otimes F+F\otimes K^m.$$

Then we have $n=-t, m=-s$ and $f(K)=a(K^{m-n}-K^{n-m})$ for some $a\in k\setminus\{0\}$.

Proof Suppose $f(K)=\sum_{j=-N}^{N} a_jK^j$, then we have

$$\begin{aligned}
&\Delta(E)\bullet\Delta(F)+\Delta(F)\bullet\Delta(E)\\
&\quad=(K^s\otimes E+E\otimes K^t)\bullet(K^n\otimes F+F\otimes K^m)+(K^n\otimes F+F\otimes K^m)\bullet(K^s\otimes E+E\otimes K^t)\\
&\quad=K^{s+n}\otimes EF+EK^n\otimes K^tF-K^sF\otimes EK^m+EF\otimes K^{t+m}+\\
&\qquad K^{n+s}\otimes FE+FK^s\otimes K^mE-K^nE\otimes FK^t+FE\otimes K^{m+t}\\
&\quad=K^{s+n}\otimes f(K)+(q^{-t}-q^n)EK^n\otimes FK^t+(q^m-q^{-s})FK^s\otimes EK^m+f(K)\otimes K^{t+m}+\\
&\quad=\sum_{j=-N}^{N} a_jK^{s+n}\otimes K^j+(q^{-t}-q^n)EK^n\otimes FK^t+\\
&\qquad(q^m-q^{-s})FK^s\otimes EK^m+\sum_{j=-N}^{N} a_jK^j\otimes K^{t+m},
\end{aligned}$$

and by Lemma 3.2, we have $\Delta(f(K))=\sum_{j=-N}^{N} a_jK^j\otimes K^j$.

Since $\Delta(E)\bullet\Delta(F)+\Delta(F)\bullet\Delta(E)=\Delta(f(K))$, and the set $\{E^iF^jK^l\}_{i,j\in\mathbb{N},\ l\in\mathbb{Z}}$ is a basis of $\mathcal{U}_q(\mathrm{osp}(1,2,f))$, by Proposition 2.5, we have that $q^m-q^{-s}=0$, $q^{-t}-q^n=0$ and thus $n=-t$, $m=-s$. Moreover,

$$\sum_{j=-N}^{N} a_jK^{s+n}\otimes K^j+\sum_{j=-N}^{N} a_jK^j\otimes K^{t+m}=\sum_{j=-N}^{N} a_jK^j\otimes K^j. \tag{3.2}$$

From the equation (3.2), we have $a_j=0$ when $j\neq n+s, m+t$. If $n+s=m+t$, then the equation (3.2) becomes

$$2a_{s+n}K^{s+n}\otimes K^{s+n}=a_{s+n}K^{s+n}\otimes K^{s+n},$$

hence, $a_{s+n}=0$, which is impossible. If $n+s\neq m+t$, then the equation (3.2) becomes

$$a_{s+n}K^{s+n}\otimes K^{s+n}+a_{m+t}K^{s+n}\otimes K^{m+t}+a_{s+n}K^{s+n}\otimes K^{m+t}+a_{m+t}K^{m+t}\otimes K^{m+t}$$

$$= a_{s+n}K^{s+n}\otimes K^{s+n} + a_{m+t}K^{m+t}\otimes K^{m+t},$$

hence, $a_{s+n}+a_{m+t}=0$. This is to say that $f(K)=a(K^{m-n}-K^{n-m})$ for some $a\in k$. □

Proposition 3.4 *Assume $f(K)$ is a non-zero Laurent polynomial in $k[K,K^{-1}]$. Then the algebra $\mathcal{U}_q(\mathrm{osp}(1,2,f))$ is a $\mathbb{Z}_2$-graded super Hopf algebra such that K,K^{-1} are group-like elements and E,F are skew primitives if and only if $f(K)=a(K^m-K^{-m})$ for some $0\neq a\in k$, $m\in\mathbb{Z}_+$, and the following relations*

$$\begin{aligned}&\Delta(K)=K\otimes K,\quad \varepsilon(K)=1,\quad S(K)=K^{-1},\\ &\Delta(K^{-1})=K^{-1}\otimes K^{-1},\quad \varepsilon(K^{-1})=1,\quad S(K^{-1})=K,\\ &\Delta(E)=E\otimes K^s+K^t\otimes E,\quad \varepsilon(E)=0,\quad S(E)=-K^{-t}EK^{-s},\\ &\Delta(F)=F\otimes K^{-t}+K^{-s}\otimes F,\quad \varepsilon(F)=0,\quad S(F)=-K^sFK^t\end{aligned}\tag{3.3}$$

hold for some $s,t\in\mathbb{Z}$ with $m=t-s$.

Proof The necessity is clear from Lemma 3.2 and Proposition 3.3. The sufficiency can be proved similarly to [5]. □

Proposition 3.5 *For all $i,j\in\mathbb{N}$ and $l\in\mathbb{Z}$, we have*

$$\begin{aligned}\Delta(E^iF^jK^l)=&\sum_{r=0}^{i}\sum_{k=0}^{j}(-1)^{k(i-r)}q^{(i-j+k-r)(rs-kt)}\binom{i}{r}_{-q^{t-s}}\binom{j}{k}_{-q^{s-t}}\times\\ &E^rF^kK^{(i-r)t-(j-k)s+l}\otimes E^{i-r}F^{j-k}K^{rs-kt+l}.\end{aligned}$$

Proof First observe that

$$\begin{aligned}\Delta(E^iF^jK^l)&=\Delta(E)^i\bullet\Delta(F)^j\bullet\Delta(K)^l\\ &=(E\otimes K^s+K^t\otimes E)^i\bullet(F\otimes K^{-t}+K^{-s}\otimes F)^j\bullet(K\otimes K)^l.\end{aligned}$$

Now,

$$(K^t\otimes E)\bullet(E\otimes K^s)=-q^{t-s}(E\otimes K^s)\bullet(K^t\otimes E),$$

and

$$(K^{-s}\otimes F)\bullet(F\otimes K^{-t})=-q^{s-t}(F\otimes K^{-t})\bullet(K^{-s}\otimes F).$$

Thus, we get

$$\begin{aligned}\Delta(E)^i&=\sum_{r=0}^{i}\binom{i}{r}_{-q^{t-s}}(E\otimes K^s)^r\bullet(K^t\otimes E)^{i-r}\\ &=\sum_{r=0}^{i}q^{(i-r)rs}\binom{i}{r}_{-q^{t-s}}(E^rK^{(i-r)t}\otimes E^{i-r}K^{rs}),\end{aligned}$$

and

$$\Delta(F)^j=\sum_{k=0}^{j}\binom{j}{k}_{-q^{s-t}}(F\otimes K^{-t})^k\bullet(K^{-s}\otimes F)^{j-k}$$

$$= \sum_{k=0}^{j} q^{(j-k)kt} \binom{j}{k}_{-q^{s-t}} (F^k K^{-(j-k)s} \otimes F^{j-k} K^{-kt}).$$

So, we have

$$\begin{aligned}
\Delta(E^i F^j K^l) =& (E \otimes K^s + K^t \otimes E)^i \bullet (F \otimes K^{-t} + K^{-s} \otimes F)^j \bullet (K \otimes K)^l \\
=& \sum_{r=0}^{i} \sum_{k=0}^{j} q^{(i-r)rs} q^{(j-k)kt} \binom{i}{r}_{-q^{t-s}} \binom{j}{k}_{-q^{s-t}} \times \\
& (E^r K^{(i-r)t} \otimes E^{i-r} K^{rs}) \bullet (F^k K^{-(j-k)s} \otimes F^{j-k} K^{-kt}) \bullet (K^l \otimes K^l) \\
=& \sum_{r=0}^{i} \sum_{k=0}^{j} q^{(i-r)rs+(j-k)kt} \binom{i}{r}_{-q^{t-s}} \binom{j}{k}_{-q^{s-t}} \times \\
& (-1)^{k(i-r)} (E^r K^{(i-r)t} F^k K^{-(j-k)s} K^l \otimes E^{i-r} K^{rs} F^{j-k} K^{-kt+l}) \\
=& \sum_{r=0}^{i} \sum_{k=0}^{j} (-1)^{k(i-r)} q^{(i-r)rs+(j-k)kt} \binom{i}{r}_{-q^{t-s}} \binom{j}{k}_{-q^{s-t}} \times \\
& q^{-k(i-r)t} q^{-rs(j-k)} E^r F^k K^{(i-r)t} K^{-(j-k)s} K^l \otimes E^{i-r} F^{j-k} K^{rs-kt+l} \\
=& \sum_{r=0}^{i} \sum_{k=0}^{j} (-1)^{k(i-r)} q^{(i-j+k-r)(rs-kt)} \binom{i}{r}_{-q^{t-s}} \binom{j}{k}_{-q^{s-t}} \times \\
& E^r F^k K^{(i-r)t-(j-k)s+l} \otimes E^{i-r} F^{j-k} K^{rs-kt+l}. \square
\end{aligned}$$

4. The center of $\mathcal{U}_q(\mathrm{osp}(1,2,f))$

Clearly, $\mathcal{U}_q(\mathrm{osp}(1,2,f))$ has also a $\mathbb{Z}$-graded structure with

$$\deg K^{\pm 1} = 0, \quad \deg E = 1, \quad \deg F = -1.$$

Thus we have $\mathcal{U}_q(\mathrm{osp}(1,2,f)) = \oplus_{m\in\mathbb{Z}} \mathcal{U}_m$, where $\mathcal{U}_m = \langle F^i K^l E^{i+m} | i \in \mathbb{N}, l \in \mathbb{Z}\rangle$. In particular, $\mathcal{U}_0 = \langle F^i K^l E^i | i \in \mathbb{N}, l \in \mathbb{Z}\rangle$. Set $\mathcal{U}^0 = k[K, K^{-1}]$, then any element x_0 in $\mathcal{U}_0$ has a unique form $x_0 = \sum_{i\in\mathbb{N}} F^i h_i E^i$, here $h_i \in \mathcal{U}_0$.

Let $r \in k$. The map $\phi_r : \mathcal{U}_0 \to \mathcal{U}_0$ defined by $\phi_r(g(K)) = g(rK)$ is an algebra isomorphism, and the image of $g(K)$ is denoted by $\phi_r g$.

Denote the center of $\mathcal{U}_q(\mathrm{osp}(1,2,f))$ by $\mathrm{Z}(\mathcal{U}_q(\mathrm{osp}(1,2,f)))$. Now, we construct an element $C_q \in \mathrm{Z}(\mathcal{U}_q(\mathrm{osp}(1,2,f)))$, which will be called the quantum Casimir element of $\mathcal{U}_q(\mathrm{osp}(1,2,f))$, and discuss its properties.

Lemma 4.1 *The element of $\mathrm{Z}(\mathcal{U}_q(\mathrm{osp}(1,2,f)))$ belongs to $\mathcal{U}_0$.*

Proposition 4.2 *Let $x = \sum_{i\in\mathbb{N}} F^i h_i E^i \in \mathcal{U}_0$. Then we have $x \in \mathrm{Z}(\mathcal{U}_q(\mathrm{osp}(1,2,f)))$ if and only if*

$$h_i = (-1)^i f_{-(i+1)}(K) h_{i+1} + (-1)^i \phi_{q^{-1}} h_i \tag{4.1}$$

holds for all $i \in \mathbb{N}$.

Proof On the one hand, by the equation (2.1), we have

$$\begin{aligned}Ex &= \sum_{i\in\mathbb{N}} EF^ih_iE^i = \sum_{i\in\mathbb{N}}((-1)^{i-1}F^{i-1}f_{-(i)}(K)+(-1)^iF^iE)h_iE^i\\ &= \sum_{i\in\mathbb{N}}(-1)^iF^if_{-(i+1)}(K)h_{i+1}E^{i+1}+\sum_{i\in\mathbb{N}}(-1)^iF^iEh_iE^i\\ &= \sum_{i\in\mathbb{N}}(-1)^iF^if_{-(i+1)}(K)h_{i+1}E^{i+1}+\sum_{i\in\mathbb{N}}(-1)^iF^i\phi_{q^{-1}}h_iE^{i+1}\\ &= F^i\sum_{i\in\mathbb{N}}((-1)^if_{-(i+1)}(K)h_{i+1}+(-1)^i\phi_{q^{-1}}h_i)E^{i+1}.\end{aligned}$$

On the other hand, we have $xE=\sum_{i\in\mathbb{N}}F^ih_iE^{i+1}$. Thus $Ex=xE$ if and only if the equation (4.1) holds.

Similarly, we can get $Fx=xF$ if and only if the equation (4.1) holds, and $xK=Kx$ holds for any $x\in\mathcal{U}_0$. □

From Propositions 2.5 and 4.2, we know that $h_1,h_2,\dots$ are uniquely determined by h_0 and that $h_i=0$ for all $i>2$ when $h_2\in k$. By the equation (4.1), we have

$$f(K)h_1+\phi_{q^{-1}}h_0=h_0, \tag{4.2}$$

$$(\phi_{q^{-1}}f(K))h_2-f(K)h_2=\phi_{q^{-1}}h_1+h_1. \tag{4.3}$$

In the following section, we assume that $f(K)=a(K^m-K^{-m})$ for some $a\in k\setminus\{0\}$ and $m>0$. Set $t=\frac{-1}{(q^{-m/2}+q^{m/2})^2}$ and assume $x=\sum_{i\in\mathrm{N}}F^ih_iE^i\in \mathrm{Z}(\mathcal{U}_q(\mathrm{osp}(1,2,f)))$ with $h_i\in\mathcal{U}^0$. First, we consider the equation (4.3):

$$\begin{aligned}\phi_{q^{-1}}h_1+h_1 &= (\phi_{q^{-1}}f(K))h_2-f(K)h_2\\ &= a(q^{-m}K^m-q^mK^{-m})h_2-a(K^m-K^{-m})h_2\\ &= a((q^{-m}-1)K^m-(q^m-1)K^{-m})h_2\\ &= a((q^{-m}+1)\frac{q^{-m}-1}{q^{-m}+1}K^m-(q^m+1)\frac{q^m-1}{q^m+1}K^{-m})h_2\\ &= a(q^m-q^{-m})((q^{-m}+1)\frac{q^{-m}-1}{(q^{-m}+1)(q^m-q^{-m})}K^m-(q^m+1)\frac{q^m-1}{(q^m+1)(q^m-q^{-m})}K^{-m})h_2\\ &= a(q^m-q^{-m})((q^{-m}+1)\frac{q^{-2m}-1}{(q^{-m}+1)^2(q^m-q^{-m})}K^m-(q^m+1)\frac{q^{2m}-1}{(q^m+1)^2(q^m-q^{-m})}K^{-m})h_2\\ &= a(q^m-q^{-m})((q^{-m}+1)\frac{-1}{(q^{-m}+1)^2q^m}K^m-(q^m+1)\frac{1}{(q^m+1)^2q^{-m}}K^{-m})h_2\\ &= a(q^m-q^{-m})((q^{-m}+1)\frac{-1}{(q^{-m/2}+q^{m/2})^2}K^m+(q^m+1)\frac{-1}{(q^{m/2}+q^{-m/2})^2}K^{-m})h_2\\ &= a(q^m-q^{-m})((q^{-m}+1)tK^m+(q^m+1)tK^{-m})h_2\\ &= a(q^m-q^{-m})t((q^{-m}K^m+q^mK^{-m})+(K^m+K^{-m}))h_2.\end{aligned}$$

So, when $h_2\in k$, we have that $P=a(q^m-q^{-m})t(K^m+K^{-m})h_2$ is an h_1 in the equation (4.3). Then, we choose h_1 to be P and consider the equation (4.2):

$$h_0-\phi_{q^{-1}}h_0=a^2(q^m-q^{-m})t(K^m-K^{-m})(K^m+K^{-m})h_2$$

$$= a^2(q^m - q^{-m})t(K^{2m} - K^{-2m})h_2$$
$$= a^2t(q^mK^{2m} - q^mK^{-2m} - q^{-m}K^{2m} + q^{-m}K^{-2m})h_2$$
$$= a^2t((q^mK^{2m} + q^{-m}K^{-2m}) - (q^{-m}K^{2m} + q^mK^{-2m}))h_2.$$

So, when $h_2 \in k$, we have that $a^2t(q^mK^{2m}+q^{-m}K^{-2m})h_2$ is an h_0 in the equation (4.2). Choose $h_2 = 1$ and we get an element C_q in $\mathrm{Z}(\mathcal{U}_q(\mathrm{osp}(1,2,f)))$,

$$C_q = F^2E^2 + at(q^m - q^{-m})F(K^m + K^{-m})E + a^2t(q^mK^{2m} + q^{-m}K^{-2m}), \tag{4.4}$$

which is called quantum Casimir element.

Let π be a map from $\mathcal{U}_0$ to $\mathcal{U}^0$ defined by $\pi(\sum_i F^ih_iE^i) = h_0$. Then π is an algebra map and is called Harish-Chandra map. The element z in $\mathcal{U}_0$ can be written as $\pi(z) + \sum_{i>0} F^ih_iE^i$. By the equation (4.1) we can easily get the following lemma.

Lemma 4.3 *$\pi|_Z$ is injective from $\mathrm{Z}(\mathcal{U}_q(\mathrm{osp}(1,2,f)))$ to $\mathcal{U}^0$.*

For any $z \in \mathcal{U}_0$, note that $\pi(z)$ is a Laurent polynomial in K, we denote its value at λ by $\pi(z)(\lambda)$.

Lemma 4.4 *Let V be the highest weight $\mathcal{U}_q(\mathrm{osp}(1,2,f))$-module with highest weight λ. Then, for any central element z of $\mathcal{U}_q(\mathrm{osp}(1,2,f))$ and any $v \in V$, we have*

$$z \cdot v = \pi(z)(\lambda)v.$$

Proof The proof is similar to that of Theorem 6.4.4 in [5]. □

In order to determine the center of $\mathcal{U}_q(\mathrm{osp}(1,2,f))$, we set $p = \sqrt{q}$ and write

$$\overline{P}(\lambda) = P(p^{-1}\lambda) = (\phi_{p^{-1}}P)(\lambda),$$

for any Laurent polynomial $P(K)$ in $k[K, K^{-1}]$.

Lemma 4.5 *For any element z in the center of $\mathcal{U}_q(\mathrm{osp}(1,2,f))$, we have*

$$\overline{\pi(z)}(\xi\lambda) = \overline{\pi(z)}(\xi\lambda^{-1}), \quad \overline{\pi(z)}(\zeta\lambda) = \overline{\pi(z)}(\zeta\lambda^{-1})$$

for all $\lambda \in k$ and ξ with $\xi^{2m} = 1$ and ζ with $\zeta^{2m} = -1$.

Proof For ξ with $\xi^{2m} = 1$ and any odd integer $n > 0$. Consider the Verma module $V(\xi p^{n-1})$. By Lemma 2.8, we have $E \cdot v_n = f_{-(n)}(\xi p^{n-1})v_{n-1} = 0$. In fact,

$$\begin{aligned} f_{-(n)}(\xi p^{n-1}) &= (n)_{-q^{-m}}(\xi p^{n-1})^m - (n)_{-q^m}(\xi p^{n-1})^{-m} \\ &= ((n)_{-q^{-m}}(\xi p^{n-1})^{2m} - (n)_{-q^m})(\xi p^{n-1})^{-m} \\ &= ((n)_{-q^{-m}}((q^m)^{n-1}) - (n)_{-q^m})(\xi p^{n-1})^{-m} \\ &= ((n)_{-q^{-m}}((-q^m)^{n-1}) - (n)_{-q^m})(\xi p^{n-1})^{-m} \\ &= 0. \end{aligned}$$

Thus, by Remark 2.9 v_n is the highest weight vector of weight $\xi p^{n-1}q^{-n} = \xi p^{n-1}p^{-2n} = \xi p^{-n-1}$.

Then, we have $\pi(z)(\xi p^{n-1}) = \pi(z)(\xi p^{-n-1})$ by Lemma 4.4. In other words, we have

$$\overline{\pi(z)}(\xi p^n) = \overline{\pi(z)}(\xi p^{-n}),$$

as what we want because of the random of n.

For ζ with $\zeta^{2m} = -1$ and any even integer $n > 0$. Consider the Verma module $V(\zeta p^{n-1})$. Similarly, we can get $\overline{\pi(z)}(\zeta\lambda) = \overline{\pi(z)}(\zeta\lambda^{-1})$. □

Lemma 4.6 *Any Laurent polynomial of $k[K, K^{-1}]$ satisfying the relations $P(\xi\lambda) = P(\xi\lambda^{-1})$ and $P(\zeta\lambda) = P(\zeta\lambda^{-1})$ for all $\lambda \in k$ and ξ with $\xi^{2m} = 1$ and ζ with $\zeta^{2m} = -1$ is a polynomial in $K^{2m} + K^{-2m}$.*

Proof It can be proved by induction. □

Theorem 4.7 *The center of $\mathcal{U}_q(\mathrm{osp}(1,2,f))$ is a polynomial algebra generated by the Casimir element C_q. The restriction of the Harish-Chandra homomorphism π to $\mathrm{Z}(\mathcal{U}_q(\mathrm{osp}(1,2,f)))$ is an isomorphism onto the subalgebra of $k[K, K^{-1}]$ generated by $q^m K^{2m} + q^{-m} K^{-2m}$.*

Proof We know that the restriction of π to the center is injective, and we are left with determining its image. By Lemmas 4.5 and 4.6 the latter is contained in the subalgebra of $k[K, K^{-1}]$ generated by $q^m K^{2m} + q^{-m} K^{-2m}$. In fact,

$$\overline{q^m K^{2m} + q^{-m} K^{-2m}} = q^m q^{-m} K^{2m} + q^{-m} q^m K^{-2m} = K^{2m} + K^{-2m}.$$

Consider the image of the Casimir element C_q defined by (4.4), we have

$$\pi(C_q) = a^2 t(q^m K^{2m} + q^{-m} K^{-2m}),$$

which implies that the image is the whole subalgebra and that C_q generates the center. The latter is a polynomial algebra because the powers of $q^m K^{2m} + q^{-m} K^{-2m}$ are linearly independent for obvious reasons of degree. □

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