

On a New Reverse Extended Hardy's Integral Inequality

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Abstract In this paper, a new reverse extended Hardy's integral inequality is proved by means of weight coefficients and the technique of real analysis. Some particular results are considered.

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1. Introduction

In 1920, Hardy [1] obtained the following Hardy's integral inequality: If $p > 1$, $f \geq 0$, $0 < \int_0^\infty f^p(x)dx < \infty$, $F(x) = \int_0^x f(t)dt$, then

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \quad (1)$$

where the constant factor $\left(\frac{p}{p-1}\right)^p$ is the best possible (also cf. Theorem 327 in [2]). And in Theorem 328 of [2], it follows that

$$\int_0^\infty F^p(x)dx < p^p \int_0^\infty (xf(x))^p dx, \quad (2)$$

where the constant factor p^p is still the best possible and $F(x) = \int_x^\infty f(t)dt$. In 1928, Hardy [3] gave best extensions of (1) and (2) by introducing an independent parameter λ as follows. For $\lambda \neq 1$, let $F(x) = \int_0^x f(t)dt$ if $\lambda > 1$ and $F(x) = \int_x^\infty f(t)dt$ if $\lambda < 1$. Then

$$\int_0^\infty x^{-\lambda} F^p(x)dx < \left(\frac{p}{|\lambda-1|}\right)^p \int_0^\infty x^{-\lambda} (xf(x))^p dx, \quad p > 1; \quad (3)$$

$$\int_0^\infty x^{-\lambda} F^p(x)dx > \left(\frac{p}{|\lambda-1|}\right)^p \int_0^\infty x^{-\lambda} (xf(x))^p dx, \quad 0 < p < 1, \quad (4)$$

where the constant factor $\left(\frac{p}{|\lambda-1|}\right)^p$ is still the best possible (also cf. Theorem 330 and Theorem 347 in [2]). For $\lambda = p$, inequality (3) reduces to (1); for $\lambda = 0$, (3) reduces to (2). Inequalities (3) and (4) are called the extended Hardy's integral inequality and the reverse extended Hardy's integral inequality, respectively. Inequalities (1)–(4) are important in analysis and its applications [2, 4]. But for $p < 0$, any inequality similar to (4) has not yet been considered since then.

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In this paper, by means of weight function and the technic of real analysis, we consider the case of (4) for $p < 0$. That is,

Theorem 1 For $p < 0$, $r < 0$, $\lambda \neq 1$, let $F_\lambda(x) = \int_0^x f(t)dt$ if $\lambda < 1$ and $F_\lambda(x) = \int_x^\infty f(t)dt$ if $\lambda > 1$. Then one has

$$\int_0^\infty x^{\frac{r}{\lambda-1}-1} F_\lambda^p(x) dx < \left| \frac{r}{\lambda-1} \right|^p \int_0^\infty x^{p(1+\frac{1-\lambda}{r})-1} f^p(x) dx, \quad (5)$$

where the constant $\left| \frac{r}{\lambda-1} \right|^p$ is the best possible. In particular, for $r = p < 0$,

$$\int_0^\infty x^{-\lambda} F_\lambda^p(x) dx < \left| \frac{p}{\lambda-1} \right|^p \int_0^\infty t^{-\lambda} (xf(x))^p dx; \quad (6)$$

for $r = -|\lambda - 1| < 0$,

$$\int_0^\infty x^{p \frac{\lambda-1}{|\lambda-1|}-1} F_\lambda^p(x) dx < \int_0^\infty x^{p(1+\frac{\lambda-1}{|\lambda-1|})-1} f^p(x) dx. \quad (7)$$

2. The preliminary theorems

First, we introduce two propositions similar to the reverses of Hilbert-type integral inequalities [5].

Proposition 1 If $p < 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $r < 0$, $\lambda < 1$, $f, g \geq 0$, $0 < \int_0^\infty t^{p(1+\frac{1-\lambda}{r})-1} f^p(t) dt < \infty$, $0 < \int_0^\infty t^{q(1-\frac{1-\lambda}{r})-1} g^q(t) dt < \infty$, then

$$\begin{aligned} I &:= \int_0^\infty \int_0^x f(y)g(x) dy dx = \int_0^\infty \int_y^\infty f(y)g(x) dx dy \\ &> \frac{r}{\lambda-1} \left\{ \int_0^\infty y^{p(1+\frac{1-\lambda}{r})-1} f^p(y) dy \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{q(1-\frac{1-\lambda}{r})-1} g^q(x) dx \right\}^{\frac{1}{q}}, \end{aligned} \quad (8)$$

where the constant factor $\frac{r}{\lambda-1}$ is the best possible.

Proof By the reverse Hölder's inequality [6], one obtains

$$\begin{aligned} I &= \int_0^\infty \int_0^x \left[\frac{y^{(1+\frac{1-\lambda}{r})/q}}{x^{(1-\frac{1-\lambda}{r})/p}} f(y) \right] \left[\frac{x^{(1-\frac{1-\lambda}{r})/p}}{y^{(1+\frac{1-\lambda}{r})/q}} g(x) \right] dy dx \\ &\geq \left\{ \int_0^\infty \int_0^x \frac{y^{(1+\frac{1-\lambda}{r})\frac{q}{p}}}{x^{1-\frac{1-\lambda}{r}}} f^p(y) dy dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \int_0^x \frac{x^{(1-\frac{1-\lambda}{r})\frac{q}{p}}}{y^{1+\frac{1-\lambda}{r}}} g^q(x) dy dx \right\}^{\frac{1}{q}} \\ &= \left\{ \int_0^\infty \left[\int_y^\infty \frac{y^{(1+\frac{1-\lambda}{r})\frac{q}{p}}}{x^{1-\frac{1-\lambda}{r}}} dx \right] f^p(y) dy \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \left[\int_0^x \frac{x^{(1-\frac{1-\lambda}{r})\frac{q}{p}}}{y^{1+\frac{1-\lambda}{r}}} dy \right] g^q(x) dx \right\}^{\frac{1}{q}} \\ &= \frac{r}{\lambda-1} \left\{ \int_0^\infty y^{p(1+\frac{1-\lambda}{r})-1} f^p(y) dy \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{q(1-\frac{1-\lambda}{r})-1} g^q(x) dx \right\}^{\frac{1}{q}}. \end{aligned} \quad (9)$$

Obviously (9) takes the strict sign-inequality. If (9) takes the form of equality, then there exist A and B , such that they are not all zero and

$$A \frac{y^{(1+\frac{1-\lambda}{r})\frac{q}{p}}}{x^{1-\frac{1-\lambda}{r}}} f^p(y) = B \frac{x^{(1-\frac{1-\lambda}{r})\frac{q}{p}}}{y^{1+\frac{1-\lambda}{r}}} g^q(x) \text{ a.e. in } D,$$

where $D = \{(x, y) | 0 < y \leq x, 0 < x < \infty\}$. It follows $Ay^{p(1+\frac{1-\lambda}{r})}f^p(y) = Bx^{q(1-\frac{1-\lambda}{r})}g^q(x) = C$ a.e. in D , and C is a constant. Suppose $A \neq 0$. For any $x > 0$, $y^{p(1+\frac{1-\lambda}{r})-1}f^p(y) = C/(Ay)$ a.e. in $(0, x]$, which is equivalent to $y^{p(1+\frac{1-\lambda}{r})-1}f^p(y) = C/(Ay)$ a.e. in $(0, \infty)$. This contradicts the fact that $0 < \int_0^\infty y^{p(1+\frac{1-\lambda}{r})-1}f^p(y)dy < \infty$. Hence one has (8). For $\varepsilon > 0$, setting $f_\varepsilon(t) = g_\varepsilon(t) = 0, t \in (0, 1)$; $f_\varepsilon(t) = t^{-1-\frac{1-\lambda}{r}-\frac{\varepsilon}{p}}, g_\varepsilon(t) = t^{-1+\frac{1-\lambda}{r}-\frac{\varepsilon}{q}}, t \in [1, \infty)$, we have

$$\begin{aligned} I_\varepsilon &= \varepsilon \int_0^\infty \int_0^x f_\varepsilon(y)g_\varepsilon(x)dydx \\ &\leq \varepsilon \int_1^\infty x^{-1+\frac{1-\lambda}{r}-\frac{\varepsilon}{q}} \left(\int_0^x y^{-1-\frac{1-\lambda}{r}-\frac{\varepsilon}{p}} dy \right) dx = \frac{rp}{p(\lambda-1)-r\varepsilon}; \end{aligned} \quad (10)$$

$$J_\varepsilon = \varepsilon \left\{ \int_0^\infty y^{p(1+\frac{1-\lambda}{r})-1} f_\varepsilon^p(y) dy \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{q(1-\frac{1-\lambda}{r})-1} g_\varepsilon^q(x) dx \right\}^{\frac{1}{q}} = 1. \quad (11)$$

If there exists $K \geq \frac{r}{\lambda-1}$, such that (8) is still valid if one replaces $\frac{r}{\lambda-1}$ by K . In particular, by (10)–(11), $\frac{rp}{p(\lambda-1)-r\varepsilon} \geq I_\varepsilon > KJ_\varepsilon = K$, and for $\varepsilon \rightarrow 0^+$, it follows $\frac{r}{\lambda-1} \geq K$. Hence $K = \frac{r}{\lambda-1}$ is the best value. The proposition is proved.

Proposition 2 If $\lambda > 1$, the other is as the assumption of Proposition 1, then

$$\begin{aligned} \tilde{I} &:= \int_0^\infty \int_x^\infty f(y)g(x)dydx = \int_0^\infty \int_0^y f(y)g(x)dx dy \\ &> \frac{r}{1-\lambda} \left\{ \int_0^\infty y^{p(1+\frac{1-\lambda}{r})-1} f^p(y) dy \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{q(1-\frac{1-\lambda}{r})-1} g^q(x) dx \right\}^{\frac{1}{q}}, \end{aligned} \quad (12)$$

where the constant factor $\frac{r}{1-\lambda}$ is the best possible.

Proof By Hölder's inequality [6], one obtains

$$\begin{aligned} \tilde{I} &= \int_0^\infty \int_0^y \left[\frac{y^{(1+\frac{1-\lambda}{r})/q}}{x^{(1-\frac{1-\lambda}{r})/p}} f(y) \right] \left[\frac{x^{(1-\frac{1-\lambda}{r})/p}}{y^{(1+\frac{1-\lambda}{r})/q}} g(x) \right] dx dy \\ &\geq \left\{ \int_0^\infty \int_0^y \frac{y^{(1+\frac{1-\lambda}{r})\frac{q}{p}}}{x^{1-\frac{1-\lambda}{r}}} f^p(y) dx dy \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \int_0^y \frac{x^{(1-\frac{1-\lambda}{r})\frac{q}{p}}}{y^{1+\frac{1-\lambda}{r}}} g^q(x) dx dy \right\}^{\frac{1}{q}} \\ &= \left\{ \int_0^\infty \left[\int_0^y \frac{y^{(1+\frac{1-\lambda}{r})\frac{q}{p}}}{x^{1-\frac{1-\lambda}{r}}} dx \right] f^p(y) dy \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \left[\int_x^\infty \frac{x^{(1-\frac{1-\lambda}{r})\frac{q}{p}}}{y^{1+\frac{1-\lambda}{r}}} dy \right] g^q(x) dx \right\}^{\frac{1}{q}} \\ &= \frac{r}{1-\lambda} \left\{ \int_0^\infty y^{p(1+\frac{1-\lambda}{r})-1} f^p(y) dy \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{q(1-\frac{1-\lambda}{r})-1} g^q(x) dx \right\}^{\frac{1}{q}}. \end{aligned}$$

Similarly, one has (12). For $0 < \varepsilon < \frac{p}{r}(\lambda-1)$, setting $f_\varepsilon, g_\varepsilon$ as Theorem 2, then we have

$$\begin{aligned} \tilde{I}_\varepsilon &= \varepsilon \int_0^\infty \int_x^\infty f_\varepsilon(y)g_\varepsilon(x)dydx \\ &\leq \varepsilon \int_1^\infty x^{-1+\frac{1-\lambda}{r}-\frac{\varepsilon}{q}} \left(\int_x^\infty y^{-1-\frac{1-\lambda}{r}-\frac{\varepsilon}{p}} dy \right) dx = \frac{pr}{p(1-\lambda)+r\varepsilon}; \\ \tilde{J}_\varepsilon &= \varepsilon \left\{ \int_0^\infty t^{p(1+\frac{1-\lambda}{r})-1} f_\varepsilon^p(t) dt \right\}^{\frac{1}{p}} \left\{ \int_0^\infty t^{q(1-\frac{1-\lambda}{r})-1} g_\varepsilon^q(t) dt \right\}^{\frac{1}{q}} = 1. \end{aligned}$$

If there exists $K \geq \frac{r}{1-\lambda}$, such that (12) is still valid if one replaces $\frac{r}{1-\lambda}$ by K . In particular, one has $\frac{pr}{p(1-\lambda)+r\varepsilon} \geq \tilde{I}_\varepsilon > K\tilde{J}_\varepsilon = K$, and for $\varepsilon \rightarrow 0^+$, it follows $\frac{r}{1-\lambda} \geq K$. Hence $K = \frac{r}{1-\lambda}$ is the best value of (12). The proposition is proved.

3. The equivalent forms

In the following, we introduce two propositions similar to the equivalent forms of Hilbert-type integral inequalities [5]:

Proposition 3 *If $p, r < 0$, $\lambda < 1$, $f \geq 0$, $0 < \int_0^\infty t^{p(1-\frac{1-\lambda}{r})-1} f^p(t) dt < \infty$, then*

$$J := \int_0^\infty x^{\frac{p}{r}(1-\lambda)-1} \left(\int_0^x f(y) dy \right)^p dx < \left(\frac{r}{\lambda-1} \right)^p \int_0^\infty t^{p(1+\frac{1-\lambda}{r})-1} f^p(t) dt, \quad (13)$$

where the constant $(\frac{r}{\lambda-1})^p$ is the best possible; (13) is equivalent to (8).

Proof If $J = 0$, then (13) is naturally valid; if $J > 0$, then there exists $n_0 \in \mathbf{N}$, such that for $n \geq n_0$, $\int_{\frac{1}{n}}^n t^{p(1+\frac{1-\lambda}{r})-1} [f(t)]_n^p dt > 0$ and $J_n := \int_{\frac{1}{n}}^n x^{\frac{p}{r}(1-\lambda)-1} \left(\int_{\frac{1}{n}}^x [f(y)]_n dy \right)^p dx > 0$, where

$$[f(t)]_n = \begin{cases} \frac{1}{n}, & f(t) < \frac{1}{n}, \\ f(y), & \frac{1}{n} \leq f(t) \leq n, \\ n, & f(t) > n. \end{cases} \quad (14)$$

One sets $g_n(x) := x^{\frac{p}{r}(1-\lambda)-1} \left(\int_{\frac{1}{n}}^x [f(y)]_n dy \right)^{p-1}$, $x \in [\frac{1}{n}, n]$, and uses (8) to obtain

$$\begin{aligned} \infty &> \int_{\frac{1}{n}}^n x^{q(1-\frac{1-\lambda}{r})-1} g_n^q(x) dx = J_n = \int_{\frac{1}{n}}^n \int_{\frac{1}{n}}^x [f(y)]_n g_n(x) dy dx \\ &> \frac{r}{\lambda-1} \left\{ \int_{\frac{1}{n}}^n y^{p(1+\frac{1-\lambda}{r})-1} [f(y)]_n^p dy \right\}^{\frac{1}{p}} \left\{ \int_{\frac{1}{n}}^n x^{q(1-\frac{1-\lambda}{r})-1} g_n^q(x) dx \right\}^{\frac{1}{q}}; \\ 0 &< \int_{\frac{1}{n}}^n x^{q(1-\frac{1-\lambda}{r})-1} g_n^q(x) dx = J_n < \left(\frac{r}{\lambda-1} \right)^p \int_0^\infty y^{p(1+\frac{1-\lambda}{r})-1} f^p(y) dy. \end{aligned}$$

Hence one confirms that $0 < \int_0^\infty x^{q(1-\frac{1-\lambda}{r})-1} g_\infty^q(x) dx < \infty$, and by (8), for $n \rightarrow \infty$, both the above forms still preserve the strict sign-inequalities. And one has (13). By the reverse Hölder's inequality, one has

$$I = \int_0^\infty \left[x^{\frac{1-\lambda}{r}-\frac{1}{p}} \int_0^x f(y) dy \right] \left[x^{\frac{1}{p}-\frac{1-\lambda}{r}} g(x) \right] dx \geq J^{\frac{1}{p}} \left\{ \int_0^\infty x^{q(1-\frac{1-\lambda}{r})-1} g^q(x) dx \right\}^{\frac{1}{q}}. \quad (15)$$

On the other hand, suppose that (13) is valid. Then by (15), one has (8), which is equivalent to (13). We confirm that the constant factor in (13) is the best possible. Otherwise, one can come to a contradiction by (15) that the constant factor in (8) is the best possible. This proves the theorem.

Proposition 4 *If $\lambda > 1$, the other is as the assumption of Proposition 3, then*

$$\tilde{J} := \int_0^\infty x^{\frac{p}{r}(1-\lambda)-1} \left(\int_x^\infty f(y) dy \right)^p dx < \left(\frac{r}{1-\lambda} \right)^p \int_0^\infty t^{p(1+\frac{1-\lambda}{r})-1} f^p(t) dt, \quad (16)$$

where the constant factor $(\frac{r}{1-\lambda})^p$ is the best possible; (16) is equivalent to (12).

Proof If $\tilde{J} = 0$, then (16) is naturally valid; if $\tilde{J} > 0$, then there exists $n_0 \in \mathbf{N}$, such that for $n \geq n_0$, $\int_{\frac{1}{n}}^n t^{p(1+\frac{1-\lambda}{r})-1} [f(t)]_n^p dt > 0$ and $\tilde{J}_n := \int_{\frac{1}{n}}^n x^{\frac{p}{r}(1-\lambda)-1} \left(\int_x^n [f(y)]_n dy \right)^p dx > 0$, where

$[f(y)]_n$ is defined by (14). One sets $g_n(x) := x^{\frac{p}{r}(1-\lambda)-1}(\int_x^n [f(y)]_n dy)^{p-1}$, $x \in [\frac{1}{n}, n]$ and uses (12) to obtain

$$\begin{aligned} \infty &> \int_{\frac{1}{n}}^n x^{q(1-\frac{1-\lambda}{r})-1} g_n^q(x) dx = \tilde{J}_n = \int_{\frac{1}{n}}^n \int_x^n [f(y)]_n g_n(x) dy dx \\ &> \frac{r}{1-\lambda} \left\{ \int_{\frac{1}{n}}^n y^{p(1+\frac{1-\lambda}{r})-1} [f(y)]_n^p dy \right\}^{\frac{1}{p}} \left\{ \int_{\frac{1}{n}}^n x^{q(1-\frac{1-\lambda}{r})-1} g_n^q(x) dx \right\}^{\frac{1}{q}}; \\ 0 &< \int_{\frac{1}{n}}^n x^{q(1-\frac{1-\lambda}{r})-1} g_n^q(x) dx = \tilde{J}_n < \left(\frac{r}{1-\lambda} \right)^p \int_0^\infty y^{p(1+\frac{1-\lambda}{r})-1} f^p(y) dy. \end{aligned}$$

Hence $0 < \int_0^\infty x^{q(1-\frac{1-\lambda}{r})-1} g_\infty^q(x) dx < \infty$ and for $n \rightarrow \infty$, both the above forms still preserve the strict sign-inequalities by (12). And one has (16). By the reverse Hölder's inequality, one has

$$\tilde{I} = \int_0^\infty [x^{\frac{1-\lambda}{r}-\frac{1}{p}} \int_x^\infty f(y) dy] [x^{\frac{1}{p}-\frac{1-\lambda}{r}} g(x)] dx \geq \tilde{J}_n^{\frac{1}{p}} \left\{ \int_0^\infty x^{q(1-\frac{1-\lambda}{r})-1} g^q(x) dx \right\}^{\frac{1}{q}}. \quad (17)$$

On the other hand, suppose that (16) is valid. Then by (17), one has (12), which is equivalent to (16). We confirm that the constant factor in (16) is the best possible. Otherwise, one can get a contradiction by (17) that the constant factor in (12) is the best possible. This proves the proposition.

Note Combining with Propositions 3–4, one has Theorem 1. In (7), one has

$$\int_0^\infty \frac{1}{x} \left(\frac{1}{x} \int_0^x f(t) dt \right)^p dx < \int_0^\infty \frac{1}{x} f^p(x) dx, \quad p < 0; \quad (18)$$

$$\int_0^\infty \frac{1}{x} \left(x \int_x^\infty f(t) dt \right)^p dx < \int_0^\infty \frac{1}{x} (x^2 f(x))^p dx, \quad p < 0. \quad (19)$$

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