Journal of Mathematical Research & Exposition
May, 2011, Vol. 31, No. 3, pp. 479–489
DOI:10.3770/j.issn:1000-341X.2011.03.013
Http://jmre.dlut.edu.cn

Augmented Lyapunov Approach to Exponential Stability of Discrete-Time Neural Networks

Zi Xin LIU[1,*], **Shu LÜ**[2], **Shou Ming ZHONG**[2], **Mao YE**[3]

1. *School of Mathematics and Statistics, Guizhou College of Finance and Economics, Guizhou* 550004*, P. R. China;*
2. *School of Mathematical Sciences, University of Electronic Science and Technology of China, Sichuan* 611731*, P. R. China;*
3. *School of Computer Science and Engineering, University of Electronic Science and Technology of China, Sichuan* 611731*, P. R. China*

Abstract This paper addresses the problem of robust stability for a class of discrete-time neural networks with time-varying delay and parameter uncertainties. By constructing a new augmented Lyapunov-Krasovskii function, some new improved stability criteria are obtained in forms of linear matrix inequality (LMI) technique. Compared with some recent results in the literature, the conservatism of these new criteria is reduced notably. Two numerical examples are provided to demonstrate the less conservatism and effectiveness of the proposed results.

Keywords discrete-time neural networks; robust exponential stability; delay-dependent criterion; time-varying delay.

Document code A
MR(2010) Subject Classification 34D20; 34D23
Chinese Library Classification O175.13

1. Introduction

Over the past few decades, recurrent neural networks (RNNs) have attracted considerable attention due to their successful applications in various areas including optimization solvers, model identification, signal processing, and other engineering areas. As is well known that any useful neural network must be a stable one. However, because of the existence of time delays, stochastic disturbances, parameter uncertainties and so on, the convergence of a neural network may often be destroyed. This makes the design or performance for the corresponding closed-loop systems become difficult. Therefore, stability analysis of delayed uncertain neural network has received much attention. Up to now, various stability conditions have been obtained, and many excellent papers and monographs have been available [1–9]. Generally speaking, these so-far obtained stability results for delayed RNNs can be mainly classified into two types: that

Received March 30, 2009; Accepted April 26, 2010
Supported by the Science and Technology Founation of Guizhou Province (Grant No. [2010]2139) and the Program for New Century Excellent Talents in University (Grant No. NCET-06-0811).
* Corresponding author
E-mail address: xinxin905@163.com (Z. X. LIU)

is, delay-independent and delay-dependent. Since the information of time delays is sufficiently considered, delay-dependent criteria may be less conservative than delay-independent ones when the size of time delay is small. For delay-dependent type, the size of the allowable upper bound of delay is always regarded as an important criterion to discriminate the quality between different criteria. Recently, free-weighting matrices method is extensively used to research the delay-dependent stability problems for RNNs with time-varying delay and parameters uncertainties [10–16]. By introducing free-weighting matrices, the conservatism of a criterion usually may be reduced effectively.

It should be pointed out that, most of these previous results have been assumed to be in continuous time, but seldom in discrete time. In practice, when implementing and applying neural networks, discrete-time neural networks play a more important role than their continuous-time counter-parts in today's digital world, such as numerical computation, and computer simulation. And they can ideally keep the dynamic characteristics, functional similarity, and even the physical or biological reality of the continuous-time networks under mild restriction. Thus, the stability analysis problems for discrete-time neural networks have received more and more interest, and some stability criteria have been proposed in the literature [10, 17–27]. In [26], Liu et al., researched a class of discrete-time RNNs with time-varying delay, and established a delay-dependent exponential stability criterion. The result obtained in [26] has been improved by Song and Wang in [20]. The results obtained in [20] were further improved in [21] by considering some useful terms. Recently, some new improved criteria are derived in [22, 23, 27], respectively.

In this paper, some new improved delay-dependent stability criteria are obtained via constructing a new augmented Lyapunov-Krasovskii function. These new conditions are less conservative than those obtained in [10, 20–23, 26, 27]. Two numerical examples are provided to illuminate the improvement of the proposed criteria.

Notation: The following notations are used in our paper unless otherwise specified. $\|\cdot\|$ denotes a vector or a matrix norm; R, R^n are real and n-dimensional real number sets, respectively; N^+ is positive integer set. I is identity matrix; $*$ represents the elements below the main diagonal of a symmetric block matrix; Real matrix $P > 0 (< 0)$ denotes P is a positive-definite (negative-definite) matrix; $N[a, b] = \{a, a+1, \ldots, b\}$; $\lambda_{\min}(\lambda_{\max})$ denotes the minimum (maximum) eigenvalue of a real matrix.

2. Preliminaries

Consider a delayed discrete-time RNNs Σ as follows

$$\Sigma : y(k+1) = C(k)y(k) + A(k)\overline{f}(y(k)) + B(k)\overline{g}(y(k-\tau(k))) + J, \tag{1}$$

where $y(k) = [y_1(k), y_2(k), \ldots, y_n(k)]^{\mathrm{T}} \in R^n$ denotes the neural state vector; $\overline{f}(y(k)) = [\overline{f}_1(y_1(k)), \overline{f}_2(y_2(k)), \ldots, \overline{f}_n(y_n(k))]^{\mathrm{T}}$, $\overline{g}(y(k-\tau(k))) = [\overline{g}_1(y_1(k-\tau(k))), \overline{g}_2(y_2(k-\tau(k))), \ldots, \overline{g}_n(y_n(k-\tau(k)))]^{\mathrm{T}}$ are the neuron activation functions; $J = [J_1, J_2, \ldots, J_n]^{\mathrm{T}}$ is the external input vector; Positive integer $\tau(k)$ represents the transmission delay satisfying $0 < \tau_m \le \tau(k) \le \tau_M$, where τ_m, τ_M are known positive integers representing the lower and upper bounds of the delay.

$C(k) = C + \triangle C(k)$, $A(k) = A + \triangle A(k)$, $B(k) = B + \triangle B(k)$; $C = \mathrm{diag}(c_1, c_2, \ldots, c_n)$ with $|c_i| < 1$ describes the rate with which the ith neuron will reset its potential to the resting state in isolation when disconnected from the networks and external inputs; $C, A, B \in R^{n\times n}$ represent the weighting matrices; $\Delta C(k), \Delta A(k), \Delta B(k)$ denote the time-varying structured uncertainties which are of the form: $[\Delta C(k)\ \ \Delta A(k)\ \ \Delta B(k)] = KF(k)[E_c\ \ E_a\ \ E_b]$, where K, E_c, E_a, E_b are known real constant matrices of appropriate dimensions; $F(k)$ is unknown time-varying matrix function satisfying $F^{\mathrm{T}}(k)F(k) \le I$, $\forall k \in N^+$.

The nominal Σ_0 of Σ can be defined as

$$\Sigma_0 : y(k+1) = Cy(k) + A\overline{f}(y(k)) + B\overline{g}(y(k-\tau(k))) + J. \tag{2}$$

For further discussion, we first introduce the following assumption and lemmas.

Assumption 1 For any $x, y \in R$, $x \ne y$,

$$l_i^- \le \frac{\overline{f}_i(x) - \overline{f}_i(y)}{x-y} \le l_i^+,\ \sigma_i^- \le \frac{\overline{g}_i(x) - \overline{g}_i(y)}{x-y} \le \sigma_i^+,\quad i \in N^+, \tag{3}$$

where l_i^-, l_i^+, σ_i^-, σ_i^+ are known constant scalars. As pointed out in [17] that, under Assumption 1, system (2) has equilibrium point. Assume $y^* = [y_1^*, y_2^*, \ldots, y_n^*]^{\mathrm{T}}$ is an equilibrium point of (2), and set $x_i(k) = y_i(k) - y_i^*$, $f_i(x_i(k)) = \overline{f}_i(x_i(k) + y_i^*) - \overline{f}_i(y_i^*)$, $g_i(x_i(k-\tau(k))) = \overline{g}_i(x_i(k-\tau(k)) + y_i^*) - \overline{g}_i(y_i^*)$. Then, system (2) can be transformed into the following form:

$$x(k+1) = Cx(k) + Af(x(k)) + Bg(x(k-\tau(k))),\quad k \in N^+, \tag{4}$$

where $x(k) = [x_1(k), x_2(k), \ldots, x_n(k)]^{\mathrm{T}}$, $f(x(k)) = [f_1(x_1(k)), f_2(x_2(k)), \ldots, f_n(x_n(k))]^{\mathrm{T}}$, $g(x(k-\tau(k))) = [g_1(x_1(k-\tau(k))), g_2(x_2(k-\tau(k))), \ldots, g_n(x_n(k-\tau(k)))]^{\mathrm{T}}$. By Assumption 1, for any $x, y \in R$, $x \ne y$, functions $f_i(\cdot), g_i(\cdot)$ satisfy

$$l_i^- \le \frac{f_i(x) - f_i(y)}{x-y} \le l_i^+,\ \sigma_i^- \le \frac{g_i(x) - g_i(y)}{x-y} \le \sigma_i^+,\ f_i(0) = 0,\ g_i(0) = 0,\ i \in N^+.$$

Definition 1 *The delayed discrete-time recurrent neural network in (4) is said to be globally exponentially stable if there exist two positive scalars $\alpha > 0$ and $0 < \beta < 1$ such that*

$$\|x(k)\| \le \alpha \cdot \beta^k \sup_{s\in N[-\tau_M, 0]} \|x(s)\|,\quad \forall k \ge 0.$$

Lemma 1 (Tchebychev Inequality [28]) *For any given vectors $v_i \in R^n$, $i \in N^+$, the following inequality holds:*

$$\Big[\sum_{i=1}^{n} v_i\Big]^{\mathrm{T}}\Big[\sum_{i=1}^{n} v_i\Big] \le n\sum_{i=1}^{n} v_i^{\mathrm{T}} v_i.$$

Lemma 2 ([29]) *For given matrices $Q = Q^{\mathrm{T}}, H, E$ and $R = R^{\mathrm{T}} > 0$ of appropriate dimensions, then*

$$Q + HFE + E^{\mathrm{T}}F^{\mathrm{T}}H^{\mathrm{T}} < 0,$$

for all F satisfying $F^{\mathrm{T}}F \le R$, if and only if there exists an $\varepsilon > 0$, such that

$$Q + \varepsilon^{-1}HH^{\mathrm{T}} + \varepsilon E^{\mathrm{T}}RE < 0.$$

Lemma 3 ([30]) *Given constant symmetric matrices* $\Sigma_1, \Sigma_2, \Sigma_3$, *where* $\Sigma_1^{\mathrm{T}} = \Sigma_1$ *and* $0 < \Sigma_2 = \Sigma_2^{\mathrm{T}}$, *then* $\Sigma_1 + \Sigma_3^{\mathrm{T}} \Sigma_2^{-1} \Sigma_3 < 0$ *if and only if*

$$\begin{bmatrix} \Sigma_1 & \Sigma_3^{\mathrm{T}} \\ \Sigma_3 & -\Sigma_2 \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^{\mathrm{T}} & \Sigma_1 \end{bmatrix} < 0.$$

Lemma 4 ([10]) *Let* N *and* E *be real constant matrices of appropriate dimensions, and matrix* $F(k)$ *satisfy* $F^{\mathrm{T}}(k)F(k) \leq I$. *Then, for any* $\epsilon > 0$, $EF(k)N + N^{\mathrm{T}}F^{\mathrm{T}}(k)E^{\mathrm{T}} \leq \epsilon^{-1}EE^{\mathrm{T}} + \epsilon N^{\mathrm{T}}N$.

3. Main results

Theorem 1 *For any given positive integers* $0 < \tau_m < \tau_M$, *then, under Assumption 1, system (4) is globally exponentially stable for any time-varying delay* $\tau(k)$ *satisfying* $\tau_m \leq \tau(k) \leq \tau_M$, *if there exist positive matrices* Q, R, H, P, *positive diagonal matrices* D_1, D_2, Z_1, Z_2, *arbitrary matrices* M_1, M_2, N_1, N_2, F_1, F_2 *of appropriate dimensions, such that the following LMI holds:*

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16} & \Xi_{17} & \Xi_{18} & \Xi_{19} & \Xi_{1,10} \\ * & \Xi_{22} & 0 & 0 & \Xi_{25} & 0 & 0 & 0 & \Xi_{29} & \Xi_{2,10} \\ * & * & \Xi_{33} & \Xi_{34} & \Xi_{35} & \Xi_{36} & \Xi_{37} & 0 & 0 & 0 \\ * & * & * & \Xi_{44} & \Xi_{45} & \Xi_{46} & \Xi_{47} & 0 & 0 & 0 \\ * & * & * & * & \Xi_{55} & \Xi_{56} & \Xi_{57} & \Xi_{58} & \Xi_{59} & \Xi_{5,10} \\ * & * & * & * & * & \Xi_{66} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Xi_{77} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Xi_{88} & 0 & 0 \\ * & * & * & * & * & * & * & * & \Xi_{99} & 0 \\ * & * & * & * & * & * & * & * & * & \Xi_{10,10} \end{bmatrix} < 0, \tag{5}$$

where $Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ * & Q_{22} & Q_{23} \\ * & * & Q_{33} \end{bmatrix}$,

$$\begin{aligned}
\Xi_{11} =& Q_{12} + Q_{12}^{\mathrm{T}} + Q_{13} + Q_{13}^{\mathrm{T}} + Q_{23} + Q_{23}^{\mathrm{T}} + Q_{22} + Q_{33} - D_1L_1 + N_1 + N_1^{\mathrm{T}} + F_1 + F_1^{\mathrm{T}} + \\
& M_1(C-I) + (C-I)^{\mathrm{T}}M_1^{\mathrm{T}} + (1 + (\tau_M - \tau_m)^{-1})R + H + P + (1+\tau_M)Z_2 + (1+\tau_m)Z_1, \\
\Xi_{12} =& N_2^{\mathrm{T}} + F_2^{\mathrm{T}} - F_1 - N_1,\ \Xi_{13} = -Q_{12} - Q_{22} - Q_{23}^{\mathrm{T}},\ \Xi_{14} = -Q_{13} - Q_{23} - Q_{33}, \\
\Xi_{15} =& Q_{11} + Q_{12} + Q_{12}^{\mathrm{T}} + Q_{13} + Q_{13}^{\mathrm{T}} + Q_{23} + Q_{23}^{\mathrm{T}} + Q_{22} + Q_{33} - M_1 + (C-I)^{\mathrm{T}}M_2^{\mathrm{T}} + N_1^{\mathrm{T}}, \\
\Xi_{16} =& Q_{22} + Q_{23}^{\mathrm{T}},\ \Xi_{17} = Q_{23} + Q_{33},\ \Xi_{18} = M_1A + D_1L_2, \Xi_{19} = M_1B, \\
\Xi_{1,10} =& N_2^{\mathrm{T}} + F_2^{\mathrm{T}} - F_1 - N_1, \Xi_{22} = -N_2 - F_2 + (-N_2 - F_2)^{\mathrm{T}} - (\tau_M - \tau_m)^{-1}R - D_2\Pi_1, \\
\Xi_{25} =& -N_1^{\mathrm{T}}, \Xi_{29} = D_2\Pi_2,\ \Xi_{2,10} = -N_2^{\mathrm{T}} - F_2^{\mathrm{T}} - F_2 - N_2, \Xi_{33} = Q_{22} - H, \Xi_{34} = -Q_{23}, \\
\Xi_{35} =& -Q_{12}^{\mathrm{T}} - Q_{23} - Q_{22}, \Xi_{36} = -Q_{22}, \Xi_{37} = -Q_{23}, \Xi_{44} = Q_{33} - P, \\
\Xi_{45} =& -Q_{13}^{\mathrm{T}} - Q_{23}^{\mathrm{T}} - Q_{33}, \Xi_{46} = -Q_{23}^{\mathrm{T}}, \Xi_{47} = -Q_{33}, \\
\Xi_{55} =& Q_{12} + Q_{12}^{\mathrm{T}} + Q_{13} + Q_{13}^{\mathrm{T}} + Q_{23} + Q_{23}^{\mathrm{T}} + Q_{11} + Q_{22} + Q_{33} - M_2^{\mathrm{T}} - M_2, \\
\Xi_{56} =& Q_{12} + Q_{22} + Q_{23}^{\mathrm{T}},\ \Xi_{57} = Q_{13} + Q_{23} + Q_{33},\ \Xi_{58} = M_2A,\ \Xi_{59} = M_2B,\ \Xi_{5,10} = -N_1,
\end{aligned}$$

$$\Xi_{66} = -\frac{Z_1}{1+\tau_m},\ \Xi_{77} = -\frac{Z_2}{1+\tau_M},\ \Xi_{88} = -D_1,$$
$$\Xi_{99} = -D_2,\ \Xi_{10,10} = -N_2^{\mathrm{T}} - N_2 - F_2^{\mathrm{T}} - F_2,$$
$$L_1 = \mathrm{diag}(l_1^+ l_1^-, \ldots, l_n^+ l_n^-), L_2 = \mathrm{diag}(\frac{l_1^+ + l_1^-}{2}, \ldots, \frac{l_n^+ + l_n^-}{2}),$$
$$\Pi_1 = \mathrm{diag}(\sigma_1^+ \sigma_1^-, \ldots, \sigma_n^+ \sigma_n^-), \Pi_2 = \mathrm{diag}(\frac{\sigma_1^+ + \sigma_1^-}{2}, \ldots, \frac{\sigma_n^+ + \sigma_n^-}{2}).$$

Proof. Constructing a new augmented Lyapunov-Krasovskii function candidate as follows:

$$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k) + V_5(k),$$
$$V_1(k) = \widehat{X}^{\mathrm{T}}(k) Q \widehat{X}(k),\ \widehat{X}^{\mathrm{T}}(k) = \Big[x^{\mathrm{T}}(k), \sum_{i=k-\tau_m}^{k} x^{\mathrm{T}}(i), \sum_{i=k-\tau_M}^{k} x^{\mathrm{T}}(i)\Big],$$
$$V_2(k) = \sum_{i=k-\tau_m}^{k-1} x^{\mathrm{T}}(i) H x(i) + \sum_{i=k-\tau_M}^{k-1} x^{\mathrm{T}}(i) P x(i),$$
$$V_3(k) = \sum_{j=k-\tau_m}^{k} \sum_{i=j}^{k-1} x^{\mathrm{T}}(i) Z_1 x(i) + \sum_{j=k-\tau_M}^{k} \sum_{i=j}^{k-1} x^{\mathrm{T}}(i) Z_2 x(i),$$
$$V_4(k) = \frac{1}{\tau_M - \tau_m} \sum_{i=k-\tau(k)}^{k-1} x^{\mathrm{T}}(i) R x(i),\ V_5(k) = \frac{1}{\tau_M - \tau_m} \sum_{j=k+1-\tau_M}^{k-\tau_m} \sum_{i=j}^{k-1} x^{\mathrm{T}}(i) R x(i).$$

Set $X^{\mathrm{T}}(k) = [x^{\mathrm{T}}(k), x^{\mathrm{T}}(k-\tau(k)), x^{\mathrm{T}}(k-\tau_m), x^{\mathrm{T}}(k-\tau_M), \eta^{\mathrm{T}}(k), \sum_{i=k-\tau_m}^{k} x^{\mathrm{T}}(i), \sum_{i=k-\tau_M}^{k} x^{\mathrm{T}}(i), f^{\mathrm{T}}(x(k)), g^{\mathrm{T}}(x(k-\tau(k)))]$, $\eta(k) = x(k+1) - x(k)$. Define $\Delta V(k) = V(k+1) - V(k)$. Then along the solution of system (4), we have

$$\begin{aligned}
\Delta V_1(k) =& \widehat{X}^{\mathrm{T}}(k+1) Q \widehat{X}(k+1) - \widehat{X}^{\mathrm{T}}(k) Q \widehat{X}(k) \\
=& x^{\mathrm{T}}(k)[Q_{12} + Q_{12}^{\mathrm{T}} + Q_{13} + Q_{13}^{\mathrm{T}} + Q_{23} + Q_{23}^{\mathrm{T}} + Q_{22} + Q_{33}]x(k) - \\
& 2x^{\mathrm{T}}(k)[Q_{12} + Q_{22} + Q_{23}^{\mathrm{T}})]x(k-\tau_m) - 2x^{\mathrm{T}}(k)[Q_{13} + Q_{23} + Q_{33})]x(k-\tau_M) + \\
& 2x^{\mathrm{T}}(k)[Q_{11} + Q_{12} + Q_{12}^{\mathrm{T}} + Q_{13} + Q_{13}^{\mathrm{T}} + Q_{23} + Q_{23}^{\mathrm{T}} + Q_{22} + Q_{33}]\eta(k) + \\
& 2x^{\mathrm{T}}(k)[Q_{22} + Q_{23}^{\mathrm{T}}] \sum_{i=k-\tau_m}^{k} x(i) + 2x^{\mathrm{T}}(k)[Q_{23} + Q_{33}] \sum_{i=k-\tau_M}^{k} x(i) + \\
& x^{\mathrm{T}}(k-\tau_m) Q_{22} x(k-\tau_m) - 2x^{\mathrm{T}}(k-\tau_m) Q_{23} x(k-\tau_M) - \\
& 2x^{\mathrm{T}}(k-\tau_m)[Q_{12}^{\mathrm{T}} + Q_{23} + Q_{22}]\eta(k) - 2x^{\mathrm{T}}(k-\tau_m) Q_{22} \sum_{i=k-\tau_m}^{k} x(i) - \\
& 2x^{\mathrm{T}}(k-\tau_m) Q_{23} \sum_{i=k-\tau_M}^{k} x(i) + x^{\mathrm{T}}(k-\tau_M) Q_{33} x(k-\tau_M) - \\
& 2x^{\mathrm{T}}(k-\tau_M) Q_{23}^{\mathrm{T}} \sum_{i=k-\tau_m}^{k} x(i) - 2x^{\mathrm{T}}(k-\tau_M)[Q_{13}^{\mathrm{T}} + Q_{23}^{\mathrm{T}} + Q_{33}]\eta(k) + \\
& 2\eta^{\mathrm{T}}(k)[Q_{13} + Q_{23} + Q_{33}] \sum_{i=k-\tau_M}^{k} x(i) - 2x^{\mathrm{T}}(k-\tau_M) Q_{33} \sum_{i=k-\tau_M}^{k} x(i) +
\end{aligned}$$

$$\eta^{\mathrm{T}}(k)[Q_{12}+Q_{12}^{\mathrm{T}}+Q_{13}+Q_{13}^{\mathrm{T}}+Q_{23}+Q_{23}^{\mathrm{T}}+Q_{11}+Q_{22}+Q_{33}]\eta(k)+$$
$$2\eta^{\mathrm{T}}(k)[Q_{12}+Q_{22}+Q_{23}^{\mathrm{T}}]\sum_{i=k-\tau_m}^{k}x(i), \tag{6}$$

$$\Delta V_2(k)=x^{\mathrm{T}}(k)(H+P)x(k)-x^{\mathrm{T}}(k-\tau_m)Hx(k-\tau_m)-x^{\mathrm{T}}(k-\tau_M)Px(k-\tau_M). \tag{7}$$

From Lemma 1, we have

$$\begin{aligned}\Delta V_3(k)=&\sum_{j=k+1-\tau_m}^{k+1}\sum_{i=j}^{k}x^{\mathrm{T}}(i)Z_1x(i)-\sum_{j=k-\tau_m}^{k}\sum_{i=j}^{k-1}x^{\mathrm{T}}(i)Z_1x(i)+\\&\sum_{j=k+1-\tau_M}^{k+1}\sum_{i=j}^{k}x^{\mathrm{T}}(i)Z_2x(i)-\sum_{j=k-\tau_M}^{k}\sum_{i=j}^{k-1}x^{\mathrm{T}}(i)Z_2x(i)\\=&\sum_{j=k-\tau_m}^{k}\sum_{i=j+1}^{k}x^{\mathrm{T}}(i)Z_1x(i)-\sum_{j=k-\tau_m}^{k}\sum_{i=j}^{k-1}x^{\mathrm{T}}(i)Z_1x(i)+\\&\sum_{j=k-\tau_M}^{k}\sum_{i=j+1}^{k}x^{\mathrm{T}}(i)Z_2x(i)-\sum_{j=k-\tau_M}^{k}\sum_{i=j}^{k-1}x^{\mathrm{T}}(i)Z_2x(i)\\\le&(1+\tau_m)x^{\mathrm{T}}(k)Z_1x(k)-\frac{1}{1+\tau_m}\Big[\sum_{j=k-\tau_m}^{k}x(j)\Big]^{\mathrm{T}}Z_1\Big[\sum_{j=k-\tau_m}^{k}x(j)\Big]+\\&(1+\tau_M)x^{\mathrm{T}}(k)Z_2x(k)-\frac{1}{1+\tau_M}\Big[\sum_{j=k-\tau_M}^{k}x(j)\Big]^{\mathrm{T}}Z_2\Big[\sum_{j=k-\tau_M}^{k}x(j)\Big].\end{aligned} \tag{8}$$

$$\begin{aligned}\Delta V_4(k)=&\frac{1}{\tau_M-\tau_m}\Big[x^{\mathrm{T}}(k)Rx(k)-x^{\mathrm{T}}(k-\tau(k))Rx(k-\tau(k)))+\\&\sum_{i=k+1-\tau(k+1)}^{k-\tau_m}x^{\mathrm{T}}(i)Rx(i)+\sum_{i=k+1-\tau_m}^{k-1}x^{\mathrm{T}}(i)Rx(i)-\sum_{i=k+1-\tau(k)}^{k-1}x^{\mathrm{T}}(i)Rx(i)\Big]\\\le&\frac{1}{\tau_M-\tau_m}[x^{\mathrm{T}}(k)Rx(k)-x^{\mathrm{T}}(k-\tau(k))Rx(k-\tau(k))]+\\&\frac{1}{\tau_M-\tau_m}\Big[\sum_{i=k+1-\tau_M}^{k-\tau_m}x^{\mathrm{T}}(i)Rx(i)\Big],\end{aligned} \tag{9}$$

$$\Delta V_5(k)=x^{\mathrm{T}}(k)Rx(k)-\frac{1}{\tau_M-\tau_m}\Big[\sum_{i=k+1-\tau_M}^{k-\tau_m}x^{\mathrm{T}}(i)Rx(i)\Big]. \tag{10}$$

For any matrices M_1, M_2 of appropriate dimensions, we have

$$2x^{\mathrm{T}}(k)M_1[(C-I)x(k)+Af(x(k))+Bg(x(k-\tau(k)))-\eta(k)]=0, \tag{11}$$
$$2\eta^{\mathrm{T}}(k)M_2[(C-I)x(k)+Af(x(k))+Bg(x(k-\tau(k)))-\eta(k)]=0. \tag{12}$$

Since $x(k)-\sum_{i=k-\tau(k)}^{k-1}\eta(i)-x(k-\tau(k))=0$, for arbitrary matrices N_1, N_2, F_1, F_2 of appropriate dimensions, we can obtain that

$$0=\widetilde{X}_1^{\mathrm{T}}\begin{bmatrix}0 & N_1\\0 & N_2\end{bmatrix}\widetilde{X}_2,\quad 0=\overline{X}_1^{\mathrm{T}}\begin{bmatrix}0 & F_1\\0 & F_2\end{bmatrix}\widetilde{X}_2, \tag{13}$$

where $\widetilde{X}_1^{\mathrm{T}}(k) = [\eta^{\mathrm{T}}(k) + x^{\mathrm{T}}(k), \sum_{i=k-\tau(k)}^{k-1} \eta^{\mathrm{T}}(i) + x^{\mathrm{T}}(k-\tau(k))]$, $\widetilde{X}_2^{\mathrm{T}} = [\eta^{\mathrm{T}}(k) + x^{\mathrm{T}}(k), x^{\mathrm{T}}(k) - \sum_{i=k-\tau(k)}^{k-1} \eta^{\mathrm{T}}(i) - x^{\mathrm{T}}(k-\tau(k))]$, $\overline{X}_1^{\mathrm{T}} = [x^{\mathrm{T}}(k), \sum_{i=k-\tau(k)}^{k-1} \eta^{\mathrm{T}}(i) + x^{\mathrm{T}}(k-\tau(k))]$.

From Assumption 1, for any positive diagonal matrices D_1, D_2 of appropriate dimensions, we have

$$\begin{aligned}
&2x^{\mathrm{T}}(k)D_1L_2f(x(k)) - x^{\mathrm{T}}(k)D_1L_1x(k) - f^{\mathrm{T}}(x(k))D_1f(x(k)) \geq 0,\\
&2x^{\mathrm{T}}(k-\tau(k))D_2\Pi_2 g(x(k-\tau(k))) - x^{\mathrm{T}}(k-\tau(k))D_2\Pi_1 x(k-\tau(k)) -\\
&\quad g^{\mathrm{T}}(x(k-\tau(k)))D_2 g(x(k-\tau(k))) \geq 0.
\end{aligned} \tag{14}$$

Combining (6)–(14), we get

$$\Delta V(k) \leq X'^{\mathrm{T}}(k)\Xi X'(k), \quad X'^{\mathrm{T}}(k) = \Big[X^{\mathrm{T}}(k), \sum_{i=k-\tau(k)}^{k-1} \eta^{\mathrm{T}}(i)\Big]. \tag{15}$$

If the LMI (5) holds, it follows that there exists a sufficient small positive scalar $\varepsilon > 0$ such that

$$\Delta V(k) \leq -\varepsilon \|x(k)\|^2. \tag{16}$$

On the other hand, it is easy to get that

$$V(k) \leq \alpha_1 \|x(k)\|^2 + \alpha_2 \sum_{i=k-\tau_M}^{k} \|x(i)\|^2, \tag{17}$$

where $\alpha_1 = \lambda_{\max}(Q)$, $\alpha_2 = (\lambda_{\max}(Q) + \lambda_{\max}(H) + \lambda_{\max}(Z_1))\tau_m + (\lambda_{\max}(Q) + \lambda_{\max}(P) + \lambda_{\max}(Z_2))\tau_M + 2\lambda_{\max}(Q) + \lambda_{\max}(Z_1) + \lambda_{\max}(Z_2) + (1 + \frac{1}{\tau_M - \tau_m})\lambda_{\max}(R)$.

For any $\theta > 1$, it follows from (17) that

$$\begin{aligned}
&\theta^{j+1}V(j+1) - \theta^j V(j) = \theta^{j+1}\Delta V(j) + \theta^j(\theta-1)V(j)\\
&\quad \leq \theta^j\Big(-\varepsilon\theta\|x(j)\|^2 + (\theta-1)\alpha_1\|x(j)\|^2 + (\theta-1)\alpha_2 \sum_{i=j-\tau_M}^{j} \|x(i)\|^2\Big).
\end{aligned} \tag{18}$$

Summing up both sides of (18) from 0 to $k-1$ we can obtain

$$\begin{aligned}
\theta^k V(k) - V(0) &\leq [\alpha_1(\theta-1) - \varepsilon\theta] \sum_{j=0}^{k-1} \theta^j \|x(j)\|^2 + \alpha_2(\theta-1) \sum_{j=0}^{k-1} \sum_{i=j-\tau_M}^{j} \theta^j \|x(i)\|^2\\
&\leq \mu_1(\theta) \sup_{j\in N[-\tau_M,0]} \|x(j)\|^2 + \mu_2(\theta) \sum_{j=0}^{k} \theta^k \|x(j)\|^2,
\end{aligned} \tag{19}$$

where $\mu_1(\theta) = \alpha_2(\theta-1)\tau_M^2\theta^{\tau_M}$, $\mu_2(\theta) = \alpha_2(\theta-1)\tau_M\theta^{\tau_M} + \alpha_1(\theta-1) - \varepsilon\theta$. Since $\mu_2(1) = -\varepsilon < 0$, there must exist a positive $\theta_0 > 1$ such that $\mu_2(\theta_0) < 0$. Then, we have

$$V(k) \leq \mu_1(\theta_0)(\frac{1}{\theta_0})^k \sup_{j\in N[-\tau_M,0]} \|x(j)\|^2 + (\frac{1}{\theta_0})^k V(0). \tag{20}$$

On the other hand, set $\sigma = \alpha_1 + (1+\tau_M)\alpha_2$, we can obtain

$$V(0) \leq \sigma \sup_{j\in N[-\tau_M,0]} \|x(j)\|^2 \ \text{ and } \ V(k) \geq \lambda_{\min}(Q)\|x(k)\|^2. \tag{21}$$

It follows that $\|x(k)\| \leq \alpha \cdot \beta^k \sup_{j \in N[-\tau_M, 0]} \|x(j)\|$, where $\beta = (\theta_0)^{-1/2}$, $\alpha = \sqrt{\frac{\mu_1(\theta_0)+\sigma}{\lambda_{\min}(Q)}}$. By Definition 1, system (4) is globally exponentially stable, which completes the proof of Theorem 1. □

Remark 1 In Theorem 1, we proposed V_1 which takes $x^{\mathrm{T}}(k)$, $\sum_{i=k-\tau_m}^{k} x^{\mathrm{T}}(i)$, $\sum_{i=k-\tau_M}^{k} x^{\mathrm{T}}(i)$ as augmented state. The proposed augmented Lyapunov function V_1 is not considered in the existing literature and may reduce the conservatism of the delay-dependent result.

Remark 2 Free-weighting matrices N_1, N_2, F_1, F_2 introduced through zero equation (13) may improve the feasibility region of delay-dependent stability criterion.

Remark 3 It is worthwhile pointing out that this new criterion can be easily extended to robust exponential stability condition. As for the robust stability of system (1), according to Lemma 2, we can obtain the following result.

Theorem 2 *For any given positive integers $0 < \tau_m < \tau_M$, then, under Assumption 1, system (1) is globally, robustly, and exponentially stable for any time-varying delay $\tau(k)$ satisfying $\tau_m \leq \tau(k) \leq \tau_M$, if there exist positive matrices Q, R, H, P, positive diagonal matrices D_1, D_2, Z_1, Z_2, arbitrary matrices M_1, M_2, N_1, N_2, F_1, F_2 of appropriate dimensions, and $\epsilon > 0$, such that the following LMI holds:*

$$\Xi' \triangleq \begin{bmatrix} \Xi & \xi_1 & \epsilon\xi_2^{\mathrm{T}} \\ * & -\epsilon I & 0 \\ * & * & -\epsilon I \end{bmatrix} < 0, \tag{22}$$

where $\xi_1^{\mathrm{T}} = [K^{\mathrm{T}} M_1^{\mathrm{T}}, 0, 0, K^{\mathrm{T}} M_2^{\mathrm{T}}, 0, 0, 0, 0, 0, 0]$, $\xi_2 = [E_c, 0, 0, 0, 0, 0, 0, E_a, E_b, 0]$.

Proof Replacing A, B, C in inequality (5) with $A + KF(t)E_a$, $B + KF(t)E_b$ and $C + KF(t)E_c$, respectively, inequality (5) for system (1) is equivalent to $\Xi + \xi_1 F(t)\xi_2 + \xi_2^{\mathrm{T}} F^{\mathrm{T}}(t)\xi_1^{\mathrm{T}} < 0$. From Lemmas 2, 3 and 4, we can easily obtain this result. The proof is completed. □

4. Numerical examples

In this section, two numerical examples will be presented to show the improvement and effectiveness of the main results derived above.

Example 1 For the convenience of comparison, consider a delayed discrete-time recurrent neural network in (4) with parameters given by

$$C = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.7 \end{bmatrix}, \quad A = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.005 \end{bmatrix}, \quad B = \begin{bmatrix} -0.1 & 0.01 \\ -0.2 & -0.1 \end{bmatrix},$$

and the activation functions are assumed to be $f_i(s) = g_i(s) = 0.5 * (|s+1| - |s-1|)$.

Obviously, $l_1^- = \sigma_1^- = -1$, $l_2^+ = \sigma_2^+ = 1$. It can be verified that the LMI (5) is feasible. For $\tau_m = 1, 4, 8, 15, 25$, Table 1 gives out the allowable upper bound τ_M of the time-varying delay for given τ_m, which shows that Theorem 1 is less conservative than these previous results obtained in [10, 20–23, 26, 27].

Cases	$\tau_m = 1$	$\tau_m = 4$	$\tau_m = 8$	$\tau_m = 15$	$\tau_m = 25$
By [10,26]	3	6	10	17	27
By [20]	12	14	16	21	29
By [21]	12	14	18	25	35
By [27]	14	17	19	26	36
By [22]	14	17	21	28	38
By [23]	20	22	26	33	43
By Theorem 3.1	$\tau_M > 0$	$\tau_M > 0$	$\tau_M > 0$	$\tau_M > 0$	$\tau_M > 0$

Table 1 Allowable upper bounds τ_M for given τ_m (Example 1)

Example 2 Consider a delayed discrete-time recurrent neural network in (1) with parameters given by

$$C = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}, \quad A = \begin{bmatrix} 0.3 & -0.1 & 0.2 \\ 0 & -0.3 & 0.2 \\ -0.1 & -0.1 & -0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ -0.2 & 0.3 & 0.1 \\ 0.1 & -0.2 & 0.3 \end{bmatrix},$$

$$K = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \quad E_c = E_a = E_b = K, \quad J = [0, 0, 0]^{\mathrm{T}},$$

$$f_1(s) = \tanh(0.2s), \quad f_2(s) = \tanh(0.4s), \quad f_3(s) = \tanh(0.2s),$$

$$g_1(s) = \tanh(0.12s), \quad g_2(s) = \tanh(0.2s), \quad g_3(s) = \tanh(0.4s).$$

It can be verified that $L_1 = \Pi_1 = 0$, $L_2 = \mathrm{diag}(0.1, 0.2, 0.1)$, $\Pi_2 = \mathrm{diag}(0.06, 0.1, 0.2)$, and the LMI (22) is feasible. For $\tau_m = 2, 4, 6, 8, 10$, Table 2 gives out the allowable upper bound τ_M of the time-varying delay for given τ_m, which implies that, for this example, the delay-dependent exponential stability result proposed in Theorem 2 in this paper provides less conservatism than those in [10, 20, 21, 23, 26].

Cases	$\tau_m = 2$	$\tau_m = 4$	$\tau_m = 6$	$\tau_m = 8$	$\tau_m = 10$
By [20]	failed	failed	failed	failed	failed
By [21]	failed	failed	failed	failed	failed
By [26]	failed	failed	failed	failed	failed
By [10]	18	20	22	24	26
By [23]	24	26	28	30	34
By Theorem 2	$\tau_M > 0$	$\tau_M > 0$	$\tau_M > 0$	$\tau_M > 0$	$\tau_M > 0$

Table 2 Allowable upper bounds τ_M for given τ_m (Example 2)

5. Conclusions

Combined with linear matrix inequality (LMI) technique, a new augmented Lyapunov-Krasovskii function is constructed, and some new improved sufficient conditions ensuring globally exponential stability or robust exponential stability are obtained. Numerical examples show that the new results are less conservative than some recent results obtained in the literature cited therein.

References

[1] QIU Jianlong, CAO Jinde. *Delay-dependent exponential stability for a class of neural networks with time delays and reaction-diffusion terms* [J]. J. Franklin Inst., 2009, **346**(4): 301–314.

[2] WANG Jiafu, HUANG Lihong, GUO Zhenyuan. *Dynamical behavior of delayed Hopfield neural networks with discontinuous activations* [J]. Appl. Math. Model., 2009, **33**(4): 1793–1802.

[3] LIU Yurong, WANG Zidong, LIU Xiaohui. *Asymptotic stability for neural networks with mixed time-delays: The discrete-time case* [J]. Neural Networks, 2009, **22**: 67–74.

[4] LIU Leipo, HAN Zhengzhi, LI Wenlin. *Global stability analysis of interval neural networks with discrete and distributed delays of neutral type* [J]. Exp. Syst. Appl., 2009, **36**: 7328–7331.

[5] KWON O M, PARK J H. *Improved delay-dependent stability criterion for neural networks with time-varying delays* [J]. Phys. Lett. A, 2009, **373**(5): 529–535.

[6] RUIZ A, OWENS D H, TOWNLEY S. *Existences, learning and replication of periodic motions in recurrent neural network* [J]. IEEE Trans. Neural Networks., 1998, **9**: 651–661.

[7] TOWNLEY S. et al. *Existences and learning of oscillations in recurrent neural networks* [J]. IEEE Trans. Neural Networks., 2000, **11**: 205–214.

[8] TAN K C, TANG Huajin, ZHANG Weinian. *Qualitative analysis for recurrent neural networks with linear threshold transfer functions* [J]. IEEE Trans. Circuits Syst. I Regul. Pap., 2005, **52**(5): 1003–1012.

[9] GAO Bo, ZHANG Weinian. *Equilibria and their bifurcations in a recurrent neural network involving iterates of a transcendental function* [J]. IEEE Trans. Neural Networks., 2008, **19**: 782–794.

[10] LIU Yurong, WANG Zidong, LIU Xiaohui. *Robust stability of discrete-time stochastic neural networks with time-varying delays* [J]. Neurocomputing, 2008, **71**: 823–833.

[11] SYED A M, BALASUBRAMANIAM P. *Stability analysis of uncertain fuzzy Hopfield neural networks with time delays* [J]. Commun. Nonlinear Sci. Numer. Simul., 2009, **14**(6): 2776–2783.

[12] XIONG Wenjun, SONG Laizhong, CAO Jinde. *Adaptive robust convergence of neural networks with time-varying delays* [J]. Nonlinear Anal. Real World Appl., 2008, **9**(4): 1283–1291.

[13] LEE S M, PARK J H. *Robust model predictive control for norm-bounded uncertain systems using new parameter dependent terminal weighting matrix* [J]. Chaos Solitons Fractals, 2008, **38**(1): 199–208.

[14] KRISHNA R K V, KAR H. *Robust stability of discrete-time state-delayed systems with saturation nonlinearities: Linear matrix inequality approach* [J]. Signal Processing, 2009, **89**: 161–173.

[15] WU Zhengguang, Su Hongye, Chu Jian, et al. *New results on robust exponential stability for discrete recurrent neural networks with time-varying delays* [J]. Neurocomputing, 2009, **72**(13-15): 3337–3342.

[16] LUO Mengzhuo, ZHONG Shouming, WANG Rongjun, et al. *Robust stability analysis for discrete-time stochastic neural networks systems with time-varying delays* [J]. Appl. Math. Comput., 2009, **209**(2): 305–313.

[17] KRISHNA R K V, KAR H. *Robust stability of discrete-time state-delayed systems employing generalized overflow nonlinearities* [J]. Nonlinear Anal., 2008, **69**(9): 2780–2787.

[18] HUANG Chengfang, CHENG Kuohua, YAN Junjun. *Robust chaos synchronization of four-dimensional energy resource systems subject to unmatched uncertainties* [J]. Commun. Nonlinear. Sci. Numer. Simulat., 2009, **14**: 2784-2792.

[19] GAO Ming, CUI Baotong. *Global robust exponential stability of discrete-time interval BAM neural networks with time-varying delays* [J]. Appl. Math. Model., 2009, **33**(3): 1270–1284.

[20] SONG Qiankun, WANG Zidong. *A delay-dependent LMI approach to dynamics analysis of discrete-time recurrent neural networks with time-varying delays* [J]. Phys. Lett. A., 2007, **368**: 134–145.

[21] ZHANG Baoyong, XU Shenyuan, ZUO Yun. *Improved delay-dependent exponential stability criteria for discrete-time recurrent neural networks with time-varying delays* [J]. Neurocomputing, 2008, **72**: 321–330.

[22] YU Jianjiang, ZHANG Kanjian, FEI Shumin. *Exponential stability criteria for discrete-time recurrent neural networks with time-varying delay* [J]. Nonlinear Anal. Real World Appl., 2010, **11**(1): 207–216.

[23] ZHANG Yijun, XU Shengyuan, ZENG Zhenping. *Novel robust stability criteria of discrete-time stochastic recurrent neural networks with time delay* [J]. Neurocomputing, 2009, **72** (13-15): 3343–3351.
[24] LIU Xinge, TANG Meilan, MARTIN R, et al. *Discrete-time BAM neural networks with variable delays* [J]. Phys. Lett. A., 2007, **367**: 322–330.
[25] SONG Qiankun, LIANG Jinling, WANG Zidong. *Passivity analysis of discrete-time stochastic neural networks with time-varying delays* [J]. Neurocomputing, 2009, **72**: 1782–1788.
[26] LIU Yurong, WANG Zidong, SERRANO A, et al. *Discrete-time recurrent neural networks with time-varying delays: Exponential stability analysis* [J]. Phys. Lett. A., 2007, **362**: 480–488.
[27] SONG Chunwei, GAO Huijun, ZHENG Weixing. *A new approach to stability analysis of discrete-time recurrent neural networks with time-varying delay* [J]. Neurocomputing, 2009, **72** (10-12): 2563–2568.
[28] LEE T N, RADOVIC U L. *General decentralized stabilization of large-scale linear continuous and discrete time-delay systems* [J]. Internat. J. Control, 1987, **46**(6): 2127–2140.
[29] XIE Lihua. *Output feedback H_∞ control of systems with parameter uncertainty* [J]. Int. J. Control., 1996, **63**: 741–750.
[30] BOYD S, EL-GHAOUI L, FERON E, et al. *Linear Matrix Inequalities in Systems and Control Theory* [M]. Philadelphia (PA): SIAM, 1994.