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A Kind of Boundary Value Problem for Hypermonogenic Function Vectors

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Abstract By the Plemelj formula and the compressed fixed point theorem, this paper discusses a kind of boundary value problem for hypermonogenic function vectors in Clifford analysis. And the paper proves the existence and uniqueness of the solution to the boundary value problem for hypermonogenic function vectors in Clifford analysis.

Keywords Clifford analysis; hypermonogenic function vector; boundary value problem.

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1. Introduction

Clifford algebra Cl_n is an associative and incommutable algebra structure. Clifford analysis is an important branch of modern analysis, which studies the properties for the functions defined on R^{n+1} with the value in Clifford algebra space [1]. Clifford analysis possesses important theoretical and applicable value and plays an important role in many fields, such as the Maxwell equation, theory of Yang-Mills field, quantum mechanics and so on [2, 3]. Since 1970, some mathematicians made great effort in real and complex Clifford analysis. In 2000, Eriksson Leutwiler first introduced hypermonogenic function and gave some properties about it [4, 5]. Some researchers such as Huang [6, 7] and Qiao [8, 9] have done many works about boundary value problem for monogenic functions and hypermonogenic functions in real Clifford analysis. In this paper we will discuss a kind of boundary value problem for hypermonogenic function vectors in Clifford analysis and prove the existence and uniqueness of the solution to the boundary value problem for hypermonogenic function vectors in Clifford analysis.

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2. Hypermonogenic function vector and its plemelj formula

Let Cl_n be a real Clifford algebra over an $(n+1)$-dimensional real vector space $\mathbf{R}^{n+1}$ with orthogonal basis $e := \{e_0, e_1, \ldots, e_n\}$, where e_0 is the unit element in Cl_n. Then Cl_n has its basis $e_0, e_1, \ldots, e_n;\ e_1e_2, \ldots, e_{n-1}e_n; \ldots; e_1 \ldots, e_n$. Hence an arbitrary element of the basis may be written as $e_A = e_{\alpha_1}, \ldots, e_{\alpha_h}$, here $A = \{\alpha_1, \ldots, \alpha_h\} \subseteq \{1, \ldots, n\}$, $1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_h \le n$ and when $A = \emptyset$ (empty set) $e_A = e_0$. So real Clifford algebra is composed of the elements having the type $a = \sum_A x_A e_A$, in which $x_A (\in \mathbf{R})$ are real numbers. In general, we have $e_0 e_i = e_i e_0 = e_i$, $e_i^2 = -1$ and $e_i e_j + e_j e_i = 0$, where $i, j = 1, \ldots, n, i \ne j$.

Let $\Omega \subset \mathbf{R}^{n+1}$ be an open connected set. The function f which is defined in Ω with values in Cl_n can be expressed as $f(x) = \sum_A e_A f_A(x)$, where the functions f_A are real-valued functions. The set of C^r-functions in Ω with values in Cl_n is denoted by $F_\Omega^{(r)} = \{f | f : \Omega \to Cl_n,\ f(x) = \sum_A f_A(x) e_A\}$, here $f_A(x)$ have continuous r-times differentials.

A function $f(y) : \partial\Omega \to Cl_n$ is said to be Hölder continuous on $\partial\Omega$, if $f(y)$ satisfies

$$|f(y_1) - f(y_1)| \le M_1 |y_1 - y_2|^\beta, \quad y_1, y_2 \in \partial\Omega,\ 0 < \beta < 1.$$

We denote the set of the Hölder continuous functions on ∂K with the index β by $H(\partial\Omega, \beta)$. For any $\varphi \in H(\partial\Omega, \beta)$, we define the module of φ as follows: $\|\varphi\|_\beta = C(\varphi, \partial\Omega) + H(\varphi, \partial\Omega, \beta)$, where $C(\varphi, \partial\Omega) = \sup_{t \in \partial\Omega} |\varphi(t)|$ and $H(\varphi, \partial\Omega, \beta) = \sup_{t_1 \ne t_2, t_1, t_2 \in \partial\Omega} \frac{|\varphi(t_1) - \varphi(t_2)|}{|t_1 - t_2|^\beta}$. It is obvious that $H(\partial\Omega, \beta)$ is a Banach space and we have $\|\bar{\varphi}\|_\beta = \|\varphi\|_\beta$. For any $f, g \in H(\partial\Omega, \beta)$, it is easy to prove $\|f + g\|_\beta \le \|f\|_\beta + \|g\|_\beta$, $\|fg\|_\beta \le J_0 \|f\|_\beta \|g\|_\beta$.

Let $F(x) = (f_1(x), f_2(x), \ldots, f_p(x))$ and $G(x) = (g_1(x), g_2(x), \ldots, g_p(x))$ be function vectors, where $f_i(x), g_i(x) \in H(\partial\Omega, \beta)$, $i = 1, \ldots, p$. We define

$$F + G = (f_1(x) + g_1(x), \ldots, f_p(x) + g_p(x)), \quad F \otimes G = (f_1(x) g_1(x), \ldots, f_p(x) g_p(x))$$

and the absolute value of $\Phi = (\varphi_1, \ldots, \varphi_p)$ as follows: $|\Phi| = (\sum_{i=1}^p |\varphi_i|^2)^{\frac{1}{2}}$. If $|\Phi(x) - \Phi(x_0)| = \{\sum_{i=1}^p |\varphi_i(x) - \varphi_i(x_0)|^2\}^{\frac{1}{2}} \le B|x - x_0|^\beta$ (B is a positive constant), the function vector $\Phi(x) = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_p(x))$ is called Hölder continuous function vector, where $x \in \partial\Omega$. By [5] we have $|F + G| \le |F| + |G|$, $|F \otimes G| \le J_0 |F||G|$. Suppose $H'(\partial\Omega, \beta)$ is the set of all bounded Hölder function vectors and its Hölder index is β $(0 < \beta < 1)$. In the paper we define that the continuous function vector means all its components are continuous. For any $\Phi \in H'(\partial\Omega, \beta)$, we define $\|\Phi\|_\beta = C(\Phi, \partial\Omega) + H'(\Phi, \partial\Omega, \beta)$, where $C(\Phi, \partial\Omega) = \sup_{t \in \partial\Omega} |\Phi(t)|$, $H'(\Phi, \partial\Omega, \beta) = \sup_{\substack{t_1, t_2 \in \partial\Omega \\ t_1 \ne t_2}} \frac{|\Phi(t_1) - \Phi(t_2)|}{|t_1 - t_2|^\beta}$. It is easy to prove that $H'(\partial\Omega, \beta)$ is a Banach space and $\|\overline{\Phi}\|_\beta = \|\Phi\|_\beta$. And for any F, $G \in H'(\partial\Omega, \beta)$, we have

$$\|F + G\|_\beta \le \|F\|_\beta + \|G\|_\beta, \quad \|F \otimes G\|_\beta \le J_0 \|F\|_\beta \|G\|_\beta. \tag{1}$$

It is easy to prove that the following conclusions are true. If $\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_p) \in H'(\partial\Omega, \beta)$, we have $\varphi_i \in H(\partial\Omega, \beta)$ $(i = 1, 2, \ldots, p)$. And if $\varphi_i \in H(\partial\Omega, \beta)$ $(i = 1, 2, \ldots, p)$, we have $\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_p) \in H'(\partial\Omega, \beta)$.

We introduce the Dirac operator: $D_l f = \sum_{i=0}^n e_i \frac{\partial f}{\partial x_i}$, $D_r f = \sum_{i=0}^n \frac{\partial f}{\partial x_i} e_i$. Then we introduce the modified Dirac operators M^l, M^r: $M_k^l f(x) = D_l f(x) + k \frac{Q' f}{x_n}$, $M_k^r f(x) = D_r f(x) + k \frac{Q' f}{x_n}$,

$k = 0, 1, \ldots, n-1$, where $f \in F_\Omega^{(r)}$ $(r > 1)$.

Definition 1 *Let $\Omega \subset \mathbf{R}_+^{n+1} = \{x = (x_0, x_1, \ldots, x_n) \,|\, x_n > 0\}$ be a connected open set. A mapping $f : \Omega \to Cl_n$ is called the left hypermonogenic (hypermonogenic in short) function, if $f \in C^1(\Omega)$ and $M_{n-1}^l f(x) = 0$ for any $x \in \Omega$. The right hypermonogenic functions are defined similarly.*

Definition 2 *Let $F(x) = (f_1(x), f_2(x), \ldots, f_p(x))$. If all f_i $(i = 1, 2, \ldots, p)$ are left hypermonogenic functions, we call $F(x)$ a left hypermonogenic function vector. The right hypermonogenic function vectors are defined similarly.*

In this paper we suppose $\Omega \subset \mathbf{R}_+^{n+1} = \{x = (x_0, x_1, \ldots, x_n) \,|\, x_n > 0\}$ is a domain and K is an $(n+1)$-chain satisfying $\overline{K} \subset \Omega$. Then we define the n-forms: $\mathrm{d}\widehat{x_i} = \mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{i-1} \wedge \mathrm{d}x_{i+1} \wedge \cdots \wedge \mathrm{d}x_n$, $i = 0, 1, \ldots, n$. A Cl_n-valued n-form is introduced by putting $\mathrm{d}\sigma_k = \frac{1}{x_n^k} \sum_{i=0}^n (-1)^{i-1} e_i \mathrm{d}\widehat{x_i}$, $k = 0, 1, \ldots, n-1$. If $\mathrm{d}S$ stands for the "classical" surface element and $\overrightarrow{m} = \sum_{i=0}^n e_i n_i$, then the Cl_n-value surface element $\mathrm{d}\sigma_0$ can be written as $\mathrm{d}\sigma_0 = \overrightarrow{m} dS$, where n_i is the i-th component of the unit outward normal vector. Furthermore the volume-element $\mathrm{d}m_k = \frac{1}{x_n^k} \mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is used.

Lemma 1 ([4, 5]) *Let Ω be as stated above, $K \subseteq \Omega$ be an arbitrary $(n+1)$-chain, $\overline{K} \subset \Omega$ and $f, g \in F_K^{(r)}$ $(r \geq 1)$. Then*

$$\int_{\partial K} g \mathrm{d}\sigma_0 f = \int_K [(M_{-k}^r g) f + g M_k^l f + \frac{k}{x_n} Q(g f')] \mathrm{d}m_0,$$

$$\int_{\partial K} g \mathrm{d}\sigma_k f = \int_K [(M_k^r g) f + g M_k^l f - \frac{k}{x_n} P(g f') e_n] \mathrm{d}m_k,$$

$$\int_{\partial K} P(g \mathrm{d}\sigma_k f) = \int_K P[(M_k^r g) f + g M_k^l f] \mathrm{d}m_k,$$

$$\int_{\partial K} Q(g \mathrm{d}\sigma_0 f) = \int_K Q[(M_{-k}^r g) f + g M_k^l f] \mathrm{d}m_0.$$

Definition 3 *Let Ω, K be the sets as stated in Lemma 1. The integral*

$$\varphi(y) = \frac{(2y_n)^n}{\omega_{n+1}} \int_{\partial K} P\left(p(x,y) \mathrm{d}\sigma_{n-1}(x) f(x)\right) + \frac{(2^n y_n^{n-1})}{\omega_{n+1}} \int_{\partial K} Q\left(q(x,y) \mathrm{d}\sigma_0(x) f(x)\right) e_n$$

is called the quasi-Cauchy-form integral, where

$$p(x,y) = \frac{x_n^{n-1}}{2y_n} \left(\frac{(x-y)^{-1} - (x-\widehat{y})^{-1}}{|x-y|^{n-1} |x-\widehat{y}|^{n-1}} \right) = x_n^{n-1} \frac{(x-y)^{-1}}{|x-y|^{n-1}} e_n \frac{(x-\widehat{y})^{-1}}{|x-\widehat{y}|^{n-1}},$$

$$q(x,y) = \frac{(x-y)^{-1} - (x-\widehat{y})^{-1}}{2|x-y|^{n-1} |x-\widehat{y}|^{n-1}} = \frac{(x-y)^{-1}}{|x-y|^{n-1}} (x - Py) \frac{(x-\widehat{y})^{-1}}{|x-\widehat{y}|^{n-1}}.$$

By [5] the quasi-Cauchy's integral has another form:

$$\varphi(y) = \frac{2^{n-1} y_n^{n-1}}{\omega_{n+1}} \left(\int_{\partial K} E(x,y) \mathrm{d}\sigma_0(x) f(x) - \int_{\partial K} M(x,y) \widehat{\mathrm{d}\sigma_0(x)} \widehat{f(x)} \right), \tag{2}$$

where

$$E(x,y) = \frac{(x-y)^{-1}}{|x-y|^{n-1} |x-\widehat{y}|^{n-1}}, \quad M(x,y) = \frac{(\widehat{x}-y)^{-1}}{|x-y|^{n-1} |x-\widehat{y}|^{n-1}}. \tag{3}$$

And ω_{n+1} is the surface measure of the unit ball in R^{n+1}. $P(\ )$ and $Q(\ )$ mean the P-part and Q-part of ( ), respectively. When $y \notin \partial K$, the integral is well defined. When $y \in \partial K$, it is a singular integral and we will give the following definition.

Definition 4 *Let Ω, K be the sets as Lemma 1 and $y \in \partial K$. Then we construct a sphere E with the center at y and radius $\delta > 0$. Then ∂K is divided into two parts by E, and the part of ∂K lying in the interior of E is denoted by λ_δ. If $\lim_{\delta\to 0}\varphi_\delta(y) = I(y)$ exists, we say that the integral is convergent and $I(y)$ is called the Cauchy principal value of the singular integral, where*

$$\varphi_\delta(y) = \frac{2^{n-1}y_n^{n-1}}{\omega_{n+1}}\Big(\int_{\partial K-\lambda_\delta} E(x,y)\mathrm{d}\sigma_0(x)f(x) - \int_{\partial K-\lambda_\delta} M(x,y)\widehat{\mathrm{d}\sigma_0(x)}\widehat{f(x)}\Big)$$

and we define $\varphi(y) = I(y)$.

Lemma 2 ([5]) *Let Ω, K be the sets as Lemma 1 and $f : \overline{K} \to Cl_n$ be a hypermonogenic function. Then when $y \in K$, $\varphi(y) = f(y)$.*

Lemma 3 ([8]) *Let Ω, K be the sets as Lemma 1 and $f : \Omega \to Cl_n$ be a hypermonogenic function. Then when $y \in R^{n+1} - \overline{K}$, $\varphi(y) = 0$.*

Lemma 4 ([5]) *Let Ω, K be the sets as Lemma 1 and $f \in C^1(\overline{K})$. Then $\varphi(y)$ is a hypermonogenic function in $R^{n+1}/\partial K$.*

By Lemma 4 and Definition 2 we can prove the following theorem.

Theorem 1 *Let Ω, K be as stated above, $F(x) = (f_1(x), f_2(x), \ldots, f_p(x))$ be a function vector and $f_i(x) \in C^1(\overline{K})$, $i = 1, 2, \ldots, p$. Then we can conclude that*

$$\Phi(y) = \frac{2^{n-1}y_n^{n-1}}{\omega_{n+1}}\Big(\int_{\partial K} E(x,y)\mathrm{d}\sigma_0(x)F(x) - \int_{\partial K} M(x,y)\widehat{\mathrm{d}\sigma_0(x)}\widehat{F(x)}\Big)$$

is a hypermonogenic function vector in $R^{n+1}/\partial K$, where

$$E(x,y) = \frac{(x-y)^{-1}}{|x-y|^{n-1}|x-\widehat{y}|^{n-1}}, \quad M(x,y) = \frac{(\widehat{x}-y)^{-1}}{|x-y|^{n-1}|x-\widehat{y}|^{n-1}}.$$

Lemma 5 ([8]) *Let $y \in \partial K$ and $f \in H^\alpha_{\partial K}$. Then*

$$\begin{aligned}\varphi(y) =& \frac{(2^{n-1}y_n^{n-1})}{\omega_{n+1}}\Big[\int_{\partial K} E(x,y)\mathrm{d}\sigma_0(x)f(x) - \int_{\partial K} M(x,y)\widehat{\mathrm{d}\sigma_0(x)}\widehat{f(x)}\Big]\\ =& \frac{(2^{n-1}y_n^{n-1})}{\omega_{n+1}}\Big(\int_{\partial K} E(x,y)\mathrm{d}\sigma_0(x)[f(x)-f(y)] -\\ & \int_{\partial K} M(x,y)\widehat{\mathrm{d}\sigma_0(x)}[\widehat{f(x)}-f(y)]\Big) + \frac{1}{2}f(y),\end{aligned} \tag{4}$$

where $E(x,y)$, $M(x,y)$ are given by equation (3).

Lemma 6 ([8]) *Let K, ∂K be as stated above, $f(x) \in H(\partial K, \beta)$, $0 < \beta < 1$ and $y_0 \in \partial K$. We denote the limits of $\varphi(x)$ by $\varphi^+(y_0)$ and $\varphi^-(y_0)$ when $y \to y_0$ in K and K^-, respectively. Then* *we have*

$$\begin{cases} \varphi^+(y_0) = \varphi(y_0) + \dfrac{1}{2} f(y_0), \\ \varphi^-(y_0) = \varphi(y_0) - \dfrac{1}{2} f(y_0); \end{cases} \qquad \begin{cases} \varphi^+(y_0) + \varphi^-(y_0) = 2\varphi(y_0), \\ \varphi^+(y_0) - \varphi^-(y_0) = f(y_0), \end{cases} \tag{5}$$

where $\varphi(y)$ is the function in equation (4).

By Lemmas 5 and 6 we have the following theorem.

Theorem 2 *Let $K, \partial K$ be as stated above and function vector $F(x) = (f_1(x), f_2(x), \ldots, f_p(x)) \in H'(\partial K, \beta)$, here $0 < \beta < 1$ and $y_0 \in \partial K$. We denote the limits of $\Phi(x)$ by $\Phi^+(y_0)$ and $\Phi^-(y_0)$ when $y \to y_0$ in K and K^-, respectively. Then we have*

$$\begin{cases} \Phi^+(y_0) = \Phi(y_0) + \dfrac{1}{2} F(y_0), \\ \Phi^-(y_0) = \Phi(y_0) - \dfrac{1}{2} F(y_0); \end{cases} \qquad \begin{cases} \Phi^+(y_0) + \Phi^-(y_0) = 2\Phi(y_0), \\ \Phi^+(y_0) - \Phi^-(y_0) = F(y_0), \end{cases} \tag{6}$$

where $\Phi(y) = (\varphi_1, \ldots, \varphi_p)$ and each φ_i $(i = 1, \ldots, p)$ is the function as in equation (4).

3. The boundary value problem for hypermonogenic function vectors

Let K, ∂K be as stated above. We will find a hypermonogenic function vector in $R^{n+1}/\partial K$ which satisfies the boundary condition

$$A(t)\Phi^+(t) + B(t)\Phi^-(t) = G(t), \tag{7}$$

where $\Phi(t)$ is the unknown function vector and $A(t)$, $B(t)$, $G(t) \in H'(\partial K, \beta)$ are known function vectors on ∂K. This boundary value problem is called Problem I.

Lemma 7 ([8]) *Let K, ∂K be as stated above, $\phi \in H(\partial K, \beta)$ and $\theta\varphi = T\varphi - \frac{\varphi}{2}$, where*

$$T\varphi(x) = \frac{2^{n-1} x_n^{n-1}}{\omega_{n+1}} \Big[\int_{\partial K} E(t,x) \mathrm{d}\sigma_0(t) \varphi(t) - M(t,x) \widehat{\mathrm{d}\sigma_0(t)} \widehat{\varphi(t)} \Big]$$

and $E(t,x)$, $M(t,x)$ are as in equation (3). Then we can conclude that $\theta\varphi$ is a hypermonogemic function and $\|\theta\varphi\|_\beta \leq J_1 \|\varphi\|_\beta$, $|T\varphi\|_\beta \leq J_2 \|\varphi\|_\beta$, $\|T\varphi + \frac{\varphi}{2}\|_\beta \leq J_2 \|\varphi\|_\beta$, where J_1, J_2 are constants independent of φ.

Theorem 3 *Let K, ∂K be as stated above, $\Phi \in H'(\partial K, \beta)$ and $\theta\Phi = T\Phi - \frac{\Phi}{2}$, where*

$$T\Phi(x) = \frac{2^{n-1} x_n^{n-1}}{\omega_{n+1}} \Big[\int_{\partial K} E(t,x) \mathrm{d}\sigma_0(t) \Phi(t) - M(t,x) \widehat{\mathrm{d}\sigma_0(t)} \widehat{\Phi(t)} \Big].$$

Then we can conclude that $\theta\Phi$ is a hypermonogemic function vector and $\|\theta\Phi\|_\beta \leq J_3 \|\Phi\|_\beta$, where J_3 is a positive constant independent of Φ.

Proof

$$\begin{aligned} \|\theta\Phi\|_\beta &\leq \sup_{t \in \Omega} \Big(\sum_{i=1}^p |\theta\varphi_i|^2 \Big)^{\frac{1}{2}} + \sup_{t_1 \neq t_2, t_1, t_2 \in \partial\Omega} \frac{|\theta\varphi_i(t_1) - \theta\varphi_i(t_2)|}{|t_1 - t_2|^\beta} \\ &\leq \sup_{t \in \Omega} \Big(\sum_{i=1}^p \|\theta\varphi_i\|_\beta^2 \Big)^{\frac{1}{2}} + \sup_{t_1 \neq t_2, t_1, t_2 \in \partial\Omega} \frac{(\sum_{i=1}^p |\theta\varphi_i(t_1) - \theta\varphi_i(t_2)|^2)^{\frac{1}{2}}}{|t_1 - t_2|^\beta} \end{aligned}$$

$$\le \sup_{t\in\Omega}\Big(\sum_{i=1}^{p}(J_1\|\varphi_i\|_\beta)^2\Big)^{\frac{1}{2}}+\sum_{i=1}^{p}\Big[\sup_{t_1\neq t_2,t_1,t_2\in\partial\Omega}\frac{(|\theta\varphi_i(t_1)-\theta\varphi_i(t_2)|^2)^{\frac{1}{2}}}{|t_1-t_2|^\beta}\Big]$$

$$\le J_4\|\Phi\|_\beta+\sum_{i=1}^{p}J_1\|\Phi\|_\beta\le J_3\|\Phi\|_\beta,$$

where J_3 is a positive constant independent of Φ. □

Theorem 4 *Under the conditions of Theorem 3, we have* $\|T\Phi\|_\beta \le J_5\|\Phi\|_\beta$, $|T\Phi+\frac{\Phi}{2}\|_\beta \le J_5\|\Phi\|_\beta$, *where* Φ *is given in Theorem 2.*

By the Plemelj formula, we can obtain $\Phi^+(x)=\frac{\Phi(x)}{2}+\Phi\phi(x)$, $\Phi^-(x)=\frac{-\Phi(x)}{2}+\Phi\varphi(x)$. Then we can change Problem I into an integral equation $Q\Phi(x)=\Phi(x)$, where $Q\Phi=(A(x)+B(x))[\frac{\Phi(x)}{2}+T\Phi(x)]+(1-B(x))\Phi(x)-G(x)$.

Theorem 5 *Let* $A(x)$, $B(x)$ *and* $G(x)\in H'(\partial K,\beta)$. *Then when*

$$J_0[\|A(x)+B(x)\|_\beta J_5+\|(1-B(x))\|_\beta]<1, \tag{8}$$

$$\frac{\|G(x)\|_\beta}{1-J_0(\|A(x)+B(x)\|_\beta J_5+\|1-B(x)\|_\beta)}<M, \tag{9}$$

there exists a unique function $\Phi_0\in T'=\{\Phi|\Phi\in H'(\partial K,\beta),\ \|\Phi\|_\beta\le M\}\subset H'(\partial K,\beta)$ *such that* $Q\Phi_0=\Phi_0$, *where* $M>0$ *is a constant. Then* Φ_0 *is the solution of Problem I.*

Proof We denote the continuous function vectors space on ∂K by $C'(\partial K)$ and T' is the function set as above. By equation (1), Theorem 4 and the condition (9), we have

$$\begin{aligned}\|Q\Phi\|_\beta &\le J_0\|A(x)+B(x)\|_\beta\|\frac{\Phi(x)}{2}+T\Phi(x)\|_\beta+J_0\|(1-B(x))\|_\beta\|\Phi\|_\beta+\|G(x)\|_\beta\\ &\le [J_0\|A(x)+B(x)\|_\beta J_5+J_0\|(1-B(x))\|_\beta]\,\|\Phi\|_\beta+\|G(x)\|_\beta\\ &\le [J_0\|A(x)+B(x)\|_\beta J_5+J_0\|(1-B(x))\|_\beta]M+\|G(x)\|_\beta\\ &\le [J_0(\|A(x)+B(x)\|_\beta J_5+\|(1-B(x))\|_\beta)-1]M+\|G(x)\|_\beta+M\le M.\end{aligned}$$

Hence Q is a mapping from T' onto itself. For any Φ_1, $\Phi_2\in H'(\partial K,\beta)$, by equation (1), Theorem 4 and the condition (8) we have

$$\begin{aligned}\|Q\Phi_1-Q\Phi_2\|_\beta \le& J_0\|A(x)+B(x)\|_\beta\|\frac{[\Phi_1(x)-\Phi_2(x)]}{2}+\\ & T[\Phi_1(x)-\Phi_2(x)]\|_\beta+J_0\|(1-B(x))\|_\beta\|\Phi_1(x)-\Phi_2(x)\|_\beta\\ \le& J_0\,[\|A+B\|_\beta J_5+\|(1-B(x))\|_\beta]\,\|\Phi_1-\Phi_2\|_\beta<\|\Phi_1-\Phi_2\|_\beta.\end{aligned}$$

Hence Q is a compressed map. By the compressed fixed point theorem we know that there exists a unique function $\Phi_0\in H'(\partial K,\beta)$, such that $Q(\Phi_0)=\Phi_0$. Then Φ_0 is a solution of boundary value Problem I. □

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