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Transcendental Meromorphic Solutions of Second-Order Algebraic Differential Equations

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Abstract Using Nevanlinna theory of the value distribution of meromorphic functions, we discuss some properties of the transcendental meromorphic solutions of second-order algebraic differential equations, and generalize some results of some authors.

Keywords meromorphic functions; transcendental meromorphic solutions; second-order algebraic differential equations.

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1. Introduction and main results

We use the standard notations and results of the Nevanlinna theory of meromorphic or algebraic function, see, e.g, [1]. In this paper we denote: $M(z,w)=\max_{|z|\le r}\{|w(z)|\}$;

$$V(z,w)=a_k(z)w^k+a_{k-1}(z)w^{k-1}+\cdots+a_0(z). \tag{1}$$

Some authors [2–6] have investigated the problems of the existence of algebraic solutions of equation, and obtained some results. Especially many investigations have been done on the form

$$(w')^n=\frac{P(z,w)}{Q(z,w)}, \tag{2}$$

and some important results were obtained as follows

Theorem A ([1]) *Let $w(z)$ be the transcendental meromorphic solution of algebraic differential equation (2). If $V(z,w)$, which is defined in* (1), *is the prime factor of $Q(z,w)$, then $V(z)=V(z,w)$ has infinite many zeros.*

Theorem B ([1]) *Let $w(z)$ be the transcendental meromorphic solution of algebraic differential equation (2). If $V(z,w)$, which is defined in (1), is the prime factor of $P(z,w)$, then $V(z)=V(z,w)$ has infinite many zeros.*

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In this paper, we mainly consider the form as follows:

$$(w'')^n = \frac{P(z,w)}{Q(z,w)}, \tag{3}$$

where $P(z,w) = \sum_{i=0}^{p} a_i(z)w^i \neq 0$ and $Q(z,w) = \sum_{i=0}^{q} b_j(z)w^j \neq 0$ are the polynomials co-prime of w, of which coefficients are rational functions. We will prove

Theorem 1 *Let $w(z)$ be the transcendental meromorphic solution of algebraic differential equation (3). If $V(z,w)$, which is defined in (1), is the prime factor of $Q(z,w)$, then $V(z) = V(z,w)$ has infinite many zeros.*

Theorem 2 *Let $w(z)$be the transcendental meromorphic solution of algebraic differential equation (3). If $V(z,w)$, which is defined in (1), is the prime factor of $P(z,w)$, then $V(z) = V(z,w)$ has infinite many zeros.*

2. Some lemmas

We wil use the following Lemmas in our proofs of the above Theorems.

Lemma 1 ([1]) *Let $w(z)$ be a transcendental meromorphic function and $V(z,w)$ be defined by (1). If $V(z) = V(z,w)$ only has a finite number of zeros. Then for all z_r in $M(r, \frac{1}{V}) = \frac{1}{|V(z_r)|}$, there is a $\beta > 0$, such that $|w(z_r)| \leq r^\beta$ when $r \geq r_0$.*

Lemma 2 ([1]) *Let $w(z)$ be a transcendental meromorphic function which has only a finite number of poles. Then for arbitrary $\alpha > 0$ and K there is $\frac{[c]^\alpha}{r^K} \to \infty$ $(r \to \infty)$.*

Lemma 3 ([1]) *Let $w(z)$ be a transcendental meromorphic function which has only a finite number of poles. Then for any $\alpha > 0$, outside a possible exception set of finite linear measure, $M(r, w') < 2^{\frac{1}{\alpha}}[M(r,w)]^{1+\alpha}$.*

Lemma 4 ([1]) *Let $w(z)$ be a transcendental meromorphic function which has only a finite number of poles, and $w(z)$ and $w'(z)$ be holomorphic in the plane. Then for any $\varepsilon > 0$, there is*

$$M(r,w) < [M(r,w')]^{1+\varepsilon}.$$

3. Proof of Theorem 1

Suppose $V(z)$ has finite number of zeros, then $y(z) = \frac{1}{V(z)}$ is transcendental meromorphic function which has only a finite number of poles. By Lemma 1, for all z_r in $M(r, \frac{1}{V}) = \frac{1}{|V(z_r)|}$, there is a $\beta > 0$, such that $|w(z_r)| \leq r^\beta$ when $r \geq r_0$. By Lemma 2, for arbitrary $\alpha > 0$ and K there holds

$$\lim_{r\to\infty} \frac{[M(r,y)]^\alpha}{r^k} = \lim_{r\to\infty} \frac{1}{|V(z)|^\alpha r^K} = 0. \tag{4}$$

$V(z,w)$ is the prime factor of $Q(z,w)$, $P(z,w)$ and $Q(z,w)$ are co-prime, so $V_w(z,w)$ and $V(z,w)$ are co-prime, and $V(z,w)$ and $P(z,w)$ are co-prime. Then there exist rational functions $P_j(z,w), Q_j(z,w)$ and $R_j \neq 0$ $(j=1,2)$ such that

$$P_1(z,w)V(z,w)+Q_1(z,w)V_w(z,w)=R_1,\ \ P_2(z,w)V(z,w)+Q_2(z,w)P(z,w)=R_2. \quad (5)$$

Through multiplied by the appropriate polynomial of z, $R_j \neq 0$ $(j=1,2)$ can be assumed to be the non-zero polynomials. Therefore, when r is large enough, there exists $\alpha > 0$, such that $|R_j| > \alpha > 0 > 0$ $(j=1,2)$. Let $y(z)=\frac{1}{V(z)}$. We have

$$y''(z)=y^3\{2V_w^2(w')^2+4V_wV_zw'+V_z^2-VV_{ww}(w')^2-2VV_{wz}w'-VV_ww''-VV_{zz}\}.$$

Then

$$\begin{aligned}
M(r,y'') &\geq |y''(z_r)| \geq |y(z_r)|^3\{|V(z_r,w(z_r))V_w(z_r,w(z_r))w^{(2)}(z_r)|-|2V_w^2(z_r,w(z_r))[w'(z_r)]^2|-\\
&\quad |4V_w(z_r,w(z_r))V_z(z_r,w(z_r))|-|V(z_r,w(z_r))V_{ww}(z_r,w(z_r))V_w(z_r,w(z_r))|-\\
&\quad |V_z^2(z_r,w(z_r))|-|2V(z_r,w(z_r))V_{wz}(z_r,w(z_r))w'(z_r)|-|V(z_r,w(z_r))V_{zz}(z_r,w(z_r))|\}\\
&=[M(r,y)]^{3+\frac{1}{n}}\{|V^{1+\frac{1}{n}}(z_r)V_w(z_r,w(z_r))w^{(2)}(z_r)|-\\
&\quad |2V^{\frac{1}{n}}(z_r)V_w^2(z_r,w(z_r))[w'(z_r)]^2|-|4V^{\frac{1}{n}}(z_r)V_w(z_r,w(z_r))V_z(z_r,w(z_r))|-\\
&\quad |V^{1+\frac{1}{n}}(z_r)V(z_r,w(z_r))V_{ww}(z_r,w(z_r))V_w(z_r,w(z_r))|-|V^{1+\frac{1}{n}}(z_r)V_z^2(z_r,w(z_r))|-\\
&\quad |2V^{1+\frac{1}{n}}(z_r)V(z_r,w(z_r))V_{wz}(z_r,w(z_r))w'(z_r)|-|V(z_r,w(z_r))V_{zz}(z_r,w(z_r))|\}. \qquad (6)
\end{aligned}$$

Let $\Im$ be the set that contains the ten types of polynomials of w:

$$\Im=\{P_1(z,w),P_2(z,w),Q_1(z,w),Q_2(z,w),\frac{Q}{V},V_z(z,w),V_w(z,w),V_{zz}(z,w),V_{zw}(z,w),V_{ww}(z,w)\}.$$

Then $\forall X(z,w)=\sum_{k=1}^n a_k(z)w^k \in \Im$, when r is large enough, there exists l such that $|a_k(z_r)|<l$. By Lemma 1, we have $|w(z_r)| \leq r^\beta$. Then there exists $\sigma > \beta$, such that $|X(z_r,w(z_r))| = |\sum_{k=1}^n a_k(z)w^k(z_r)| < r^\sigma$, namely, $\Im=\{X(z,w)||X(z_r,w(z_r))|<r^\sigma\}$.

1^0. By $P_1(z,w(z_r)) \in \Im, Q_1(z,w) \in \Im$, we obtain $|P_1(z_r,w(z_r))|<r^\sigma$, $|Q_1(z_r,w(z_r))|<r^\sigma$. By (4), there is $|P_1(z_r,w(z_r))||V(z_r)|<r^\sigma|V(z_r)|<\frac{a}{2}$. From (5), we have

$$|V_w(z_r,w(z_r))|=|\frac{R_1(z_r)-P_1(z_r,w(z_r))V(z_r)}{Q_1(z_r,w(z_r))}| \geq \frac{|R_1(z_r)|-|P_1(z_r,w(z_r))V(z_r)|}{|Q_1(z_r,w(z_r))|} \geq \frac{a}{2r^\sigma}. \quad (7)$$

Similarly, we get $|P(z_r,w(z_r))| \geq \frac{|R_2(z_r)|-|P_2(z_r,w(z_r))V(z_r)|}{|Q_2(z_r,w(z_r))|} \geq \frac{a}{2r^\sigma}$. By $\frac{Q(z,w}{V(z,w} \in \Im$, we have $|\frac{Q(z_r,w(z_r}{V(z_r,w(z_r}|<r^\sigma$. Since $w(z)$ is the transcendental meromorphic solution of algebraic differential equation (3), $\frac{a}{2r^\sigma} \leq |P(z_r,w(z_r))|=|w''(z_r)|^n|Q(z_r,w(z_r))| \leq |w''(z_r)|^n|V(z_r)|r^\sigma$. Then

$$|w''(z_r)||V(z_r)|^{\frac{1}{n}} \geq (\frac{a}{2})^{\frac{1}{n}}r^{\frac{-2\sigma}{n}} \geq (\frac{a}{2})^{\frac{1}{n}}r^{-2\sigma}. \qquad (8)$$

Combining the inequalities (7) and (8), we obtain

$$|V_w(z_r,w(z_r))||w''(z_r)||V(z_r)|^{\frac{1}{n}} \geq (\frac{a}{2})^{1+\frac{1}{n}}r^{-3\sigma}. \qquad (9)$$

2^0. By Lemma 3, when $\alpha=1$, then $|w'(z_r)| \leq M(r,w') < 2[M(r,w)]^2=2|w(z_r)|^2 \leq 2r^{2\beta} \leq 2r^{2\sigma}$. By (4), when $\alpha=\frac{1}{n}$, $K=9\sigma$, and r is large enough, then $|V(z_r)|^{\frac{1}{n}} \leq \frac{1}{64}\frac{a}{2})^{1+\frac{1}{n}}r^{-9\sigma}$. By $V_w(z,w) \in \Im$, we have $|V_w(z_r,w(z_r))|<r^\sigma$, then

$$|2V^{\frac{1}{n}}(z_r)V_w^2(z_r,w(z_r))[w'(z_r)]^2| \leq \frac{1}{8}(\frac{a}{2})^{1+\frac{1}{n}}r^{-3\sigma}. \qquad (10)$$

3^0. By $V_z(z,w) \in \Im$, we have $|V_z(z_r, w(z_r))| < r^\sigma$. It follows from the above

$$|V^{\frac{1}{n}}(z_r)V_z(z_r, w(z_r))w'(z_r)| \le \frac{1}{32}(\frac{a}{2})^{1+\frac{1}{n}}r^{-3\sigma}, \tag{11}$$

$$|V^{1+\frac{1}{n}}(z_r)V_z^2(z_r, w(z_r))| \le \frac{1}{64}(\frac{a}{2})^{1+\frac{1}{n}}r^{-3\sigma}. \tag{12}$$

4^0. By $V_{ww}(z,w) \in \Im$, we have $|V_{ww}(z_r, w(z_r))| < r^\sigma$, and it follows from the above

$$|w'(z_r)|^2|V^{1+\frac{1}{n}}(z_r)V_{ww}(z_r, w(z_r))| \le \frac{1}{16}(\frac{a}{2})^{1+\frac{1}{n}}r^{-3\sigma}. \tag{13}$$

5^0. By $V_{wz}(z,w) \in \Im$, we have $|V_{wz}(z_r, w(z_r))| < r^\sigma$, and it follows from the above

$$2|w'(z_r)||V^{1+\frac{1}{n}}(z_r)V_{wz}(z_r, w(z_r))| \le \frac{1}{16}(\frac{a}{2})^{1+\frac{1}{n}}r^{-3\sigma}. \tag{14}$$

6^0. By $V_{zz}(z,w) \in \Im$, we have $|V_{zz}(z_r, w(z_r))| < r^\sigma$, and it follows from the above

$$|V^{1+\frac{1}{n}}(z_r)V_{zz}(z_r, w(z_r))| \le \frac{1}{64}(\frac{a}{2})^{1+\frac{1}{n}}r^{-3\sigma}. \tag{15}$$

Combining the inequalities (9)–(15) and (6) gives

$$\begin{aligned} M(r,y'') \ge & [M(r,y)]^{3+\frac{1}{n}}\{(\frac{a}{2})^{1+\frac{1}{n}}r^{-3\sigma} - \frac{1}{8}(\frac{a}{2})^{1+\frac{1}{n}}r^{-3\sigma} - \frac{1}{8}(\frac{a}{2})^{1+\frac{1}{n}}r^{-3\sigma} - \\ & \frac{1}{64}(\frac{a}{2})^{1+\frac{1}{n}}r^{-3\sigma} - \frac{1}{16}(\frac{a}{2})^{1+\frac{1}{n}}r^{-3\sigma} - \frac{1}{16}(\frac{a}{2})^{1+\frac{1}{n}}r^{-3\sigma} - \frac{1}{64}(\frac{a}{2})^{1+\frac{1}{n}}r^{-3\sigma}\} \\ \ge & [M(r,y)]^{3+\frac{1}{n}}\frac{9}{16}(\frac{a}{2})^{1+\frac{1}{n}}r^{-3\sigma}. \end{aligned} \tag{16}$$

By Lemma 3, when $\alpha = 1$, outside a possible exception set of finite linear measure, we have

$$M(r,y'') < 2[M(r,y')]^2 < 8[M(r,y)]^4.$$

Combining the inequality (16), we have $\frac{[M(r,y)]^{\frac{1}{n}-1}}{r^{-3\sigma}} \le \frac{128}{9}(\frac{a}{2})^{-(1+\frac{1}{n})} < \infty$. Cleary, this and (4) result in a contradiction. So $V(z) = V(z,w)$ has infinite many zeros. This completes the proof of Theorem 1. □

4. Proof of Theorem 2

If $V(z)$ has finite number of zeros, then $y(z) = \frac{1}{V(z)}$ is a transcendental meromorphic function which has only a finite number of poles. If value z_r satisfies $M(r,y) = |y(z_r)| = \frac{1}{|V(z_r)|}$, then for any $\alpha > 0$ and K, by Lemma 2,

$$\lim_{r\to\infty}\frac{[M(r,y)]^\alpha}{r^K} = \lim_{r\to\infty}\frac{1}{|V(z_r)|^\alpha r^K} = \infty \Rightarrow \lim_{r\to\infty}|V(z_r)|^\alpha r^K = 0. \tag{17}$$

From $y(z_r) = \frac{1}{V(z_r)}$ and (6), we have

$$\begin{aligned} M(r,y'') \ge & |y''(z_r)| \\ \ge & |y(z_r)|^3\{|V(z_r, w(z_r))V_w(z_r, w(z_r))w^{(2)}(z_r)| - |2V_w^2(z_r, w(z_r))[w'(z_r)]^2| - \\ & |4V_w(z_r, w(z_r))V_z(z_r, w(z_r))| - |V(z_r, w(z_r))V_{ww}(z_r, w(z_r))V_w(z_r, w(z_r))| - \\ & |V_z^2(z_r, w(z_r))| - |2V(z_r, w(z_r))V_{wz}(z_r, w(z_r))w'(z_r)| - |V(z_r, w(z_r))V_{zz}(z_r, w(z_r))| \\ \ge & [M(r,y)]^3\{|V_z^2(z_r, w(z_r))| - |V(z_r)V_w(z_r, w(z_r))w''(z_r)| - |2V_w^2(z_r, w(z_r))[w'(z_r)]^2| - \end{aligned}$$

$$|4V_w(z_r,w(z_r))V_z(z_r,w(z_r))w'(z_r)|-|V(z_r)V_{ww}(z_r,w(z_r))[w'(z_r)]^2|-$$
$$|2V(z_r)V_{wz}(z_r,w(z_r))w'(z_r)|-|V(z_r)V_{zz}(z_r,w(z_r))|\}. \tag{18}$$

Since $V(z,w)$ is the prime factor of $P(z,w)$, $V(z,w)$ and $V_z(z,w)$ are co-prime. Then there exist two polynomials $P_1(z,w)$ and $Q_1(z,w)$ in w and rational function $R(z)\neq 0$ such that

$$P_1(z,w)V(z,w)+Q_1(z,w)V_z(z,w)=R(z). \tag{19}$$

When r is large enough, there exists $b>0$, such that $|R(z)|>b>0$.

By Lemma 1, there exists $\tau>0$, such that $|w(z_r)|\leq r^\tau$. Similarly to the proof of Theorem 1, let $\Re$ be the set that contains the eight types of polynomials of w

$$\Re=\{P_1(z,w),Q_1(z,w),\frac{P(z,w}{V(z,w},V_z(z,w),V_w(z,w),V_{zz}(z,w),V_{zw}(z,w),V_{ww}(z,w)\}.$$

Then $\forall X(z,w)=\sum_{k=1}^n a_k(z)w^k\in\Re$, when r is large enough, there exists l such that $|a_k(z_r)|<r^l$. By Lemma 1, we have $|w(z_r)|\leq r^\tau$. Then there exists $\upsilon>\tau$, such that $|X(z_r,w(z_r))|=|\sum_{k=1}^n a_k(z)w^k(z_r)|<r^\tau$, namely, $\Re=\{X(z,w)||X(z_r,w(z_r))|<r^\tau\}$. Then by $P_1(z,w)\in\Re, Q_1(z,w)\in\Re$, we get $|P_1(z_r,w(z_r))|<r^\tau, |Q_1(z_r,w(z_r))|<r^\tau$. By (17), we obtain

$$|P_1(z_r,w(z_r))||V(z_r)|<r^\tau|V(z_r)|<\frac{b}{2}.$$

From (19), we have $|V_z(z_r,w(z_r))|=\frac{|R(z_r)|-|P_1(z_r,w(z_r))||V(z_r)|}{|Q_1(z_r,w(z_r))|}\geq\frac{b}{2r^\upsilon}$. Then

$$|V_z(z_r,w(z_r))|^2\geq(\frac{b}{2})^2r^{-2\upsilon}. \tag{20}$$

Because $Q(z)$ is rational function, when r is large enough, $|Q(z)|<r^\upsilon$. By $\frac{P(z,w}{V(z,w}\in\Re$, then $|\frac{P(z,w}{V(z,w}|<r^\tau$. Since $w(z)$ is the transcendental meromorphic solution of algebraic differential equation (3), we have $|w''(z_r)|^n=\frac{1}{|Q(z)|}\frac{|P(z_r,w(z_r))|}{|V(z_r)|}|V(z_r)|<r^{2\upsilon}|V(z_r)|$. By (17) and $\alpha=\frac{1}{n}, K=4\upsilon+\frac{2\upsilon}{n}$, when r is large enough, there is $|V^{\frac{1}{n}}(z_r)|\leq\frac{1}{8}(\frac{b}{2})^2r^{-(4\upsilon+\frac{2\upsilon}{n})}$. So we obtain $|w''(z_r)|<\frac{1}{8}\frac{b}{2}r^{-4\upsilon}<\frac{1}{8}(\frac{b}{2})^2r^{-3\upsilon}$. By $V_w(z,w)\in\Re$, then $|V_w(z_r,w(z_r))|<r^\tau$, also $|V(z_r)|\leq[\frac{1}{8}(\frac{b}{2})^2r^{-(4\upsilon+\frac{2\upsilon}{n})}]^2\leq 1$, we get

$$|V(z_r)||V_w(z_r,w(z_r))||w''(z_r)|<\frac{1}{8}(\frac{b}{2})^2r^{-2\upsilon}. \tag{21}$$

By Lemma 4, when $\varepsilon=1$, then $|w'(z_r)|\leq M(r,w')<M^2(r,w'')=|w''(z_r)|^2<[\frac{1}{8}\frac{b}{2}r^{-4\upsilon}]^2<\frac{1}{64}(\frac{b}{2})^2r^{-4\upsilon}$. By $V_w(z,w)\in\Re$, then $|V_w(z_r,w(z_r))|<r^\tau$, and it follows

$$2|V_w^2(z_r,w(z_r))||w'(z_r)|^2<\frac{1}{32}(\frac{b}{2})^2r^{-2\upsilon}. \tag{22}$$

Similarly, we obtain

$$|4V_w(z_r,w(z_r))V_z(z_r,w(z_r))w'(z_r)|<\frac{1}{2}(\frac{b}{2})^2r^{-2\upsilon}, \tag{23}$$

$$|2V_w^2(z_r,w(z_r))[w'(z_r)]^2|<\frac{1}{32}(\frac{b}{2})^2r^{-2\upsilon}, \tag{24}$$

$$|2V(z_r)V_{wz}(z_r,w(z_r))w'(z_r)|<\frac{1}{4}(\frac{b}{2})^2r^{-2\upsilon}. \tag{25}$$

By $|V(z_r)V_{zz}(z_r,w(z_r))|<[\frac{1}{8}\frac{b}{2}r^{-(4\upsilon+\frac{2\upsilon}{n})}]^n r^\upsilon$, when $n=2$, then

$$|V(z_r)V_{zz}(z_r,w(z_r))|<\frac{1}{64}(\frac{b}{2})^2r^{-2\upsilon}. \tag{26}$$

Combining the inequalities (20)–(26) and (18), we have

$$\begin{aligned}M(r,y'')\geq&[M(r,y)]^3\{(\frac{b}{2})^2r^{-2\upsilon}-\frac{1}{8}(\frac{b}{2})^2r^{-2\upsilon}-\frac{1}{32}(\frac{b}{2})^2r^{-2\upsilon}-\frac{1}{2}(\frac{b}{2})^2r^{-2\upsilon}-\\&\frac{1}{64}(\frac{b}{2})^2r^{-2\upsilon}-\frac{1}{4}(\frac{b}{2})^2r^{-2\upsilon}-\frac{1}{64}(\frac{b}{2})^2r^{-2\upsilon}\}\\\geq&[M(r,y)]^3\frac{1}{8}(\frac{b}{2})^2r^{-2\upsilon}.\end{aligned} \tag{27}$$

By Lemma 3, when $\alpha=\frac{1}{2}$, outside a possible exception set of finite linear measure, we have

$$M(r,y'')<4[M(r,y')]^{\frac{3}{2}}<4(4[M(r,y)]^{\frac{3}{2}})^{\frac{3}{2}}=32[M(r,y)]^{\frac{9}{4}}.$$

Combining the inequality (27) gives

$$[M(r,y)]^3\frac{1}{8}(\frac{b}{2})^2r^{-2\upsilon}<32[M(r,y)]^{\frac{9}{4}}\Rightarrow\frac{[M(r,y)]^{\frac{3}{4}}}{r^{2\upsilon}}<256(\frac{b}{2})^2\to\infty.$$

Clearly, this and (17) lead to a contradiction. So $V(z)=V(z,w)$ has infinite many zeros. The proof of Theorem 2 is completed. □

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