Journal of Mathematical Research & Exposition
May, 2011, Vol. 31, No. 3, pp. 503–508
DOI:10.3770/j.issn:1000-341X.2011.03.016
Http://jmre.dlut.edu.cn

Existence of Nonoscillatory Solutions for a Second-Order Nonlinear Neutral Delay Differential Equation

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Abstract A new second-order nonlinear neutral delay differential equation

$$\Big(r(t)\big(x(t)+P(t)x(t-\tau)\big)'+cr(t)\big(x(t)-x(t-\tau)\big)\Big)'+ F\big(t,x(t-\sigma_1),x(t-\sigma_2),\ldots,x(t-\sigma_n)\big)=G(t),\quad t\geq t_0,$$

where $\tau>0,\sigma_1,\sigma_2,\ldots,\sigma_n\geq 0, P,r\in C([t_0,+\infty),\mathbb{R}), F\in C([t_0,+\infty)\times\mathbb{R}^n,\mathbb{R}), G\in C([t_0,+\infty),\mathbb{R})$ and c is a constant, is studied in this paper, and some sufficient conditions for existence of nonoscillatory solutions for this equation are established and expatiated through five theorems according to the range of value of function $P(t)$. Two examples are presented to illustrate that our works are proper generalizations of the other corresponding results. Furthermore, our results omit the restriction of $Q_1(t)$ dominating $Q_2(t)$ (See condition C in the text).

Keywords nonoscillatory solution; second-order neutral delay differential equation; contraction mapping.

Document code A
MR(2010) Subject Classification 34K15; 34C10
Chinese Library Classification O177.91

1. Introduction and preliminaries

It is well known that the oscillatory and asymptotic behaviour of solutions for second-order neutral and nonneutral delay differential equations has been studied widely by many authors [1, 3–5, 9–11, 13]. But the nonoscillatory of solutions for second-order neutral delay differential equations received much less attention, which is mainly due to the technical difficulties arising in its analysis [3, 8, 10–12, 14]. For further knowledge of existence and uniqueness of solutions of neutral delay differential equations, one can refer to [2, 6, 7].

In 1998, Kulenovic and Hadziomerspahic [8] studied the following second-order linear neutral delay differential equation with positive and negative coefficients:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big(x(t)+px(t-\tau)\big)+Q_1(t)x(t-\sigma_1)-Q_2(t)x(t-\sigma_2)=0,\quad t\geq t_0, \tag{E$_{1.1}$}$$

where $\tau>0,\sigma_1,\sigma_2\geq 0, p\in\mathbb{R}=(-\infty,+\infty),\ Q_1,Q_2\in C([t_0,+\infty),\mathbb{R}^+)$ and $\mathbb{R}^+=[0,+\infty)$, and gave some sufficient conditions which guarantee the existence of nonoscillatory solutions for (E$_{1.1}$). In 2004, Cheng and Annie [3] established the existence of solutions for (E$_{1.1}$) under

Received May 4, 2009; Accepted September 15, 2009
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weaker conditions and improved the works of Kulenovic and Hadziomerspahic [8]. In 2005, Yu and Wang [14] extended the results of Kulenovic and Hadziomerspahic [8] and investigated the existence of nonoscillatory solutions for the following second-order nonlinear neutral delay differential equation with positive and negative coefficients:

$$\Big(r(t)\big(x(t)+P(t)x(t-\tau)\big)'\Big)'+Q_1(t)f\big(x(t-\sigma_1)\big)-Q_2(t)g\big(x(t-\sigma_2)\big)=0,\quad t\ge t_0,\qquad (\mathrm{E}_{1.2})$$

where $\tau>0,\sigma_1,\sigma_2\ge 0,P,r\in C([t_0,+\infty),\mathbb{R}),\ Q_1,Q_2\in C([t_0,+\infty),\mathbb{R}^+),\ f,g\in C(\mathbb{R},\mathbb{R})$. But the results in [8] and [14] need the condition

$$aQ_1(t)-Q_2(t) \text{ is eventually nonnegative for each } a>0. \qquad (\mathrm{C})$$

The purpose of this paper is to investigate the following second-order nonlinear neutral delay differential equation

$$\begin{aligned}\Big(r(t)\big(x(t)+P(t)x(t-\tau)\big)'+cr(t)\big(x(t)-x(t-\tau)\big)\Big)'+\\ F\big(t,x(t-\sigma_1),x(t-\sigma_2),\dots,x(t-\sigma_n)\big)=G(t),\quad t\ge t_0,\end{aligned}\qquad (\mathrm{E}_{1.3})$$

where $\tau>0,\sigma_1,\sigma_2,\dots,\sigma_n\ge 0,P,r\in C([t_0,\infty),\mathbb{R}),F\in C([t_0,+\infty)\times\mathbb{R}^n,\mathbb{R}),G\in C([t_0,+\infty),\mathbb{R})$ and c is a constant. The existence of nonoscillatory solutions for $(\mathrm{E}_{1.3})$ is established under some new conditions without the condition (C) in Section 2. Clearly, $(\mathrm{E}_{1.1})$ and $(\mathrm{E}_{1.2})$ are special cases of $(\mathrm{E}_{1.3})$. In Section 3, two examples are provided to illustrate that our results extend, improve and generalize properly the corresponding results in [3, 8, 14]. A solution of $(\mathrm{E}_{1.3})$ is said to be oscillatory if it has arbitrarily large zeros, and otherwise it is called nonoscillatory. Let $u\in C([t_0-\rho,+\infty),\mathbb{R})$, where $\rho=\max\{\tau,\sigma_i: i=1,2,\dots,n\}$, be a given function and y_0 a given constant. From the method of steps, it follows that $(\mathrm{E}_{1.3})$ has a unique solution $x\in C([t_0-\rho,+\infty),\mathbb{R})$ if $x(t)+P(t)x(t-\tau)$, $r(t)\big(x(t)+P(t)x(t-\tau)\big)'$ and $r(t)\big(x(t)+P(t)x(t-\tau)\big)'+cr(t)\big(x(t)-x(t-\tau)\big)$ are continuously differentiable for $t\ge t_0$, $x(t)$ satisfies the $(\mathrm{E}_{1.3})$ and

$$x(s)=u(s)\quad\text{for}\quad s\in[t_0-\rho,t_0],$$
$$\big(x(t)+P(t)x(t-\tau)\big)'_{t=t_0}=y_0.$$

Throughout this paper, we assume that X denotes the set of all continuous and bounded functions on $[t_0,+\infty)$ with the sup norm and $S=\big\{x\in X: M\le x(t)\le N, t\ge t_0\big\}$ for $N>M>0$. Obviously, S is a nonempty closed convex subset of the Banach space X. For $P\in C([t_0,+\infty),\mathbb{R})$, put $\overline{P}=\limsup_{t\to+\infty}P(t)$ and $\underline{P}=\liminf_{t\to+\infty}P(t)$.

2. Existence of nonoscillatory solutions

In this section, a few sufficient conditions of the existence of nonoscillatory solutions for $(\mathrm{E}_{1.3})$ will be given.

Theorem 2.1 *Assume that there exist constants P_0, c with $|c|<\frac{1}{\tau}$, M and N with $N>\frac{1-|c|\tau}{1-2P_0-|c|\tau}M>0$ and functions $h,q,r\in C([t_0,+\infty),\mathbb{R}^+)$ and $P\in C([t_0,+\infty),\mathbb{R})$ such that*

$$|P(t)|\le P_0<\frac{1-|c|\tau}{2},\ \textit{eventually},\qquad (2.1)$$

$$\big|F(t,u_1,u_2,\ldots,u_n)-F(t,v_1,v_2,\ldots,v_n)\big|\le h(t)\max\big\{|u_i-v_i|:1\le i\le n\big\},$$
$$t\in[t_0,\infty),\ u_i,v_i\in[M,N],\ 1\le i\le n, \tag{2.2}$$

$$\big|F(t,u_1,u_2,\cdots,u_n)\big|\le q(t),\ t\in[t_0,\infty),\ u_i\in[M,N],\ 1\le i\le n, \tag{2.3}$$

$$\begin{gathered} r(t)>0, R(t)=\int_{t_0}^{t}\frac{1}{r(s)}\mathrm{d}s,\ \forall t\in[t_0,+\infty)\ \textit{and}\\ \int_{t_0}^{+\infty}R(t)\max\big\{|G(t)|,h(t),q(t)\big\}\mathrm{d}t<+\infty. \end{gathered}\tag{2.4}$$

Then ($E_{1.3}$) *has a nonoscillatory solution.*

Proof By (2.1) and (2.4), a sufficiently large $l>t_0+\rho$ can be chosen such that

$$|P(t)|\le P_0<\frac{1-|c|\tau}{2},\quad \forall t\ge l, \tag{2.5}$$

$$\int_{l}^{+\infty}R(s)\Big(q(s)+\big|G(s)\big|\Big)\mathrm{d}s\le\frac{(1-2P_0-|c|\tau)N-(1-|c|\tau)M}{2} \tag{2.6}$$

and

$$\int_{l}^{+\infty}R(s)h(s)\mathrm{d}s<1-P_0-|c|\tau. \tag{2.7}$$

Define a mapping $T:S\to X$ by

$$(Tx)(t)=\begin{cases}\dfrac{(1+c\tau)(M+N)}{2}-P(t)x(t-\tau)-c\displaystyle\int_{t-\tau}^{t}x(s)\mathrm{d}s+\\ \quad R(t)\displaystyle\int_{t}^{+\infty}\big[F\big(s,x(s-\sigma_1),x(s-\sigma_2),\ldots,x(s-\sigma_n)\big)-G(s)\big]\mathrm{d}s+\\ \quad\displaystyle\int_{l}^{t}R(s)\big[F\big(s,x(s-\sigma_1),x(s-\sigma_2),\ldots,x(s-\sigma_n)\big)-G(s)\big]\mathrm{d}s, & t\ge l\\ (Tx)(l), & t_0\le t<l\end{cases}$$

for all $x\in S$. Clearly, Tx is continuous for each $x\in S$. Two cases have to be considered with respect to c.

Case 1 $c\ge 0$. For every $x\in S$ and $t\ge l$, it follows from (2.3), (2.5) and (2.6) that

$$\begin{aligned}(Tx)(t)\ge&\frac{(1+c\tau)(M+N)}{2}-P_0N-c\int_{t-\tau}^{t}N\mathrm{d}s-\\ &R(s)\int_{t}^{+\infty}\Big(\big|F\big(s,x(s-\sigma_1),x(s-\sigma_2),\ldots,x(s-\sigma_n)\big)\big|+\big|G(s)\big|\Big)\mathrm{d}s=\\ &\int_{l}^{t}R(s)\Big(\big|F\big(s,x(s-\sigma_1),x(s-\sigma_2),\ldots,x(s-\sigma_n)\big)\big|+\big|G(s)\big|\Big)\mathrm{d}s\\ \ge&\frac{(1+c\tau)(M+N)}{2}-P_0N-c\tau N-\\ &\int_{l}^{+\infty}R(s)\Big(\big|F\big(s,x(s-\sigma_1),x(s-\sigma_2),\ldots,x(s-\sigma_n)\big)\big|+\big|G(s)\big|\Big)\mathrm{d}s\\ \ge&\frac{(1+c\tau)(M+N)}{2}-P_0N-c\tau N-\int_{l}^{+\infty}R(s)\Big(q(s)+\big|G(s)\big|\Big)\mathrm{d}s\end{aligned}$$

$$\geq M$$

and

$$(Tx)(t) \leq \frac{(1+c\tau)(M+N)}{2} + P_0N - c\tau M + \int_l^{+\infty} R(s)\Big(q(s) + \big|G(s)\big|\Big)\mathrm{d}s \leq N.$$

Case 2 $c < 0$. By virtue of (2.3), (2.5) and (2.6), it is derived that

$$(Tx)(t) \geq \frac{(1+c\tau)(M+N)}{2} - P_0N - c\tau M - \int_l^{+\infty} R(s)\Big(q(s) + \big|G(s)\big|\Big)\mathrm{d}s \geq M$$

and

$$(Tx)(t) \leq \frac{(1+c\tau)(M+N)}{2} + P_0N - c\tau N + \int_l^{+\infty} R(s)\Big(q(s) + \big|G(s)\big|\Big)\mathrm{d}s \leq N$$

for every $x \in S$ and $t \geq l$.

That is, $TS \subseteq S$, no matter c is positive or negative. It is claimed that T is a contraction mapping on S. In fact, (2.2), (2.5) and (2.7) guarantee that for any $x, y \in S$ and $t \geq l$

$$\begin{aligned}
&\big|(Tx)(t) - (Ty)(t)\big| \\
&\leq |P(t)|\big|x(t-\tau) - y(t-\tau)\big| + |c|\int_{t-\tau}^{t} \big|x(t-\tau) - y(t-\tau)\big|\mathrm{d}s + \\
&\quad R(t)\int_t^{+\infty} \Big|F\big(s, x(s-\sigma_1), x(s-\sigma_2), \ldots, x(s-\sigma_n)\big) - \\
&\quad F\big(s, y(s-\sigma_1), y(s-\sigma_2), \ldots, y(s-\sigma_n)\big)\Big|\mathrm{d}s + \\
&\quad \int_l^t R(s)\Big|F\big(s, x(s-\sigma_1), x(s-\sigma_2), \ldots, x(s-\sigma_n)\big) - \\
&\quad F\big(s, y(s-\sigma_1), y(s-\sigma_2), \ldots, y(s-\sigma_n)\big)\Big|\mathrm{d}s \\
&\leq P_0\|x-y\| + |c|\int_{t-\tau}^{t} \|x-y\|\mathrm{d}s + \\
&\quad \int_l^{+\infty} R(s)\Big|F\big(s, x(s-\sigma_1), x(s-\sigma_2), \ldots, x(s-\sigma_n)\big) - \\
&\quad F\big(s, y(s-\sigma_1), y(s-\sigma_2), \ldots, y(s-\sigma_n)\big)\Big|\mathrm{d}s \\
&\leq P_0\|x-y\| + |c|\tau\|x-y\| + \\
&\quad \int_l^{+\infty} R(s)h(s)\max\big\{|x(s-\sigma_i) - y(s-\sigma_i)| : 1 \leq i \leq n\big\}\mathrm{d}s \\
&\leq P_0\|x-y\| + |c|\tau\|x-y\| + \|x-y\|\int_l^{+\infty} R(s)h(s)\mathrm{d}s \\
&= k\|x-y\|,
\end{aligned}$$

where $k = P_0 + |c|\tau + \int_l^{+\infty} R(s)h(s)\mathrm{d}s < 1$. This implies that

$$\|Tx - Ty\| \leq k\|x-y\|, \quad \forall x, y \in S,$$

that is, T is a contraction mapping on S. Consequently T has a unique fixed point $x \in S$, which is a nonoscillatory solution of $(\mathrm{E}_{1.3})$. This completes the proof. □

Similarly to the proof of Theorem 2.1, we have other 4 theorems:

Theorem 2.2 *Assume that there exist constants c with $|c| < \frac{1}{\tau}$, M and N with $N > M > 0$ and functions $h, q, r \in C([t_0, +\infty), \mathbb{R}^+)$ and $P \in C([t_0, +\infty), \mathbb{R})$ satisfying (2.2)–(2.4) and*

$$P(t) \geq 0, \quad \text{eventually, and } \ 0 \leq \underline{P} \leq \overline{P} < 1 - |c|\tau. \tag{2.8}$$

Then ($E_{1.3}$) has a nonoscillatory solution in S.

Theorem 2.3 *Assume that there exist constants c with $|c| < \frac{1}{\tau}$, M and N with $N > M > 0$ and functions $h, q, r \in C([t_0, +\infty), \mathbb{R}^+)$ and $P \in C([t_0, +\infty), \mathbb{R})$ satisfying (2.2)–(2.4) and*

$$P(t) \leq 0, \quad \text{eventually, and } \ -1 + |c|\tau < \underline{P} \leq \overline{P} \leq 0. \tag{2.9}$$

Then ($E_{1.3}$) has a nonoscillatory solution in S.

Theorem 2.4 *Assume that there exist constants c with $|c| < \frac{\underline{P}-1}{\tau}$, M and N with $N > M > 0$ and functions $h, q, r \in C([t_0, +\infty), \mathbb{R}^+)$ and $P \in C([t_0, +\infty), \mathbb{R})$ satisfying (2.2)–(2.4) and*

$$P(t) > 1, \quad \text{eventually, } \ 1 < \underline{P} \ \text{ and } \ \overline{P} < \frac{\underline{P}^2}{1 + |c|\tau} < +\infty. \tag{2.10}$$

Then ($E_{1.3}$) has a nonoscillatory solution in S.

Theorem 2.5 *Assume that there exist constants c with $|c| < -\frac{\overline{P}^2 + \overline{P}}{\underline{P}\tau}$, M and N with $N > M > 0$ and functions $h, q, r \in C([t_0, +\infty), \mathbb{R}^+)$ and $P \in C([t_0, +\infty), \mathbb{R})$ satisfying (2.2)–(2.4) and*

$$P(t) < -1, \quad \text{eventually, } \ -\infty < \underline{P} \ \text{ and } \ \overline{P} < -1. \tag{2.11}$$

Then ($E_{1.3}$) has a nonoscillatory solution in S.

3. Examples

In this section, two examples are presented to illustrate the advantage of the above results.

Example 3.1 Consider the following second-order nonlinear neutral delay differential equation:

$$\begin{gathered}\Big(\frac{1}{t^2}\big(x(t) + \frac{(t\cos t)x(t-\tau)}{2+3t^2}\big)' + \frac{1}{2\tau t^2}\big(x(t) - x(t-\tau)\big)\Big)' + \\ \frac{x(t-\sigma_1)x(t-\sigma_2)x(t-\sigma_3)}{t^5} = \frac{\ln t}{t^6}, \quad t \geq t_0 = 1,\end{gathered} \tag{E$_{3.1}$}$$

where $\tau > 0, \sigma_1, \sigma_2, \sigma_3 \geq 0$. Choose positive constants M and N with $N > \frac{1}{\sqrt{6}-2}M$. Put

$$\begin{aligned}&r(t) = \frac{1}{t^2}, \ R(t) = \int_1^t \frac{1}{r(s)}\mathrm{d}s = \frac{t^3-1}{3}, \ P(t) = \frac{t\cos t}{2+3t^2}, \\ &P_0 = \frac{1}{2\sqrt{6}}, \ F(t,u,v,w) = \frac{uvw}{t^5}, \ G(t) = \frac{\ln t}{t^6}, \ h(t) = \frac{N^2}{t^5} \ \text{ and} \\ &q(t) = \frac{N^3}{t^5}, \quad \forall t \geq 1, u, v, w \in \mathbb{R}.\end{aligned}$$

It is easy to verify that the conditions of Theorem 2.1 are satisfied. Therefore Theorem 2.1 ensures that ($E_{3.1}$) has a nonoscillatory solution. However, the results in [3, 8, 14] are not applicable for ($E_{3.1}$).

Example 3.2 Consider the following second-order nonlinear neutral delay differential equation:

$$\Big((1+t^2)(1+4t^2)\big(x(t)+\frac{t^2+\cos^2 t}{4t^2-\sin^2 t}x(t-\tau)\big)'-\frac{1}{2\tau}(1+t^2)(1+4t^2)\big(x(t)-x(t-\tau)\big)\Big)'+$$
$$\frac{(\sin^2 t)x(t-\sigma_1)}{t^2+1}-\frac{(\cos^2 t)x^2(t-\sigma_2)}{t^2+1}=0,\quad t\ge t_0=2, \tag{E$_{3.2}$}$$

where $\tau>0,\sigma_1,\sigma_2\ge 0$. Select positive constants M and N with $N>M$. Put

$$r(t)=(1+t^2)(1+4t^2),$$
$$R(t)=\int_2^t\frac{1}{r(s)}\mathrm{d}s=\frac{1}{3}\big(2\arctan 2t-\arctan t\big)-\frac{1}{3}\big(2\arctan 4-\arctan 2\big),$$
$$P(t)=\frac{t^2+\cos^2 t}{4t^2-\sin^2 t},\ F(t,u,v)=\frac{(\sin^2 t)u}{t^2+1}-\frac{(\cos^2 t)v^2}{t^2+1},\ G(t)=0,$$
$$h(t)=\frac{\sin^2 t}{t^2+1}+\frac{2N\cos^2 t}{t^2+1}\text{ and }q(t)=\frac{N\sin^2 t}{t^2+1}+\frac{N^2\cos^2 t}{t^2+1},\quad\forall t\ge 2,\ u,v\in\mathbb{R}.$$

Clearly, the assumptions of Theorem 2.2 are fulfilled. It follows from Theorem 2.2 that ($E_{3.2}$) has a nonoscillatory solution. Set

$$Q_1(t)=\frac{\sin^2 t}{t^2+1}\text{ and }Q_2(t)=\frac{\cos^2 t}{t^2+1},\quad\forall t\ge 2.$$

Obviously, the condition (C) does not hold. Hence the results in [3, 8, 14] are inapplicable for ($E_{3.2}$).

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