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Convergence of Wavelet Expansions in the Orlicz Space

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Abstract In this paper we study the convergence in norm and pointwise convergence of wavelet expansion in the Orlicz spaces, and prove that, under certain conditions on the wavelet, the wavelet expansion converges in the Orlicz-norm and also converges almost everywhere.

Keywords pointwise convergence; wavelet; Orlicz spaces.

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1. Introduction

An orthonormal wavelet on $\mathbb{R}$ is a function $\psi \in L^2(\mathbb{R})$ such that $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$, where

$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k), \quad j,k \in \mathbb{Z}.$$

All the wavelets we will use in this note are assumed to arise from a multiresolution analysis (MRA).

A multiresolution analysis (MRA) consists of a sequence of closed subspaces V_j, $j \in \mathbb{Z}$ of $L^2(\mathbb{R})$ satisfying

1) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
2) $f \in V_j$ if and only if $f(2(\cdot)) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
3) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
4) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$;
5) There exists a function $\varphi \in V_0$, such that $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0.

The function φ in 5) is called a scaling function of the given MRA.

Let W_j be the orthogonal complement of V_j in V_{j+1}, that is, $V_{j+1} = V_j \bigoplus W_j$, $j \in \mathbb{Z}$.

The scaling function φ satisfies the equation

$$\hat{\varphi}(2\xi) = \hat{\varphi}(\xi)\sum_{k\in\mathbb{Z}} \alpha_k e^{ik\xi} = \hat{\varphi}(\xi)m(\xi),$$

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where $m(\xi) = \sum_{k\in\mathbb{Z}} \alpha_k e^{ik\xi}$ is called the low pass filter associated with φ.

We construct a wavelet ψ from the MRA by using equation

$$\hat{\psi}(2\xi) = e^{i\xi}\overline{m(\xi+\pi)}\hat{\varphi}(\xi).$$

For an orthonormal wavelet ψ and a function $f \in L^2(\mathbb{R})$, we have

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k}$$

with convergence in the $L^2(\mathbb{R})$ norm, and

$$\|f\|_{L^2(\mathbb{R})} = \Big(\sum_{j,k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2\Big)^{\frac{1}{2}},$$

where $\langle f, \psi_{j,k}\rangle = \int_{\mathbb{R}} f(x)\overline{\psi_{j,k}(x)}\mathrm{d}x$.

In [1, 3–5], the convergence in norm and pointwise convergence for the wavelet expansions in $L^p(\mathbb{R})$, $1 < p < \infty$, were treated.

The purpose of this note is to consider the Orlicz spaces and study the convergence in norm and pointwise convergence of wavelet expansions.

The standard theory of Orlicz spaces can be found in Kokilashvili and Krbec [2]. Let ρ be a nondecreasing right continuous function on $[0, \infty)$ with $\rho(0+) = 0$, and define Φ to be a Young's function $\Phi(t) = \int_0^t \rho(s)\mathrm{d}s$. Note that a Young's function is convex on the interval where it is finite. Associated to ρ, we have the function ϱ defined by $\varrho(t) = \sup\{s : \ \rho(s) \le t\}$, which has the same aforementioned properties of ρ. The Young's function Ψ defined by $\Psi = \int_0^t \varrho(s)\mathrm{d}s$ is called the complementary function of Φ. In what follows we shall always assume that Φ, together with its complementary function Ψ, satisfy the Δ_2, i.e., there exists a constant C such that for all $t > 0$

$$\Phi(2t) \le C\Phi(t), \quad \Psi(2t) \le C\Psi(t). \tag{1}$$

We need the following elementary properties of the Young's function Φ and its complementary Ψ (see [2]):

$$\Phi(t) \sim t\rho(t), \quad \Psi(t) \sim t\varrho(t). \tag{2}$$

We write $f \sim g$, if there is a constant $C > 0$ such that both $f \le Cg$ and $g \le Cf$ hold.

Define

$$q_\Phi = \lim_{\lambda\to 0+} \frac{\log h(\lambda)}{\log\lambda}, \quad p_\Phi = \lim_{\lambda\to+\infty} \frac{\log h(\lambda)}{\log\lambda},$$

where $h(\lambda) = \sup_{t>0}[\Phi(\lambda t)/\Phi(t)]$.

We have

$$1 < q_\Phi \le p_\Phi < \infty, \tag{3}$$

$$\Phi(ut) \le K_1 u^p \Phi(t), \quad \forall t \ge 0,\ 0 \le u \le 1,\ 1 < p < q_\Phi, \tag{4}$$

$$\Phi(ut) \le K_2 u^q \Phi(t), \quad \forall t \ge 0,\ u > 1,\ p_\Phi < q < \infty,$$

where K_1, K_2 are constants, independent of u, t.

The Luxemberg norm $\|\cdot\|_\Phi$ on the Orlicz spaces

$$L^\Phi(\mathbb{R}) = \{f : \int_{\mathbb{R}} \Phi(|f(x)|)\mathrm{d}x < \infty\}$$

is given by

$$\|f\|_\Phi = \inf\Big\{\lambda > 0 : \int_{\mathbb{R}} \Phi(|f(x)|/\lambda)\mathrm{d}x \le 1\Big\}.$$

We have

$$\|f\|_\Phi \le \sup\Big\{|\int_{\mathbb{R}} f(x)g(x)\mathrm{d}x| : \|g\|_\Psi \le 1\Big\} \le 2\|f\|_\Phi, \tag{5}$$

and the Hölder inequality

$$\Big|\int_{\mathbb{R}} f(x)g(x)\mathrm{d}x\Big| \le C\|f\|_\Phi\|g\|_\Psi, \tag{6}$$

Examples (a) If $\rho(s) = ps^{p-1}$, where $1 \le p < \infty$, then $\Phi(t) = t^p$. The Orlicz space L^Φ in this case is the Lebesgue space L^p.

(b) If $\rho(s) = 0$, $(0 \le s \le 1)$, and $\rho(s) = 1 + \log s$ $(1 < s < \infty)$, then $\Phi(t) = t\log^+ t$ $(0 \le t < \infty)$. The Orlicz space L^Φ in this case is the Zygmund space $L\log L$.

For the sake of simplicity, we denote by $|E|$ the Lebesgue measure of set E, and by χ_E the characteristic function of E. The letter C always denotes a positive constant which may change from one step to the next.

2. Convergence in norm of wavelet expansions convergence

Given a function g defined on $\mathbb{R}$, we say that a bounded function W is a radial decreasing L^1-majorant of g if $|g(x)| \le W(|x|)$ and W satisfies the following conditions:

- $W \in L^1([0,\infty))$;
- W is decreasing.

Suppose that we have a wavelet ψ that arises from an MRA with scaling function φ. Associated with the increasing sequence of subspaces $\{V_j\}_{j\in\mathbb{Z}}$, we have the orthogonal projections of $L^2(\mathbb{R})$ onto V_j given by

$$P_j f = \sum_{k\in\mathbb{Z}} \langle f, \varphi_{j,k}\rangle \varphi_{j,k} \quad \text{for} \quad f \in L^2(\mathbb{R}), \tag{7}$$

the projections from $L^2(\mathbb{R})$ onto W_j given by

$$Q_j f = \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k} \quad \text{for} \quad f \in L^2(\mathbb{R}), \tag{8}$$

and the partial sum of the wavelet expansion of f:

$$S^\sigma_{j,k} f(x) = P_j f(x) + \sum_{m=1}^{k} \langle f, \psi_{j,\sigma(m)}\rangle \psi_{j,\sigma(m)}, \tag{9}$$

where σ is any permutation of $\mathbb{Z}$.

Hernández and Weiss proved that the wavelet expansion converges in the $L^p(\mathbb{R})$-norm, $1 < p < \infty$ (see [1, pp. 221–224]).

For the Orlicz spaces we have the following result.

Theorem 1 *Suppose that the wavelet ψ arises from an MRA with scaling function φ.*

(a) Suppose that φ has a radial decrease L^1-majorant. Then there exists $C > 0$, independent of j, such that for all $f \in L^\Phi(\mathbb{R})$,

$$\|P_j f\|_\Phi \le C\|f\|_\Phi, \tag{10}$$

$$\lim_{j\to\infty} \|P_j f - f\|_\Phi = 0. \tag{11}$$

(b) Suppose that ψ has a radial decrease L^1-majorant. Then there exists $C > 0$, independent of j, such that for all $f \in L^\Phi(\mathbb{R})$,

$$\|Q_j f\|_\Phi \le C\|f\|_\Phi.$$

(c) Suppose that φ and ψ have radial decrease L^1-majorant respectively. Then there exists $C > 0$, independent of j, k and σ, such that for all $f \in L^\Phi(\mathbb{R})$,

$$\|S^\sigma_{j,k} f\|_\Phi \le C\|f\|_\Phi,$$

$$\lim_{j\to\infty} \|S^\sigma_{j,k} f - f\|_\Phi = 0 \quad \textit{for all} \quad k = 1, 2, \ldots.$$

Proof Firstly we prove (a). By (3), choose r and s such that $1 < r < q_\Phi \le p_\Phi < s < \infty$.

Given $f \in L^\Phi(\mathbb{R})$ and $\alpha > 0$, we split $f = f_1 + f_2$, where $f_1(x) = f(x)$ if $|f(x)| \ge \alpha$, and $f_1(x) = 0$ otherwise. Then we have $f_1 \in L^r$ and $f_2 \in L^s$.

By the L^p result [1, p.221], the operators P_j are bounded on $L^p(\mathbb{R})$, $1 \le p \le \infty$. It follows that there is a constant C such that

$$|\{x \in \mathbb{R} :\ |P_j f_1(x)| > \alpha\}| \le C\alpha^{-r} \int_{\mathbb{R}} |f_1(x)|^r \mathrm{d}x,$$

$$|\{x \in \mathbb{R} :\ |P_j f_2(x)| > \alpha\}| \le C\alpha^{-s} \int_{\mathbb{R}} |f_2(x)|^s \mathrm{d}x.$$

Since $|P_j f(x)| \le C(|P_j f_1(x)| + |P_j f_2(x)|)$, we have

$$\begin{aligned}
\int_{\mathbb{R}} \Phi(|P_j f(x)|)\mathrm{d}x \le & C \int_0^\infty |\{x \in \mathbb{R} :\ |P_j f(x)| > \alpha\}|\rho(\alpha)\mathrm{d}\alpha \\
\le & C \int_0^\infty |\{x \in \mathbb{R} :\ |P_j f_1(x)| > \alpha/(2C)\}|\rho(\alpha)\mathrm{d}\alpha + \\
& C \int_0^\infty |\{x \in \mathbb{R} :\ |P_j f_2(x)| > \alpha/(2C)\}|\rho(\alpha)\mathrm{d}\alpha \\
\le & C \int_0^\infty \big(\alpha^{-r} \int_{\mathbb{R}} |f_1|^r \mathrm{d}x\big)\rho(\alpha)\mathrm{d}\alpha + \\
& C \int_0^\infty \big(\alpha^{-s} \int_{\mathbb{R}} |f_2|^s \mathrm{d}x\big)\rho(\alpha)\mathrm{d}\alpha \\
\le & C \int_{\mathbb{R}} |f(x)|^r \big(\int_0^{|f(x)|} \alpha^{-r}\rho(\alpha)\mathrm{d}\alpha\big)\mathrm{d}x + \\
& C \int_{\mathbb{R}} |f(x)|^s \big(\int_{|f(x)|}^\infty \alpha^{-s}\rho(\alpha)\mathrm{d}\alpha\big)\mathrm{d}x.
\end{aligned} \tag{12}$$

For the first integral, if p, q are such that $r < p < q_\Phi$, $p_\Phi < q < s$, we have, from (2) and (4)

$$\begin{aligned}\int_0^{|f(x)|} \alpha^{-r}\rho(\alpha)\mathrm{d}\alpha &\le C' \int_0^{|f(x)|} \alpha^{-1-r}\Phi(\alpha)\mathrm{d}\alpha \\ &= C' \int_0^{|f(x)|} \alpha^{-1-r}\Phi\big(\frac{\alpha}{|f(x)|}|f(x)|\big)\mathrm{d}\alpha \\ &\le C'K_1 \int_0^{|f(x)|} \alpha^{-1-r}\big(\frac{\alpha}{|f(x)|}\big)^p \Phi(|f(x)|)\mathrm{d}\alpha \\ &= \frac{C'K_1}{p-r}|f(x)|^{-r}\Phi(|f(x)|), \end{aligned} \tag{13}$$

where the constant $C' > 1$ is such that $t\rho(t) < C'\Phi(t)$, $t > 0$. Similarly for the second integral, we have

$$\int_{|f(x)|}^{\infty} \alpha^{-s}\rho(\alpha)\mathrm{d}\alpha \le \frac{CK_2}{s-q}|f(x)|^{-s}\Phi(|f(x)|). \tag{14}$$

Thus combining it with (13) and (12) gives

$$\int_{\mathbb{R}} \Phi(|P_j f(x)|)\mathrm{d}x \le C \int_{\mathbb{R}} \Phi(|f(x)|)\mathrm{d}x. \tag{15}$$

Let $\lambda = \|f\|_\Phi$. Then $\int_{\mathbb{R}} \Phi(|f(x)|/\lambda)\mathrm{d}x \le 1$. Hence by (15) and the convexity of Φ, we have

$$\begin{aligned}\int_{\mathbb{R}} \Phi(\frac{|P_j f(x)|}{C\lambda})\mathrm{d}x = \int_{\mathbb{R}} \Phi(|P_j(\frac{f(x)}{C\lambda})|)\mathrm{d}x &\le C \int_{\mathbb{R}} \Phi(\frac{|f(x)|}{C\lambda})\mathrm{d}x \\ &\le \int_{\mathbb{R}} \Phi(\frac{|f(x)|}{\lambda})\mathrm{d}x \le 1,\end{aligned}$$

where the constant $C > 1$ is the same as that in (15). It follows that

$$\|P_j f\|_\Phi \le C\lambda = C\|f\|_\Phi.$$

Suppose that W is a radial decrease L^1-majorant of φ. By the same argument as in [1, pp. 219–224], we have

$$|P_j f(x) - f(x)| \le C \int_{\mathbb{R}} 2^j W(2^{j-1}|t|)|f(x-t) - f(x)|\mathrm{d}t. \tag{16}$$

Suppose $g \in L^\Psi(\mathbb{R})$. Using Minkowshi's inequality and the Hölder inequality (6) yields

$$\begin{aligned}&\int_{\mathbb{R}} |P_j f(x) - f(x)||g(x)| \\ &\le C \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} 2^j W(2^{j-1}|t|)|f(x-t) - f(x)|\mathrm{d}t\Big)|g(x)|\mathrm{d}x \\ &\le C \int_{\mathbb{R}} W(\frac{1}{2}|t|)\Big(\int_{\mathbb{R}} |f(x - 2^{-j}t) - f(x)||g(x)|\mathrm{d}x\Big)\mathrm{d}t \\ &\le C \int_{\mathbb{R}} W(\frac{1}{2}|t|)\|f(\cdot - 2^{-j}t) - f\|_\Phi \|g\|_\Psi \mathrm{d}t \\ &\le C \int_{\mathbb{R}} W(\frac{1}{2}|t|)\|f(\cdot - 2^{-j}t) - f\|_\Phi \mathrm{d}t.\end{aligned}$$

Taking the supremum over all $g \in L^\Psi(\mathbb{R})$ such that $\|g\|_\Psi \le 1$, we deduce from (5)

$$\|P_j - f\|_\Phi \le C \int_{\mathbb{R}} W(\frac{1}{2}|t|)\|f(\cdot - 2^{-j}t) - f\|_\Phi \mathrm{d}t.$$

Since $\|f(\cdot - 2^{-j}t) - f\|_\Phi \to 0$ as $j \to \infty$, by the Lebesgue dominated convergence theorem, we get

$$\lim_{j\to\infty} \|P_j f - f\|_\Phi = 0.$$

The proofs for (b) and (c) are similar. □

3. Pointwise convergence of wavelet expansions in the Orlicz spaces

We now consider almost everywhere convergence of above operators P_j, $S^\sigma_{j,k}$.

Theorem 2 *(a) Suppose that φ is the scaling function of an MRA and that φ has a radial decreasing L^1-majorant. If $f \in L^\Phi(\mathbb{R})$, then*

$$\lim_{j\to\infty} P_j f(x) = f(x)$$

for every x in the Lebesgue set of f. In particular, $P_j f(x)$ converges to $f(x)$ almost everywhere as $j \to \infty$.

(b) Suppose that the wavelet ψ and the scaling function φ of an MRA each has a radial decrease L^1-majorant. If $f \in L^\Phi(\mathbb{R})$, then, for $k = 1, 2, \ldots$

$$\lim_{j\to\infty} S^\sigma_{j,k} f(x) = f(x)$$

for every x in the Lebesgue set of f. In particular, the partial sums $S^\sigma_{j,k} f(x)$ converge to $f(x)$ almost everywhere as $j \to \infty$ and $k = 1, 2, \ldots$.

Proof We first prove (a). If x is a point in the Lebesgue set of f, for every $\varepsilon > 0$, there exists an $\eta > 0$ such that

$$\frac{1}{r}\int_{|t|\le r} f(x-t) - f(x)|\mathrm{d}t \le \varepsilon \quad \text{when} \quad 0 < r \le \eta.$$

Suppose that W is a radial decrease L^1-majorant of φ. By (16), we have

$$\begin{aligned} C^{-1}|P_j f(x) - f(x)| \le & \int_{|t|<\eta} 2^j W(2^{j-1}|t|)|f(x-t) - f(x)|\mathrm{d}t + \\ & \int_{|t|\ge\eta} 2^j W(2^{j-1}|t|)|f(x-t) - f(x)|\mathrm{d}t \\ =& \mathrm{I} + \mathrm{II}. \end{aligned}$$

Arguing as in [1, p.226], we have

$$I \le C\varepsilon, \tag{17}$$

where the constant C depends only on W and does not depend on j.

In order to estimate II, we let χ_η be the characteristic function of the set $\{t \in \mathbb{R} : |t| \ge \eta\}$. If Ψ denotes the complementary function of Φ, we use the Hölder's inequality to write

$$\mathrm{II} \le \|f\|_\Phi \|\chi_\eta 2^j W(2^{j-1}|\cdot|)\|_\Psi + |f(x)| \int_{\mathbb{R}} |\chi_\eta(t) 2^j W(2^{j-1}|t|)|\mathrm{d}t. \tag{18}$$

But

$$\int_{\mathbb{R}} |\chi_\eta(t) 2^j W(2^{j-1}|t|)|\mathrm{d}t = 2\int_{|s|\ge 2^{j-1}\eta} W(|s|)\mathrm{d}s,$$

which tends to zero as $j \to \infty$.

Choosing $g \in L^{\Phi}(\mathbb{R})$ and $\|g\|_{\Phi} \le 1$ gives

$$\Big|\int_{\mathbb{R}} \chi_\eta(t) 2^j W(2^{j-1}|t|) g(t) \mathrm{d}t\Big| \le \sup_{|t|\ge\eta} |2^j W(2^{j-1}|t|)| \int_{|t|\ge\eta} |g(t)|\mathrm{d}t. \tag{19}$$

If $\int_{|t|\ge\eta} |g(t)|\mathrm{d}t > 1$, using the convexity of Φ, we get

$$\begin{aligned}\Phi(1)\int_{|t|\ge\eta} |g(t)|\mathrm{d}t \le \Phi\Big(\int_{|t|\ge\eta} |g(t)|\mathrm{d}t\Big) &\le \int_{|t|\ge\eta} \Phi(|g(t)|)\mathrm{d}t \\ &\le \int_{\mathbb{R}} \Phi(|g(t)|)\mathrm{d}t \le 1,\end{aligned}$$

where the last inequality follows from $\|g\|_{\Phi} \le 1$. Hence, $\int_{|t|\ge\eta} |g(t)|\mathrm{d}t$ is bounded, that is, there exists a constant M, independent of g, such that $\int_{|t|\ge\eta} |g(t)|\mathrm{d}t \le M$. By (19) we get

$$\Big|\int_{\mathbb{R}} \chi_\eta(t) 2^j W(2^{j-1}|t|) g(t) \mathrm{d}t\Big| \le 2^j W(2^{j-1}\eta) M.$$

Taking the supremum over all $g \in L^{\Phi}(\mathbb{R})$ such that $\|g\|_{\Phi} \le 1$, we obtain

$$\|\chi_\eta 2^j W(2^{j-1}|\cdot|)\|_{\Psi} \le 2^j W(2^{j-1}\eta) M.$$

Since $rW(r) \to 0$, as $r \to \infty$, we have that $\|\chi_\eta 2^j W(2^{j-1}|\cdot|)\|_{\Psi}$ tends to zero as $j \to \infty$. Hence, choosing j large enough, we deduce from (18) that II can be made smaller than ε. This, together with inequality (17), proves the desired result.

The proof for part (b) is similar. □

Remark Our result is an extension of the classical result of the spaces L^p, and holds also for the Zygmund space $L \log L$.

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