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A Weak Convergence Theorem for Equilibrium Problems, Variational Inequalities and Fixed Point Problems in 2-Uniformly Convex Banach Spaces

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Abstract In this paper, we introduce a new iterative scheme for finding the common element of the set of solutions of an equilibrium problem, the set of solutions of variational inequalities for an α-inversely strongly monotone operator and the set of fixed points of relatively nonexpansive mappings in a real uniformly smooth and 2-uniformly convex Banach space. Some weak convergence theorems are obtained, to extend the previous work.

Keywords relatively nonexpansive mapping; α-inversely strongly monotone operator; equilibrium problem; variational inequality; weak convergence; fixed point.

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1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$, let E^* denote the dual of E and let $\langle x, f^*\rangle$ denote the value of $f \in E^*$ at $x \in E$. Let $J : E \to 2^{E^*}$ be the normalized duality mapping which is defined as follows:

$$Jx := \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\}, \quad \forall x \in E.$$

Assume that C is a nonempty closed convex subset of E and $U : C \to C$ is a self-mapping. We use $F(U) = \{x \in D(U) : Ux = x\}$ to denote the set of fixed points of U. And, we use "$\to$" and "$\rightharpoonup$" to denote strong and weak convergences either in E or in E^*, respectively.

Recall that the functional $\varphi : E \times E \to R^+$ (see [1]) is called Lyapunov functional if it is defined as follows:

$$\varphi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \text{for } \forall x, y \in E.$$

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A point $p \in C$ is said to be an asymptotic fixed point of $S : C \to C$ (see [2]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}(x_n - Sx_n) = 0$. The set of asymptotic fixed points of S will be denoted by $\widehat{F}(S)$. A mapping S from C into itself is said to be relatively nonexpansive [2–6] if $\widehat{F}(S) = F(S)$ and $\varphi(p, Sx) \leq \varphi(p, x)$, for $\forall x \in C$ and $p \in F(S)$.

Finding iterative schemes to approximate fixed points of relatively nonexpansive mappings is a hot topic during recent years since it is widely used in many practical problems. In 2005, Matsushita and Takahashi proposed the following hybrid iterative scheme [2] to approximate the fixed point of a relatively nonexpansive mapping S in a uniformly smooth and uniformly convex Banach space E:

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrarily,} \\ Jy_n = \alpha_n Jx_n + (1-\alpha_n)JSx_n, \\ H_n = \{v \in C : \varphi(v, y_n) \leq \varphi(v, x_n)\}, \\ W_n = \{v \in C : \langle x_n - v, Jx_1 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \bigcap W_n} x_1, \quad n = 1, 2, \ldots, \end{cases} \tag{1.1}$$

where Π_K is the generalized projection operator from E onto a closed convex subset K of E. They proved that under some conditions, $\{x_n\}$ generated by (1.1) converges strongly to $\Pi_{F(S)}x_1$. Much work has then been done under the frame of (1.1), see [3–6].

An operator A of C into E^* is said to be monotone if

$$\langle x - y, Ax - Ay\rangle \geq 0,$$

for $\forall x, y \in C$. For a monotone operator A, we use $A^{-1}0$ to denote the set of zero points of A. An operator A of C into E^* is said to be α-inversely strongly monotone [7] if α is a positive real number such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2,$$

for all $x, y \in C$. If A is α-inversely strongly monotone, then it is obvious that A is $\frac{1}{\alpha}$ Lipschitz continuous. An operator A is said to be a strongly monotone operator if for each $x, y \in C$, there exists $k \in (0, 1)$ such that

$$\langle x - y, Ax - Ay\rangle \geq k\|Ax - Ay\|^2.$$

A monotone operator A is said to be maximal monotone if its graph, $G(A) = \{(x, y) : x \in D(A), y \in Ax\}$, is not properly contained in the graph of any other monotone operators. The classical variational inequality problem for a monotone operator $A : C \to E^*$ is to find a point $u \in C$ such that

$$\langle y - u, Au\rangle \geq 0, \tag{1.2}$$

for all $y \in C$. The set of solutions of the variational inequality (1.2) is denoted by $\mathrm{VI}(C, A)$.

Finding solutions of variational inequalities is also a hot topic. In 2008, Iiduka and Takahashi designed the following projection iterative scheme in [7] for finding a zero point of an α-inversely strongly monotone operator A in a uniformly smooth and 2-uniformly convex Banach space E :

$$\begin{cases} x_1 \in C, \quad \text{chosen arbitrarily}, \\ x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \quad n = 1, 2, \dots . \end{cases} \tag{1.3}$$

Moreover, the additional assumption that $\|Ax\| \leq \|Ax - Ap\|$, for all $x \in C$ and $p \in \mathrm{VI}(C, A) \neq \emptyset$ is needed. It is proved that $\{x_n\}$ converges weakly to an element z in $\mathrm{VI}(C, A)$, where $z = \lim_{n\to\infty} \Pi_{\mathrm{VI}(C,A)}(x_n)$.

Let f be a bifunction of $C \times C$ into R, where R is the set of real numbers. The equilibrium problem for f is to find $x \in C$ such that

$$f(x, y) \geq 0, \quad \text{for all} \;\; y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by $\mathrm{EP}(f)$. Given a mapping $T : C \to E^*$, let $f(x, y) = \langle y - x, Tx\rangle$ for all $x, y \in C$. Then $z \in \mathrm{EP}(f)$ if and only if $\langle y - z, Tz\rangle \geq 0$, for all $y \in C$, i.e., z is a solution of the variational inequality. Many problems in physics, optimization and economics can be ultimately converted to find a solution of (1.4). To solve equilibrium problems (1.4), much work has been done in Hilbert spaces, for instance, Blum-Oettli [8] and Combettes-Hirstoaga [9], etc.

Recently, in [10], Takahashi and Zembayashi extended the work of equilibrium problem in Hilbert space to the following one in the uniformly smooth and uniformly convex Banach space:

$$\begin{cases} x_0 \in C, \quad \text{chosen arbitrarily}, \\ u_n \in C, \quad \text{such that} \;\; f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = J^{-1}(\alpha_n Ju_n + (1 - \alpha_n)JSu_n), \quad n = 0, 1, 2, \dots . \end{cases} \tag{1.5}$$

They showed that $\{x_n\}$ converges weakly to $z \in \mathrm{EP}(f) \bigcap F(S)$, where

$$z = \lim_{n\to\infty} \Pi_{F(S) \bigcap \mathrm{EP}(f)}(x_n)$$

under some conditions. Although only the result of weak convergence is obtained in [10], they provided a new idea of dealing with such problems compared with the already existing work of hybrid iterative method, such as (1.1).

Motivated and inspired by the ideas of (1.3) and (1.5), we shall present the following iterative scheme in a uniformly smooth and 2-uniformly convex Banach space E:

$$\begin{cases} x_0 \in C, \quad \text{chosen arbitrarily}, \\ u_n \in C, \quad \text{such that} \;\; f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \Pi_C J^{-1}(Ju_n - \lambda_n Au_n), \\ x_{n+1} = J^{-1}(\alpha_n Jy_n + (1 - \alpha_n)JSu_n), \quad n = 0, 1, 2, \dots , \end{cases} \tag{1.6}$$

where $A : C \to E^*$ is an α-inversely strongly monotone operator, and $S : C \to C$ is a relatively nonexpansive mapping. And, we shall show that under some conditions, $\{x_n\}$ converges weakly to $z \in F(S) \bigcap \mathrm{VI}(C, A) \bigcap \mathrm{EP}(f)$, where $z = \lim_{n\to\infty} \Pi_{F(S) \bigcap \mathrm{VI}(C,A) \bigcap \mathrm{EP}(f)}(x_n)$, which extends, complements or improves some work done in [7, 10] and some others.

2. Preliminaries

Let E be a Banach space. The modulus of smoothness of E is the function $\rho_E : [0, +\infty) \to [0, +\infty)$ defined by

$$\rho_E(\tau) := \sup\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| \leq \tau\}.$$

The space E is said to be smooth if $\rho_E(\tau) > 0$, for $\forall \tau > 0$. And E is said to be uniformly smooth if and only if $\lim_{\tau \to 0^+} \frac{\rho_E(\tau)}{\tau} = 0$.

The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf\{1 - \frac{\|x+y\|}{2} : x, y \in E, \|x\| = \|y\| = 1, \|x-y\| \geq \epsilon\}.$$

The space E is said to be uniformly convex if and only if $\delta_E(\epsilon) > 0$, for $\forall \epsilon \in (0, 2]$. Let p be a fixed real number with $p \geq 2$. Then E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$, for $\forall \epsilon \in (0, 2]$. Every p-uniformly convex space is a uniformly convex space.

Lemma 2.1 ([7]) *Let E be a 2-uniformly convex Banach space. Then for all $x, y \in E$, we have*

$$\|x - y\| \leq \frac{2}{c^2}\|Jx - Jy\|, \tag{2.1}$$

where J is the normalized duality mapping from E into E^* and $0 < c \leq 1$.

Lemma 2.2 ([5]) *The normalized duality mapping J from E into E^* has the following properties: (i) If E is a real reflexive and smooth Banach space, then $J : E \to E^*$ is single-valued; (ii) For $\forall x \in E$ and $\forall \lambda \in R$, $J(\lambda x) = \lambda Jx$; (iii) If E is a real uniformly convex and uniformly smooth Banach space, then $J^{-1} : E^* \to E$ is also a duality mapping. Moreover, both J and J^{-1} are uniformly continuous on each bounded subset of E or E^*, respectively.*

Lemma 2.3 ([1]) *Let E be a real reflexive, strictly convex and smooth Banach space, let C be a nonempty closed and convex subset of E and $x \in E$. Then there exists a unique element $x_0 \in C$ such that $\varphi(x_0, x) = \min\{\varphi(z, x) : z \in C\}$.*

In this case, the mapping Π_C of E onto C defined by $\Pi_C x = x_0$, for all $x \in E$, is called the generalized projection operator.

Lemma 2.4 ([1]) *Let E be a real reflexive, strictly convex and smooth Banach space, let C be a nonempty closed and convex subset of E and $x \in E$. Then, for $\forall y \in C$, $\varphi(y, \Pi_C x) + \varphi(\Pi_C x, x) \leq \varphi(y, x)$.*

Lemma 2.5 ([4]) *Let E be a real smooth and uniformly convex Banach space, let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\varphi(x_n, y_n) \to 0$, as $n \to \infty$, then $x_n - y_n \to 0$, as $n \to \infty$.*

Lemma 2.6 ([1]) *Let E be a real smooth Banach space, let C be a convex subset of E, let $x \in E$ and $x_0 \in C$. Then $\varphi(x_0, x) = \inf\{\varphi(z, x) : z \in C\}$ if and only if $\langle z - x_0, Jx_0 - Jx\rangle \geq 0$, $\forall z \in C$.*

Lemma 2.7 ([2]) *Let E be a real smooth and uniformly convex Banach space, let C be a nonempty closed and convex subset of E and let $S : C \to C$ be a relatively nonexpansive mapping. Then $F(S)$ is a closed and convex subset of C.*

Lemma 2.8 ([7]) *Let E be a Banach space, let C be a nonempty closed and convex subset of E and let $A : C \to E^*$ be a monotone and hemicontinuous operator. Let $B : E \to 2^{E^*}$ be defined as follows:*

$$Bv := \begin{cases} Av + N_C(v), & v \in C \\ \emptyset, & v \overline{\in} C, \end{cases} \tag{2.2}$$

where $N_C(v)$ is the normal cone for C at a point $v \in C$, that is, $N_C(v) := \{x^* \in E^* : \langle v - y, x^* \rangle \geq 0, \forall y \in C\}$. Then B is maximal monotone and $B^{-1}0 = \mathrm{VI}(C, A)$, which ensures in our later discussion that $\mathrm{VI}(C, A)$ is a closed convex subset of C.

We shall introduce the following function $V : E \times E^* \to R$ which is defined by:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2, \quad \forall x \in E, x^* \in E^*.$$

In the context of Section 1, we have $V(x, x^*) = \varphi(x, J^{-1}x^*)$, for $\forall x \in E$ and $x^* \in E^*$. And, we also have the following Lemma:

Lemma 2.9 ([1]) *Let E be a real reflexive, strictly convex and smooth Banach space with E^* as its dual. Then*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^* \rangle \leq V(x, x^* + y^*), \quad \forall x \in E, \ x^*, y^* \in E^*.$$

For solving the equilibrium problem for a bifunction $f : C \times C \to R$, let us assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$, for all $x \in C$;

(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$, for all $x, y \in C$;

(A3) $\limsup_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y)$, for all $x, y, z \in C$;

(A4) For each $x \in C$, $y \to f(x, y)$ is convex and lower semicontinuous.

Lemma 2.10 ([10]) *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let f be a bifunction of $C \times C$ to R satisfying (A1)-(A4). Let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \text{for} \quad \forall y \in C.$$

Lemma 2.11 ([10]) *Let C be a nonempty closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E and let f be a bifunction of $C \times C$ to R satisfying (A1)–(A4). For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r(x) = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C\}.$$

Then, the following conclusions hold:

(1) T_r is single-valued;

(2) T_r is firmly nonexpansive-type mapping , i.e., for any $x, y \in E$,

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \le \langle T_r x - T_r y, Jx - Jy\rangle;$$

(3) $F(T_r) = \mathrm{EP}(f)$;

(4) $\mathrm{EP}(f)$ is a closed and convex subset of C.

Lemma 2.12 ([10]) *Let C be a nonempty closed convex subset of a real smooth, strictly convex and reflexive Banach space E and let f be a bifunction of $C \times C$ to R satisfying (A1)–(A4). Let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$, we have:*

$$\varphi(q, T_r x) + \varphi(T_r x, x) \le \varphi(q, x).$$

Lemma 2.13 ([11]) *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|),$$

for all $t \in [0, 1]$ and $x, y \in B_r$, where $B_r := \{z \in E : \|z\| \le r\}$.

Lemma 2.14 ([12]) *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and $g(\|x - y\|) \le \varphi(x, y)$, for all $x, y \in B_r$, where B_r is the same as that in Lemma 2.13.*

3. Main results

Theorem 3.1 *Let E be a real uniformly smooth and 2-uniformly convex Banach space with dual E^* and C be a nonempty closed and convex subset of E. Suppose the duality mapping $J : E \to E^*$ is weakly sequentially continuous. Let $f : C \times C \to R$ be a bifunction satisfying (A1)–(A4), let $S : C \to C$ be a relatively nonexpansive mapping, and let $A : C \to E^*$ be an $\alpha-$inversely strongly monotone operator. Suppose that $D := F(S) \bigcap \mathrm{VI}(C, A) \bigcap \mathrm{EP}(f) \neq \emptyset$. Moreover, assume that $\|Ax\| \le \|Ax - Ap\|$, for $\forall x \in C$ and $p \in \mathrm{VI}(C, A)$. Let the sequence $\{x_n\}$ be generated by the iterative scheme (1.6).*

If $\{\alpha_n\} \subset [0, 1)$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, $\lambda_n \in [a, b]$ for some a, b with $0 < a < b \le \frac{\alpha c^2}{2}$, where $c \in (0, 1]$ is the same constant as that in (2.1), and $\{r_n\} \subset [d, +\infty)$, where d is a positive constant, then $\{x_n\}$ converges weakly to z, where $z = \lim_{n\to\infty} \Pi_D x_n$.

Proof We split the proof into four steps.

Step 1. $\{x_n\}$ is a bounded sequence.

To observe this, take $p \in D$. Then we have:

$$\varphi(p, x_{n+1}) \le \alpha_n\varphi(p, y_n) + (1 - \alpha_n)\varphi(p, Su_n) \le \alpha_n\varphi(p, y_n) + (1 - \alpha_n)\varphi(p, u_n). \tag{3.1}$$

By using Lemmas 2.1, 2.4 and 2.9, and the assumption that $\|Ax\| \le \|Ax - Ap\|$, $\forall x \in C$ and $p \in \mathrm{VI}(C, A)$, we have:

$$\varphi(p, y_n) \le \varphi(p, J^{-1}(Ju_n - \lambda_n Au_n)) = V(p, Ju_n - \lambda_n Au_n)$$

$$\begin{aligned}
&\leq V(p,(Ju_n-\lambda_nAu_n)+\lambda_nAu_n)-2\langle J^{-1}(Ju_n-\lambda_nAu_n)-p,\lambda_nAu_n\rangle\\
&=\varphi(p,u_n)-2\lambda_n\langle u_n-p,Au_n\rangle-2\lambda_n\langle J^{-1}(Ju_n-\lambda_nAu_n)-J^{-1}Ju_n,Au_n\rangle\\
&\leq\varphi(p,u_n)-2\lambda_n\langle u_n-p,Au_n-Ap\rangle-2\lambda_n\langle u_n-p,Ap\rangle+\frac{4\lambda_n^2}{c^2}\|Au_n-Ap\|^2\\
&\leq\varphi(p,u_n)-(2\lambda_n\alpha-\frac{4\lambda_n^2}{c^2})\|Au_n-Ap\|^2\leq\varphi(p,u_n). \qquad (3.2)
\end{aligned}$$

From Lemma 2.11, we know that $u_n = T_{r_n}x_n$. Using Lemma 2.12, we have:

$$\varphi(p,u_n)\leq\varphi(p,x_n). \qquad (3.3)$$

From (3.1),(3.2) and (3.3), we know that

$$\varphi(p,x_{n+1})\leq\varphi(p,x_n). \qquad (3.4)$$

Therefore, $\lim_{n\to\infty}\varphi(p,x_n)$ exists and then $\{x_n\}$, $\{y_n\}$, $\{Su_n\}$ and $\{\Pi_Dx_n\}$ are all bounded. Moreover, $\lim_{n\to\infty}\varphi(p,x_n)=\lim_{n\to\infty}\varphi(p,u_n)$.

Step 2. $\lim_{n\to\infty}\Pi_Dx_n$ exists.

For this, let $w_n=\Pi_Dx_n$ for all $n\geq 0$. Since $w_n\in D$, (3.4) implies that

$$\varphi(w_n,x_{n+1})\leq\varphi(w_n,x_n). \qquad (3.5)$$

From Lemma 2.4, we know that

$$\varphi(w_{n+1},x_{n+1})\leq\varphi(w_n,x_{n+1})-\varphi(w_n,w_{n+1})\leq\varphi(w_n,x_{n+1}).$$

Combining this with (3.5), we have $\varphi(w_{n+1},x_{n+1})\leq\varphi(w_n,x_n)$. Therefore, $\lim_{n\to\infty}\varphi(w_n,x_n)$ exists. From (3.4) and (3.5), actually we have: for $\forall k\in N$, $\varphi(w_n,x_{n+k})\leq\varphi(w_n,x_n)$.

Noticing that $w_{n+k}=\Pi_Dx_{n+k}$ and using Lemma 2.4 again, we have:

$$\varphi(w_n,w_{n+k})+\varphi(w_{n+k},x_{n+k})\leq\varphi(w_n,x_{n+k})\leq\varphi(w_n,x_n),$$

and then

$$\varphi(w_n,w_{n+k})\leq\varphi(w_n,x_n)-\varphi(w_{n+k},x_{n+k}).$$

Let $r=\sup_{n\in N}\|w_n\|$. From Lemma 2.14, we know that there exists a continuous, strictly increasing, and convex function g with $g(0)=0$ such that $g(\|w_n-w_{n+k}\|)\leq\varphi(w_n,w_{n+k})\leq\varphi(w_n,x_n)-\varphi(w_{n+k},x_{n+k})\to 0$, as $n\to\infty$. So, $\{w_n\}$ is a Cauchy sequence and then $\lim_{n\to\infty}\Pi_Dx_n$ exists.

Step 3. $\omega(x_n)\subset D$, where $\omega(x_n)$ denotes the set of weak convergent points of all the weak convergent subsequences of $\{x_n\}$.

For $\forall x\in\omega(x_n)$, there exists a subsequence of $\{x_n\}$, which is still denoted by $\{x_n\}$, such that $x_n\rightharpoonup x$, as $n\to\infty$.

First we shall show that $x\in F(S)$.

Noticing (3.2), we have for $p\in D$,

$$\begin{aligned}
\varphi(p,x_{n+1})&\leq\alpha_n\varphi(p,y_n)+(1-\alpha_n)\varphi(p,u_n)\\
&\leq\alpha_n[\varphi(p,x_n)+2b(\frac{2b}{c^2}-\alpha)\|Au_n-Ap\|^2]+(1-\alpha_n)\varphi(p,x_n). \qquad (3.6)
\end{aligned}$$

Since $\lim_{n\to\infty}\varphi(p,x_n)$ exists, we have

$$-\alpha_n 2b(\frac{2b}{c^2}-\alpha)\|Au_n-Ap\|^2 \le \varphi(p,x_n)-\varphi(p,x_{n+1})\to 0,$$

which implies that $\|Au_n-Ap\|\to 0$, as $n\to\infty$.

Notice that

$$\begin{aligned}
\varphi(u_n,y_n) &= \varphi(u_n,\Pi_C J^{-1}(Ju_n-\lambda_n Au_n))\\
&\le \varphi(u_n,J^{-1}(Ju_n-\lambda_n Au_n))) = V(u_n,Ju_n-\lambda_n Au_n)\\
&\le V(u_n,(Ju_n-\lambda_n Au_n)+\lambda_n Au_n)-2\langle J^{-1}(Ju_n-\lambda_n Au_n)-u_n,\lambda_n Au_n\rangle\\
&= \varphi(u_n,u_n)-2\lambda_n\langle J^{-1}(Ju_n-\lambda_n Au_n)-u_n,Au_n\rangle\\
&= 2\langle J^{-1}(Ju_n-\lambda_n Au_n)-u_n,-\lambda_n Au_n\rangle \le \frac{4\lambda_n^2}{c^2}\|Au_n-Ap\|^2\to 0,
\end{aligned}\tag{3.7}$$

which implies that $u_n-y_n\to 0$, as $n\to\infty$.

Let $r^*=\sup_{n\ge N}\{\|y_n\|,\|Su_n\|\}$. By using Lemmas 2.12 and 2.13, we have:

$$\begin{aligned}
\varphi(p,u_{n+1}) &= \varphi(p,T_{r_{n+1}}x_{n+1}) \le \varphi(p,x_{n+1}) = \varphi(p,J^{-1}(\alpha_n Jy_n+(1-\alpha_n)JSu_n))\\
&= \|p\|^2-2\alpha_n\langle p,Jy_n\rangle-2(1-\alpha_n)\langle p,JSu_n\rangle+\|\alpha_n Jy_n+(1-\alpha_n)JSu_n\|^2\\
&\le \|p\|^2-2\alpha_n\langle p,Jy_n\rangle-2(1-\alpha_n)\langle p,JSu_n\rangle+\alpha_n\|y_n\|^2+(1-\alpha_n)\|Su_n\|^2-\\
&\quad \alpha_n(1-\alpha_n)g(\|Jy_n-JSu_n\|)\\
&= \alpha_n\varphi(p,y_n)+(1-\alpha_n)\varphi(p,Su_n)-\alpha_n(1-\alpha_n)g(\|Jy_n-JSu_n\|)\\
&\le \alpha_n\varphi(p,u_n)+(1-\alpha_n)\varphi(p,u_n)-\alpha_n(1-\alpha_n)g(\|Jy_n-JSu_n\|)\\
&= \varphi(p,u_n)-\alpha_n(1-\alpha_n)g(\|Jy_n-JSu_n\|).
\end{aligned}\tag{3.8}$$

Since $\lim_{n\to\infty}\varphi(p,u_n)$ exists and $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$, we have

$$\alpha_n(1-\alpha_n)g(\|Jy_n-JSu_n\|)\le\varphi(p,u_n)-\varphi(p,u_{n+1})\to 0,$$

which implies that $Ju_n-JSu_n\to 0$, as $n\to\infty$. Using Lemma 2.2, we have $u_n-Su_n\to 0$, as $n\to\infty$.

Since $u_n=T_{r_n}x_n$, and $\lim_{n\to\infty}\varphi(p,x_n)=\lim_{n\to\infty}\varphi(p,u_n)$, by Lemma 2.12, we have for $p\in D$,

$$\varphi(u_n,x_n)\le\varphi(p,x_n)-\varphi(p,u_n)\to 0.$$

Therefore,

$$u_n-x_n\to 0,\tag{3.9}$$

as $n\to\infty$. So $u_n\rightharpoonup x$ and from $u_n-Su_n\to 0$, we know that $x\in F(S)$.

Secondly, we shall show that $x\in \mathrm{EP}(f)$.

Notice again that $u_n=T_{r_n}x_n$, we have:

$$f(u_n,y)+\frac{1}{r_n}\langle y-u_n,Ju_n-Jx_n\rangle\ge 0,\quad \forall y\in C.$$

From (A2), we know that

$$\frac{1}{r_n}\langle y-u_n,Ju_n-Jx_n\rangle\ge -f(u_n,y)\ge f(y,u_n),\quad \forall y\in C.\tag{3.10}$$

Let $n \to \infty$ in (3.10). Then from (A4) and (3.9) we have:

$$f(y, x) \leq 0, \quad \forall y \in C.$$

For $t : 0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)x$. Since $y \in C$ and $x \in C$, we have $y_t \in C$ and hence $f(y_t, x) \leq 0$. So, from (A1) and (A4), we have

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, x) \leq tf(y_t, y).$$

Therefore, $f(y_t, y) \geq 0$, $\forall y \in C$. Letting $t \downarrow 0$, from (A3), we have $f(x, y) \geq 0$, $\forall y \in C$. Thus, $x \in \mathrm{EP}(f)$.

Finally, we shall show that $x \in \mathrm{VI}(C, A)$.

Let $B : E \to 2^{E^*}$ be defined as that in (2.2). Then in view of Lemma 2.8, B is maximal monotone and $B^{-1}0 = \mathrm{VI}(C, A)$. So, to show that $x \in \mathrm{VI}(C, A)$ is equivalent to showing that $x \in B^{-1}0$.

Let $(v, w) \in G(B)$. Since $w \in Bv = Av + N_C(v)$, we have $w - Av \in N_C(v)$. Since $y_n \in C$, we get

$$\langle v - y_n, w - Av \rangle \geq 0. \tag{3.11}$$

On the other hand, from $y_n = \Pi_C J^{-1}(Ju_n - \lambda_n Au_n)$ and Lemma 2.6, we obtain:

$$\langle v - y_n, Jy_n - (Ju_n - \lambda_n Au_n) \rangle \geq 0,$$

and then

$$\langle v - y_n, \frac{Ju_n - Jy_n}{\lambda_n} - Au_n \rangle \leq 0. \tag{3.12}$$

It follows from (3.11) and (3.12):

$$\begin{aligned}
&\langle v - y_n, w \rangle \geq \langle v - y_n, Av \rangle \geq \langle v - y_n, Av \rangle + \langle v - y_n, \frac{Ju_n - Jy_n}{\lambda_n} - Au_n \rangle \\
&= \langle v - y_n, \frac{Ju_n - Jy_n}{\lambda_n} + Av - Au_n \rangle \\
&= \langle v - y_n, Av - Ay_n \rangle + \langle v - y_n, Ay_n - Au_n \rangle + \langle v - y_n, \frac{Ju_n - Jy_n}{\lambda_n} \rangle \\
&\geq -\|v - y_n\| \|Ay_n - Au_n\| - \|v - y_n\| \| \frac{Ju_n - Jy_n}{\lambda_n} \| \\
&\geq -\|v - y_n\| \frac{\|y_n - u_n\|}{\alpha} - \|v - y_n\| \| \frac{Ju_n - Jy_n}{\lambda_n} \|.
\end{aligned} \tag{3.13}$$

Hence we have $\langle v - x, w \rangle \geq 0$ by letting $n \to \infty$ in (3.13). Since B is maximal monotone, $x \in B^{-1}0$. Therefore, $x \in D$.

Step 4. $x_n \rightharpoonup z = \lim_{n\to\infty} \Pi_D x_n$.

From Step 2, we know that there exists $z \in D$ such that $z = \lim_{n\to\infty} \Pi_D x_n$. Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \widehat{z}$, as $j \to \infty$. From Step 3, we have $\widehat{z} \in D$. Then from Lemma 2.6, we have:

$$\langle \Pi_D x_{n_j} - \widehat{z}, Jx_{n_j} - J\Pi_D x_{n_j} \rangle \geq 0. \tag{3.14}$$

Since J is weakly sequentially continuous, letting $j \to \infty$ in (3.14), we have $\langle z - \widehat{z}, J\widehat{z} - Jz\rangle \geq 0$. Therefore, $\langle z - \widehat{z}, J\widehat{z} - Jz\rangle = 0$. Since E is strictly convex, we have $z = \widehat{z}$.

Suppose there exists another subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $x_{n_l} \rightharpoonup \overline{z}$, as $l \to \infty$. Then $\overline{z} \in D$ and $Jx_{n_l} \rightharpoonup J\overline{z}$, as $l \to \infty$. Repeating the above proof, we can also know that $\overline{z} = z$. Therefore, all of the weak convergent subsequences of $\{x_n\}$ converges weakly to the same element z, which ensures that $x_n \rightharpoonup z$, as $n \to \infty$.

This completes the proof. $\square$

Corollary 3.2 *Let E, C, J, f, S satisfy the same conditions as those in Theorem 3.1. Let $A : C \to E^*$ be a strongly monotone operator with coefficient k and also be a Lipschitz continuous mapping with Lipschitz constant $L > 0$. Suppose that $F(S) \bigcap \mathrm{VI}(C, A) \bigcap \mathrm{EP}(f) \neq \emptyset$. Moreover, assume that $\|Ax\| \leq \|Ax - Ap\|$, $\forall x \in C$ and $p \in \mathrm{VI}(C, A)$. Let the sequence $\{x_n\}$ be generated by the iterative scheme (1.6).*

If $\{\alpha_n\} \subset [0, 1)$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, $\lambda_n \in [a, b]$ for some a, b with $0 < a < b \leq \frac{kc^2}{2L^2}$, where $c \in (0, 1]$ is the same constant as that in (2.1), and $\{r_n\} \subset [d, +\infty)$, where d is a positive constant, then $\{x_n\}$ converges weakly to $z = \lim_{n\to\infty} \Pi_{F(S) \bigcap \mathrm{VI}(C,A) \bigcap \mathrm{EP}(f)} x_n$.

Proof In this case, we can easily know that A is $\frac{k}{L^2}-$inversely strongly monotone. Therefore, the conclusion follows from Theorem 3.1. $\square$

Corollary 3.3 *Let E be a real uniformly smooth and 2-uniformly convex Banach space with dual E^* and C be a nonempty closed and convex subset of E. Let $S : C \to C$ be a relatively nonexpansive mapping, and let $A : C \to E^*$ be an α-inversely strongly monotone operator. Suppose that $F(S) \bigcap \mathrm{VI}(C, A) \neq \emptyset$. Moreover, assume that $\|Ax\| \leq \|Ax - Ap\|$, $\forall x \in C$ and $p \in \mathrm{VI}(C, A)$. Let the sequence $\{x_n\}$ be generated by the following iterative scheme:*

$$\begin{cases} x_0 \in C, \quad \text{chosen arbitrarily}, \\ y_n = \Pi_C(J^{-1}(Jx_n - \lambda_n Ax_n)), \\ x_{n+1} = J^{-1}(\alpha_n Jy_n + (1 - \alpha_n)JS\Pi_C x_n), \quad n = 0, 1, 2, \ldots . \end{cases} \tag{3.15}$$

If $\{\alpha_n\} \subset [0, 1)$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, $\lambda_n \in [a, b]$ for some a, b with $0 < a < b \leq \frac{\alpha c^2}{2}$, where $c \in (0, 1]$ is the same constant as that in (2.1), then $\{x_n\}$ converges weakly to $z = \lim_{n\to\infty} \Pi_{F(S) \bigcap \mathrm{VI}(C,A)} x_n$.

Proof Let $f(x, y) \equiv 0$, for $\forall x, y \in C$, and $r_n \equiv 1$, for $\forall n \geq 0$ in Theorem 3.1. Then we know that $u_n = \Pi_C x_n$, for $n \geq 0$. Then the result follows from Theorem 3.1. $\square$

Remark 3.4 Our main results of Theorem 3.1 and Corollary 3.2 can be considered as the combination of the discussion of variational inequalities, fixed point problems for relatively nonexpansive mappings and equilibrium problems.

Corollary 3.5 *Let H be a real Hilbert space and C be a nonempty closed and convex subset of H. Let $f : C \times C \to R$ be a bifunction satisfying (A1)–(A4), $S : C \to C$ be a nonexpansive mapping, and $A : C \to H$ be an $\alpha-$inversely strongly monotone operator. Suppose that*

$D := F(S)\bigcap \mathrm{VI}(C,A)\bigcap \mathrm{EP}(f)\neq\emptyset$. Moreover, assume that $\|Ax\|\le\|Ax-Ap\|$, $\forall x\in C$ and $p\in \mathrm{VI}(C,A)$. Let the sequence $\{x_n\}$ be generated by the following iterative scheme:

$$\begin{cases} x_0\in C, \quad \textit{chosen arbitrarily},\\ u_n\in C, \quad \textit{such that} \;\; f(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0, \quad \forall y\in C,\\ y_n=P_C(u_n-\lambda_nAu_n),\\ x_{n+1}=\alpha_ny_n+(1-\alpha_n)Su_n, \quad n=0,1,2,\ldots. \end{cases} \tag{3.16}$$

If $\{\alpha_n\}\subset[0,1)$ such that $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$, $\lambda_n\in[a,b]$ for some a,b with $0<a<b\le\frac{\alpha c^2}{2}$, where $c\in(0,1]$ is the same constant as that in (2.1), and $\{r_n\}\subset[d,+\infty)$, where d is a positive constant, then $\{x_n\}$ converges weakly to $\lim_{n\to\infty}P_Dx_n$.

Remark 3.6 If in Corollary 3.5, $f(x,y)\equiv 0$, for $\forall x,y\in C$ and $r_n\equiv 1$, for $\forall n\ge 0$, then (3.16) reduces to the following one:

$$\begin{cases} x_0\in C, \quad \text{chosen arbitrarily},\\ y_n=P_C(x_n-\lambda_nAx_n),\\ x_{n+1}=\alpha_ny_n+(1-\alpha_n)SP_Cx_n, \quad n=0,1,2,\ldots. \end{cases} \tag{3.17}$$

References

[1] ALBER Y I. *Metric and Generalized Projection Operators in Banach Spaces: Properties and Applications* [M]. Dekker, New York, 1996.

[2] MATSUSHITA S, TAKAHASHI W. *A strong convergence theorem for relatively nonexpansive mappings in a Banach space* [J]. J. Approx. Theory, 2005, **134**(2): 257–266.

[3] QIN Xiaolong, SU Yongfu. *Strong convergence theorems for relatively nonexpansive mappings in a Banach space* [J]. Nonlinear Anal., 2007, **67**(6): 1958–1965.

[4] PLUBTIENG S, UNGCHITTRAKOOL K. *Strong convergence theorems for a common fixed point of two relatively nonexpansive mappings in a Banach space* [J]. J. Approx. Theory, 2007, **149**(2): 103–115.

[5] WEI Li, CHO Y J, ZHOU Haiyun. *A Strong convergence theorem for common fixed points of two relatively nonexpansive mappings and its applications* [J]. J. Appl. Math. Comput., 2009, **29**(1-2): 95–103.

[6] WEI Li, SU Yongfu, ZHOU Haiyun. *Iterative convergence theorems for maximal monotone operators and relatively nonexpansive mappings* [J]. Appl. Math. J. Chinese Univ. Ser. B, 2008, **23**(3): 319–325.

[7] IIDUKA H, TAKAHASHI W. *Weak convergence of a projection algorithm for variational inequalities in a Banach space* [J]. J. Math. Anal. Appl., 2008, **339**(1): 668–679.

[8] BLUM E, OETTLI W.*From optimization and variational inequalities to equilibrium problems* [J]. Math. Student, 1994, **63**(1-4): 123–145.

[9] COMBETTES P L, HIRSTOAGA S A. *Equilibrium programming in Hilbert spaces* [J]. J. Nonlinear Convex Anal., 2005, **6**(1): 117–136.

[10] TAKAHASHI W, ZEMBAYASHI K . *Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces* [J]. Nonlinear Anal., 2009, **70**(1): 45–57.

[11] XU Hongkun. *Inequalities in Banach spaces with applications* [J]. Nonlinear Anal., 1991, **16**(12): 1127–1138.

[12] KAMIMURA S, TAKAHASHI W. *Strong convergence of a proximal-type algorithm in a Banach space* [J]. SIAM J. Optim., 2002, **13**(3): 938–945.