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The Bessel Numbers and Bessel Matrices

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Abstract In this paper, using exponential Riordan arrays, we investigate the Bessel numbers and Bessel matrices. By exploring links between the Bessel matrices, the Stirling matrices and the degenerate Stirling matrices, we show that the Bessel numbers are special case of the degenerate Stirling numbers, and derive explicit formulas for the Bessel numbers in terms of the Stirling numbers and binomial coefficients.

Keywords Bessel number of the first kind; Bessel number of the second kind; exponential Riordan array; Stirling numbers; Bessel matrix.

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1. Introduction

The Bessel polynomials [1, 2] are the unique polynomial solutions of the second-order differential equations $x^2y_n'' + (2x+2)y_n' = n(n+1)y_n$ (for natural numbers n) that are normalized to have unit constant term. The coefficients of these polynomials are known as Bessel coefficients. Let the Bessel number of the first kind $b(n,k)$ be the coefficient of x^{n-k} in $-y_{n-1}(-x)$, and let the Bessel number of the second kind $B(n,k)$ be the number of set partitions of $[n] := \{1,2,3,\ldots,n\}$ into k blocks of size one or two. Choi et al. [3] investigated the analogies between Stirling numbers and Bessel numbers. In [4], it was shown that Bessel numbers satisfy two properties of Stirling numbers: The two kinds of Bessel numbers are related by inverse formulas, and both Bessel numbers of the first kind and those of the second kind form log-concave sequences. Shapiro et al. [5] introduced the concept of a Riordan array in 1991, then the concept is generalized to the exponential Riordan array by many authors [6–8]. In this paper, we obtain the exponential generating functions for the Bessel numbers. By using exponential Riordan arrays, we deduce the inverse formulas for the Bessel numbers, and find relations between the Bessel matrices, the Catalan matrix and the Fibonaaci matrix. By exploring links between the Bessel matrices, the

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Stirling matrices and the degenerate Stirling matrices, we show that the Bessel numbers are special case of the degenerate Stirling numbers, and derive explicit formulas for the Bessel numbers in terms of the Stirling numbers and binomial coefficients.

2. Generating functions for the Bessel numbers

For each nonnegative integer n, the Bessel polynomial $y_n(x)$ is defined to be the (unique) polynomial of degree n, with unit constant term, satisfying second-order differential equation

$$x^2y_n'' + (2x+2)y_n' = n(n+1)y_n. \tag{1}$$

From the differential equation (1) one can easily derive the formula

$$y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{2^k k!(n-k)!}x^k.$$

The coefficient of x^{n-k} in the $(n-1)$-th Bessel polynomial $y_{n-1}(x)$ is called the signless Bessel number of the first kind, denoted by $a(n,k)$, and $b(n,k) = (-1)^{n-k}a(n,k)$ is called the Bessel number of the first kind. By convention, we put $a(0,k) = b(0,k) = \delta_{0,k}$. So the number $b(n,k)$ was given by [4]

$$b(n,k) = \begin{cases} (-1)^{n-k}\dfrac{(2n-k-1)!}{2^{n-k}(k-1)!(n-k)!}, & \text{if } 1 \le k \le n, \\ 0, & \text{if } 1 \le n < k. \end{cases} \tag{2}$$

Define the Bessel matrix of the first kind to be the infinite lower-triangle matrix $\mathbf{b}$ whose (n,k)-th element is equal to $b(n,k)$:

$$\mathbf{b} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 3 & -3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & -15 & 15 & -6 & 1 & 0 & 0 & \cdots \\ 0 & 105 & -105 & 45 & -10 & 1 & 0 & \cdots \\ 0 & -945 & 945 & -420 & 105 & -15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Lemma 2.1 *The Bessel numbers of the first kind $b(n,k)$ have the following vertical exponential generating function*

$$\sum_{n=k}^{\infty} b(n,k)\frac{t^n}{n!} = \frac{1}{k!}\left(\sqrt{1+2t}-1\right)^k. \tag{3}$$

Proof Recall that the generalized binomial power series $B_z(t)$ was defined in [9, 10] as $B_z(t) = \sum_{n=0}^{\infty}\binom{zn+1}{n}\frac{t^n}{zn+1}$. The generalized binomial power series satisfies $B_z(t) = 1 + tB_z(t)^z$. When z

is any real or complex number, exponentiation of this series is known to yield

$$B_z(t)^x = \sum_{n=0}^{\infty} \binom{zn+x}{n} \frac{xt^n}{zn+x}. \tag{4}$$

The special case $z = 2$ gives

$$B_2(t) = \frac{1-\sqrt{1-4t}}{2t} = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + \cdots, \tag{5}$$

in which the coefficients are the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. From (4) and (5), we have $\sqrt{1+2t} - 1 = tB_2(-\frac{t}{2}) = t\sum_{n=0}^{\infty}\binom{2n+1}{n}\frac{1}{2n+1}(-\frac{t}{2})^n$, and $\frac{1}{k!}\left(\sqrt{1+2t}-1\right)^k = \frac{t^k}{k!}B_2(-\frac{t}{2})^k = \sum_{n=k}^{\infty}(-1)^{n-k}\binom{2n-k}{n-k}\frac{kt^n}{2^{n-k}k!(2n-k)} = \sum_{n=k}^{\infty} b(n,k)\frac{t^n}{n!}$. □

For nonnegative integers n and k, the Bessel number of the second kind $B(n,k)$ is defined to be the number of partitions of $[n] := \{1, 2, \ldots, n\}$ into k nonempty blocks of size at most 2. A block of size 1 is called a singleton, and a block of size 2 is called a pair. The Bessel numbers of the second kind satisfy the recursion

$$B(n,k) = B(n-1,k-1) + (n-1)B(n-2,k-1) \tag{6}$$

with initial conditions $B(1,1) = 1$ and $B(0,k) = \delta_{0,k}$. Equation (6) is proved as follows: To obtain a partition of $[n]$ into k blocks of size 1 or 2, we can put n in a block itself and partition $[n-1]$ into $k-1$ block of size 1 or 2 in $B(n-1,k-1)$ ways, or we can put n and some $x \in [n-1]$ in a block of size 2 in $n-1$ ways and then partition $[n-1]-\{x\}$ into $k-1$ blocks of size 1 or 2 in $B(n-2,k-1)$ ways. Hence (6) follows.

Lemma 2.2 *The Bessel numbers of the second kind $B(n,k)$ have the following vertical exponential generating function*

$$\sum_{n=k}^{\infty} B(n,k)\frac{t^n}{n!} = \frac{1}{k!}\left(t + \frac{t^2}{2}\right)^k. \tag{7}$$

Proof Define $f_k(t) = \sum_{n=0}^{\infty} B(n,k)\frac{t^n}{n!}$. Multiply (6) on both sides by $\frac{t^{n-1}}{(n-1)!}$ and sum on n. The result is that $f_k'(t) = (1+t)f_{k-1}(t)$. Since $f_0(t) = 1$ and $f_k(0) = 0$ for $k \geq 1$, we find successively that $f_1(t) = t + \frac{t^2}{2}$, $f_2(t) = \frac{1}{2}(t + \frac{t^2}{2})$, and so forth. The result follows by induction. □

By Lemma 2.2, it is easy to see that the number $B(n,k)$ is given by

$$B(n,k) = \begin{cases} \dfrac{n!}{2^{n-k}(2k-n)!(n-k)!}, & \text{if } \lceil \dfrac{n}{2} \rceil \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

From (2) and (8), $B(n,k) = a(k+1, 2k-n+1)$, or equivalently $a(n,k) = B(2n-k-1, n-1)$.

Define the Bessel matrix of the second kind to be the infinite lower-triangle matrix $\mathbf{B}$ whose

(n,k)-th element is equal to $B(n,k)$:

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 6 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 15 & 10 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 15 & 45 & 15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The Bessel numbers of the second kind can also be interpreted in terms of matchings of the complete graph [11, 12]. Let G be a loopless graph with n vertices. A set M of k edges of G is called a k-matching in G if no two edges in M are adjacent in G. A matching M saturates a vertex v, and v is said to be M-saturated, if some edge of M is incident with v; otherwise, v is M-unsaturated. Let $m(G,k)$ denote the number of k-matchings in G. We take $m(G,0)=1$ and $m(G,-1)=0$. It is clear that $m(G,1)$ is the number of edges in G and that for $k > \lfloor \frac{n}{2} \rfloor$, $m(G,k)=0$, where n is the number of vertices of G. The complete graph K_n with n vertices is a simple graph each pair of whose distinct vertices are adjacent. Let $m(n,k)$ denote the number of k-matchings in the complete graph K_n. We call $m(n,k)$ a matching number. Assume that all vertices of K_n are labeled with the integers in $[n]$. Assume a partition of $[n]$ into k nonempty blocks, each of size at most 2, has c singletons and b pairs. Then we have $c+b=k$ and $c+2b=n$. Consequently, $c=2k-n$ and $b=n-k$. Thus a partition of $[n]$ into k nonempty blocks, each of size at most 2, can be identified with an $(n-k)$-matching in K_n canonically. Hence $m(n,k)=B(n,n-k)$.

3. Inverse formulas for the Bessel numbers

An exponential Riordan array is defined by a pair of exponential generating functions $g(t) = 1+\sum_{n=1}^{\infty} g_n \frac{t^n}{n!}$ and $f(t)=\sum_{n=1}^{\infty} f_n \frac{t^n}{n!}$ with $f_1 \neq 0$, such that their j-th column has exponential generating function $\frac{1}{j!} g(t) f(t)^j$ (the first column being indexed by 0). As usual, the matrix corresponding to the pair $g(t)$, $f(t)$ is denoted by $[g(t), f(t)]$ or $[g,f]$. The exponential Riordan group is the set of all exponential Riordan arrays with the operation being matrix multiplication. The group law is then given by

$$[g(t), f(t)][h(t), l(t)] = [g(t)h(f(t)), l(f(t))]. \tag{9}$$

The identity for this law is $I=[1,t]$ and the inverse of $[g(t), f(t)]$ is $[1/g(\bar{f}(t)), \bar{f}(t)]$, where $\bar{f}(t)$ is compositional inverse of $f(t)$. Let $a=(a_0,a_1,a_2,\ldots)^{\mathrm{T}}$ be a real sequence with exponential generating function $A(t)$. Then the sequence $[g(t), f(t)]a$ has exponential generating function

$g(t)A(f(t))$, i.e.,

$$[g(t), f(t)]A(t) = g(t)A(f(t)). \tag{10}$$

From Lemmas 2.1 and 2.2, we get the following theorem.

Theorem 3.1 *The Bessel matrix of the first kind is the inverse to the Bessel matrix of the second kind, and they can be represented by exponential Riordan array as* $\mathbf{b} = [1, \sqrt{1+2t}-1]$, *and* $\mathbf{B} = [1, t+\frac{t^2}{2}]$, *respectively.*

The compositional inverse of $f(t) = t + \frac{t^2}{2}$ is $\bar{f}(t) = \sqrt{1+2t} - 1$, hence the inverse of the exponential Riordan array $[1, t+\frac{t^2}{2}]$ is $[1, \sqrt{1+2t}-1]$. Therefore, we derive the following inverse formulas for the Bessel numbers which were deduced in [4] by using the Lagrange Inversion Formula:

$$\sum_{k\le j\le n}^{n} B(n,j)b(j,k) = \delta_{n,k}, \tag{11}$$

$$\sum_{k\le j\le n}^{n} b(n,j)B(j,k) = \delta_{n,k}. \tag{12}$$

Theorem 3.2 *The row sums of the Bessel matrix of the second kind* $\mathbf{B}$ *are given by* $u_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{(2k)!}{2^k k!}$, *and satisfy the recurrence equation*

$$u_n = u_{n-1} + (n-1)u_{n-2}$$

with $u_0 = 1$, $u_1 = 1$.

Proof It follows from the fact that exponential generating function of the sequence of row sums is given by $[1, t+\frac{t^2}{2}]e^t = e^{t+\frac{t^2}{2}}$. $\square$

The sequence of row sums of $\mathbf{B}$ can be represented by the Hermite polynomial as $u_n = (\frac{\sqrt{-1}}{\sqrt{2}})^n H_n(-\frac{\sqrt{-1}}{\sqrt{2}})$, here the Hermite polynomials $H_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n!}{k!(n-2k)!}(2x)^{n-2k}$, which can be defined by exponential generating function

$$\sum_{k=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{2xt-t^2}. \tag{13}$$

The combinatorial interpretations of the n-th row sum u_n are (see Stanley [13]):

(a) The number of matchings in the complete graph K_n.

(b) The number of involutions in the symmetric group S_n.

(c) The number of partitions of an n-set with blocks of size 1 or 2.

(d) The number of symmetric $n \times n$ permutation matrices.

We shall show that the Bessel matrices have a tight connection with the Catalan matrix $\mathbf{C}$ which has many important applications in Combinatorics and Number Theory [14–17]. The

Catalan matrix $\mathbf{C}$ is given by

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & -2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & -5 & 5 & -3 & 1 & 0 & 0 & \cdots \\ 0 & 14 & -14 & 9 & -4 & 1 & 0 & \cdots \\ 0 & -42 & 42 & -28 & 14 & -5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and its inverse is the Fibonacci matrix $\mathbf{F}$ defined by

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 3 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3 & 4 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 6 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $\mathbf{G} = \mathrm{diag}(1, \frac{1!}{2}, \frac{2!}{2^2}, \frac{3!}{2^3}, \cdots)$. Then we have following theorem.

Theorem 3.3 The connection between the Bessel matrices and Catalan matrix is given by $\mathbf{b} = G\mathbf{C}G^{-1}$; $\mathbf{B} = G\mathbf{F}G^{-1}$.

Proof The general term of the matrix $\mathbf{C}$ is $\mathbf{C}(n,k) = (-1)^{n+k}\binom{2n-k}{n-k}\frac{k}{2n-k}$. From the proof of Lemma 2.1, $b(n,k) = \frac{(-1)^{n-k}n!}{2^{n-k}k!}\binom{2n-k}{n-k}\frac{k}{2n-k} = \frac{n!}{2^n}\mathbf{C}(n,k)\frac{2^k}{k!}$. Hence $\mathbf{b} = G\mathbf{C}G^{-1}$, and $\mathbf{B} = \mathbf{b}^{-1} = G\mathbf{C}^{-1}G^{-1} = G\mathbf{F}G^{-1}$.

4. Connection with Stirling numbers

The well-known Stirling numbers of the first kind $s(n,k)$ and the second kind $S(n,k)$ are defined as connection coefficients:

$$(t)_n = \sum_{k=0}^{n} s(n,k)t^k, \tag{14}$$

$$t^n = \sum_{k=0}^{n} S(n,k)(t)_k, \tag{15}$$

where $(t)_0 = 1$ and $(t)_n = t(t-1)(t-2)\cdots(t-n+1)$ for any integer $n > 0$.

It is known that the exponential generating functions for the Stirling numbers are

$$\sum_{n=k}^{\infty} s(n,k)\frac{t^n}{n!} = \frac{1}{k!}\left(\log(1+t)\right)^k, \tag{16}$$

$$\sum_{n=k}^{\infty} S(n,k)\frac{t^n}{n!} = \frac{1}{k!}\left(e^t-1\right)^k. \tag{17}$$

The Stirling matrix of the first kind $\widetilde{\mathbf{s}}$ and the Stirling matrix of the second kind $\widetilde{\mathbf{S}}$ are defined respectively by [18, 19]

$$\widetilde{s}(i,j) = \begin{cases} s(i,j), & \text{if } i \geq j \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$\widetilde{S}(i,j) = \begin{cases} S(i,j), & \text{if } i \geq j \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that the infinite lower triangular matrix $\widetilde{\mathbf{s}}$ and $\widetilde{\mathbf{S}}$ are exponential Riordan arrays, namely $\widetilde{\mathbf{s}} = [1, \log(1+t)]$, and $\widetilde{\mathbf{S}} = [1, e^t - 1]$.

For any nonzero real number x, the Stirling function matrices of the first kind $\widetilde{s}[x]$ and the second kind $\widetilde{S}[x]$ are defined respectively by [18]

$$\widetilde{s}[x](i,j) = \begin{cases} x^{i-j}s(i,j), & \text{if } i \geq j \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$\widetilde{S}[x](i,j) = \begin{cases} x^{i-j}S(i,j), & \text{if } i \geq j \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The infinite lower triangular matrices $\widetilde{s}[x]$ and $\widetilde{S}[x]$ can be expressed by exponential Riordan arrays as $\widetilde{s}[x] = [1, \frac{1}{x}\log(1+xt)]$, and $\widetilde{S}[x] = [1, \frac{1}{x}(e^{xt}-1)]$.

Theorem 4.1 *The Bessel matrices are connected with the Stirling matrices by*

$$\mathbf{b}\widetilde{s} = \widetilde{s}[2], \quad \widetilde{S}\mathbf{B} = \widetilde{S}[2].$$

Equivalently, the following identities hold

$$\sum_{j=0}^{n} b(n,j)s(j,k) = s(n,k)2^{n-k}, \quad \sum_{j=0}^{n} S(n,j)B(j,k) = S(n,k)2^{n-k}.$$

Proof By the multiplication rule of the exponential Riordan arrays (9), we obtain

$$\mathbf{b}\widetilde{s} = [1, \sqrt{1+2t}-1][1, \log(1+t)] = [1, \frac{1}{2}\log(1+2t)] = \widetilde{s}[2],$$

$$\widetilde{S}\mathbf{B} = [1, e^t-1][1, t+\frac{t^2}{2}] = [1, \frac{1}{2}(e^{2t}-1)] = \widetilde{S}[2]. \quad \square$$

Using Theorem 4.1, we can obtain

$$\mathbf{b} = \widetilde{s}[2]\widetilde{S}, \quad \mathbf{B} = \widetilde{s}\widetilde{S}[2].$$

Thus the Bessel numbers may be expressed in terms of Stirling numbers as:

$$b(n,k) = \sum_{j=k}^{n} 2^{n-j} s(n,j)S(j,k), \tag{18}$$

$$B(n,k) = \sum_{j=k}^{n} 2^{j-k} s(n,j)S(j,k). \tag{19}$$

The degenerate Stirling numbers of the first kind $s(n,k|x)$ and the second kind $S(n,k|x)$ are defined by [20–22]

$$\left(\frac{(1+t)^x - 1}{x}\right)^k = k! \sum_{n=k}^{\infty} s(n,k|x)\frac{t^n}{n!}, \tag{20}$$

$$\left((1+xt)^{\frac{1}{x}} - 1\right)^k = k! \sum_{n=k}^{\infty} S(n,k|x)\frac{t^n}{n!}. \tag{21}$$

Now, we define the infinite lower triangular matrices $\overline{s}[x]$ and $\overline{S}[x]$ by

$$\overline{s}[x](i,j) = s(i,j|x),\ \overline{S}[x](i,j) = S(i,j|x) \text{ for } \ i,j = 0,1,2,\ldots.$$

Then they can be written as exponential Riordan array as $\overline{s}[x] = [1, \frac{1}{x}((1+t)^x - 1)]$, $\overline{S}[x] = [1, (1+xt)^{\frac{1}{x}} - 1]$.

Theorem 4.2 *The Stirling function matrices are connected with the Stirling matrices by*

$$\overline{S}[x]\widetilde{s} = \widetilde{s}[x], \quad \widetilde{S}\overline{s}[x] = \widetilde{S}[x].$$

Equivalently, the following identities hold

$$\sum_{j=0}^{n} S(n,j|x)s(j,k) = s(n,k)x^{n-k}, \tag{22}$$

$$\sum_{j=0}^{n} S(n,j)s(j,k|x) = S(n,k)x^{n-k}. \tag{23}$$

Proof By the multiplication rule of the exponential Riordan arrays (9), we have

$$\overline{S}[x]\widetilde{s} = [1, (1+xt)^{\frac{1}{x}} - 1][1, \log(1+t)] = [1, \frac{1}{x}\log(1+xt)] = \widetilde{s}[x],$$

$$\widetilde{S}\overline{s}[x] = [1, e^t - 1][1, \frac{1}{x}((1+t)^x - 1)] = [1, \frac{1}{x}(e^{xt} - 1)] = \widetilde{S}[x]. \quad \square$$

Using Theorem 4.2, we can derive expressions for the degenerate Stirling numbers in terms of Stirling numbers [20]:

$$S(n,k|x) = \sum_{j=k}^{n} s(n,j)S(j,k)x^{n-j}, \tag{24}$$

$$s(n,k|x)=\sum_{j=k}^{n}s(n,j)S(j,k)x^{j-k}. \tag{25}$$

Setting $x=-1$, we have

$$\overline{s}[-1]=[1,\frac{t}{1+t}]=\left((-1)^{i+j}\frac{i!}{j!}\binom{i-1}{j-1}\right)_{i,j\geq 0},$$

$$\overline{S}[-1]=[1,\frac{t}{1-t}]=\left(\frac{i!}{j!}\binom{i-1}{j-1}\right)_{i,j\geq 0},$$

which are Lah matrices defined by Tan [23]. From (22) and (23), we get

$$\sum_{j=0}^{n}\frac{n!}{j!}\binom{n-1}{j-1}s(j,k)=(-1)^{n-k}s(n,k), \tag{26}$$

$$\sum_{j=0}^{n}(-1)^{j-k}S(n,j)\frac{j!}{k!}\binom{j-1}{k-1}=(-1)^{n-k}S(n,k). \tag{27}$$

Theorem 4.3 We have the following explicit formulas for the two kinds of Bessel numbers

$$b(n,k)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j(j-2)(j-4)\cdots(j-2n+2), \tag{28}$$

$$B(n,k)=\frac{1}{2^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}2j(2j-1)(2j-2)\cdots(2j-n+1). \tag{29}$$

Proof Comparing the exponential generating functions (3), (7) and (20), (21), we find that $b(n,k)=S(n,k|2)$, $B(n,k)=s(n,k|2)$. Carlitz [20] proved that

$$s(n,k|x)=\frac{1}{x^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}jx(jx-1)(jx-2)\cdots(jx-n+1), \tag{30}$$

$$S(n,k|x)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j(j-x)(j-2x)\cdots(j-nx+x). \tag{31}$$

Substituting $x=2$ into (30) and (31), respectively, we obtain the desired results. $\square$

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