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k-Walk-Regular Digraphs

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Abstract In this paper, we define a class of strongly connected digraph, called the k-walk-regular digraph, study some properties of it, provide its some algebraic characterization and point out that the 0-walk-regular digraph is the same as the walk-regular digraph discussed by Liu and Lin in 2010 and the D-walk-regular digraph is identical with the weakly distance-regular digraph defined by Comellas et al in 2004.

Keywords k-walk-regular digraph; predistance polynomial; the crossed uv-local multiplicity.

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1. Introduction

Let $\Gamma = (V, E)$ be a strongly connected digraph, with V denoting the set of vertices and E the set of arcs. If $(u, v) \in E$, we say that u is adjacent to v (or v is adjacent from u). A walk of length l in Γ is a sequence $(u_0, u_1, \ldots, u_l)$ of vertices such that $(u_{i-1}, u_i) \in E$, $i = 1, 2, \ldots, l$, and a walk is closed if its first and last vertices are the same. If the vertices in a walk are distinct, we call it a path. We say a digraph is strongly connected if any two vertices can be joined by a path. The number of arcs traversed in the shortest walk from u to v is called the distance from u to v, denoted by $\partial(u, v)$, and we call the value $D := \max\{\partial(u, v) | u, v \in V\}$ the diameter of Γ. Notice that what we consider is the oriented graphs, so $\partial(u, v)$ and $\partial(v, u)$ may not be equal. For any fixed integer $0 \le k \le D$, we will denote by $\Gamma_k^+(u)$ (respectively, $\Gamma_k^-(u)$) the set of vertices at distance k from u (respectively, the set of vertices from which u is at distance k). Sometimes it is written, for short, $\Gamma^+(u)$ or $\Gamma^-(u)$ instead of $\Gamma_1^+(v)$ or $\Gamma_1^-(v)$, respectively. Thus the out-valency and in-valency of u are $k_+(u) := |\Gamma^+(u)|$ and $k_-(u) := |\Gamma^-(u)|$. The digraph Γ is k-regular if $k_+(u) = k_-(u) = k$ for every $u \in V$.

The adjacency matrix A and the distance-k matrix A_k, where $0 \le k \le D$, of Γ are defined

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by

$$A_{uv} := \begin{cases} 1, & \text{if } (u,v) \in E, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$(A_k)_{uv} := \begin{cases} 1, & \text{if } \partial(u,v) = k, \\ 0, & \text{otherwise,} \end{cases}$$

respectively. If $AA^{\mathrm{T}} = A^{\mathrm{T}}A$, then Γ is said to be normal. As for normal matrices, there are the following properties:

Theorem 1 *Let A be an $n \times n$ complex matrix with eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$. Then A is normal if and only if any of the following assertions holds:*

(a) $U^*AU = \Lambda$ for some matrix U such that $UU^* = I$, and $\Lambda = \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$, where U^* is the transpose of U's conjugate;

(b) $A^* = p(A)$ for some polynomial $p \in \mathbf{C}[x]$;

(c) $\mathrm{tr}(AA^*) = \sum_{i=0}^{n-1} |\lambda_i|^2$.

Now assume that A has $d+1$ distinct eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_d$ and $m(x) = (x-\lambda_0)(x-\lambda_1)\cdots(x-\lambda_d)$ is the minimal polynomial of A. The spectrum of the digraph Γ, denoted by $sp\Gamma$ consists of the eigenvalues of A, which might be not real since A is not symmetric, together with their (algebraic) multiplicities:

$$sp\Gamma = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, \ldots, \lambda_d^{m(\lambda_d)}\},$$

where the superscripts stand for the multiplicities $m_i = m(\lambda_i)$ for $i = 0, 1, \ldots, n-1$ and λ_0 is the maximal eigenvalue and $\lambda_i \neq \lambda_j$ if $i \neq j$. In particular, $m(\lambda_0) = 1$ and $m(\lambda_0) + m(\lambda_1) + \cdots + m(\lambda_d) = n$ (for details the readers can see [1]).

By the Perron-Frobenius theorem, λ_0 is simple and has a positive eigenvector ν. If Γ is k-regular, then we may pick $\nu = \mathbf{j}$, where $\mathbf{j}$ denotes the all 1-vector, and $\lambda_0 = k$.

The adjacency algebra of Γ (also called Bose-Mesner algebra when it is closed under the Hadamard-or componentwise-product) is defined by $\mathscr{A}(\Gamma) := \{p(A) | p \in \mathbf{C}[x]\}$, where A is the adjacency matrix of Γ. The dimension of $\mathscr{A}(\Gamma)$, as a $\mathbf{C}$-vector space, equals the degree of the minimum polynomial $m(x)$. It is obvious that $\{I, A, \ldots, A^d\}$ is a basis of the adjacency or Bose-Mesner algebra $\mathscr{A}(\Gamma)$ and the dimension of the Bose-Mesner algebra is at least $D+1$ since the powers $I, A, A^2, \ldots, A^D$ are linearly independent. By Theorem 1(a), we know that the eigenvectors of a normal $n \times n$ square matrix constitute an orthogonal basis of the vector space $\mathbf{C}^n$, with inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^*\mathbf{v}$. For each polynomial $p \in \mathbf{C}[x]$ we define p operates on the vector $\mathbf{v} \in \mathbf{C}^n$ by $p\mathbf{v} = p(A)\mathbf{v}$. For each λ_i, let U_i be the matrix whose columns form an orthonormal basis of the eigenspace $V_i := \mathrm{Ker}(A - \lambda_i I)$. Then the orthogonal projection onto V_i is represented by the matrix $E_i = U_i U_i^*$, or alternatively, $E_i = \frac{1}{\phi_i} \prod_{j=0, j\neq i}^{d} (A - \lambda_j I)$, where $\phi_i = \prod_{j=0, j\neq i}^{d} (\lambda_i - \lambda_j)$. These matrices are called the principal idempotents of A and satisfy the following properties: $E_i E_j = \delta_{ij} E_i$, $AE_i = \lambda_i E_i$. Also $\{E_0, E_1, \ldots, E_d\}$ is a basis of $\mathscr{A}(\Gamma)$. Then we can give the orthogonal decomposition of the unitary vector $\mathbf{e}_u$ representing vertex u

as follows:

$$\mathbf{e}_u = \mathbf{z}_u^0 + \mathbf{z}_u^1 + \ldots + \mathbf{z}_u^d, \tag{1}$$

where $\mathbf{z}_u^i = E_i \mathbf{e}_u$, $i = 0, 1, \ldots, d$.

Throughout this paper, we assume that Γ is a strongly connected digraph with order $n = |V|$, size $m = |E|$, diameter D, normal adjacency matrix A and $d+1$ distinct eigenvalues.

2. Main results

Definition 1 $\Gamma = (V, E)$ *is said to be a k-walk-regular digraph, for a given integer k $(0 \le k \le D)$, if the number of walks of length l, $a_{uv}^l = (A^l)_{uv}$, from vertex u to vertex v only depends on the distance from u to v, provided that this distance does not exceed k. In this case we just denote the number by a_k^l.*

Thus, in particular, the 0-walk-regular digraph coincides with the walk-regular digraph, where the number of cycles of length l rooted at a given vertex is a constant through all the digraph defined by Liu and Lin in [5] and the D-walk-regular digraph is the same as weakly distance-regular digraph defined by Comellas in [2].

For a given digraph Γ with adjacency matrix A, we consider the following scalar product in $\mathbf{C}[x]$:

$$\langle p, q\rangle = \frac{1}{n}\mathrm{tr}(p(A)q(A)^*).$$

This product is well defined in the quotient ring $\mathbf{C}[x]/(m(x))$, where $(m(x))$ is the ideal generated by the minimum polynomial of Γ, $m(x)$.

Proposition 1 *If Γ is a normal digraph with spectrum $sp\Gamma$ defined above, then we have*

$$\langle p, q\rangle = \frac{1}{n}\mathrm{tr}(p(A)q(A)^*) = \frac{1}{n}\sum_{k=0}^{d} m_k(p(\lambda_k)\overline{q(\lambda_k)}).$$

Proof A can be diagonalized by means of a unitary matrix, that is, $U^*AU = D$ for some matrix U such that $U^*U = I$ and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

$$\begin{aligned}\langle p, q\rangle &= \frac{1}{n}\mathrm{tr}(p(A)q(A)^*) = \frac{1}{n}\mathrm{tr}(Up(\Lambda)U^*(Uq(\Lambda)U^*)^*) = \frac{1}{n}\mathrm{tr}(Up(\Lambda)U^*Uq(\Lambda)^*U^*)\\ &= \frac{1}{n}\mathrm{tr}(Up(\Lambda)\overline{q(\Lambda)}U^*) = \frac{1}{n}\mathrm{tr}(p(\Lambda)\overline{q(\Lambda)}U^*U) = \frac{1}{n}\mathrm{tr}(p(\Lambda)\overline{q(\Lambda)})\\ &= \frac{1}{n}\sum_{k=0}^{d} m_k(p(\lambda_k)\overline{q(\lambda_k)}). \ \Box\end{aligned}$$

Notice that $1, x, x^2, \ldots, x^d$ are linearly independent in $\mathbf{C}_d[x]$, then by using the Gram-Schmidt method and normalizing appropriately, one can immediately prove the existence and the uniqueness of an orthogonal system of polynomials $\{p_k\}_{0\le k\le d}$ called predistance polynomials which, for any $0 \le h$, $k \le d$, satisfy:

(1) $dgr(p_k) = k$;

(2) $\langle p_h, p_k\rangle = 0$, if $h \ne k$;

(3) $\| p_k \|^2 = p_k(\lambda_0)$.

Recall that, in a weakly distance-regular digraph, we have $D = d$ and such polynomials satisfy $p_k(A) = A_k$ $(0 \le k \le d)$, where A_k stands for the distance-k matrix.

From the decomposition (1) we define the crossed uv-local multiplicity of eigenvalue λ_k as $m_{uv}(\lambda_k) = (E_k)_{uv}$ (which is similar to the definition in [3]). Furthermore,

$$m_{uv}(\lambda_k) = (E_k)_{uv} = \langle E_k \mathbf{e}_u, \mathbf{e}_v \rangle = \langle E_k \mathbf{e}_u, E_k \mathbf{e}_v \rangle = \langle \mathbf{z}_u^k, \mathbf{z}_v^k \rangle, \quad u, v \in V.$$

Now, for a given k, $0 \le k \le d$, if the crossed uv-local multiplicities of λ_h, $m_{uv}(\lambda_h)$, only depend on the distance from u to v, provided that $\partial(u,v) = i \le k$, we denote $m_{uv}(\lambda_h) \triangleq m_{ih}$.

Theorem 2 *Let Γ be a strongly connected normal digraph with predistance polynomials $p_0, p_1, \ldots, p_d$. Then the following statements are equivalent.*

(i) Γ is k-walk-regular;

(ii) The uv-local multiplicities of λ_h, $m_{uv}(\lambda_h)$, only depend on the distance from u to v, provided that $\partial(u,v) = i \le k$, that is, $m_{uv}(\lambda_h) = m_{ih}$.

Proof Since A^l can be expressed as a linear combination of the idempotents E_k : $A^l = \sum_{h=0}^{d} \lambda_h^l E_h$, we have that the number of walks a_{uv}^l can be computed in terms of the crossed uv-local multiplicities as

$$a_{uv}^l = (A^l)_{uv} = \sum_{h=0}^{d} \lambda_h^l (E_h)_{uv} = \sum_{h=0}^{d} m_{uv}(\lambda_h) \lambda_h^l.$$

Then if $m_{uv}(\lambda_h) = m_{ih}$ for any $u, v \in V$ such that $\partial(u,v) = i \le k$, and $l \ge 0$, $a_{uv}^l = \sum_{h=0}^{d} m_{ih} \lambda_h^l$ is independent of u, v, provided that $\partial(u,v) = i \le k$ and Γ is k-walk-regular.

Conversly, the crossed uv-local multiplicities are $m_{uv}(\lambda_h) = (E_h)_{uv} = (P_h(A))_{uv}$, where $P_h(A) = \frac{1}{\phi_h} \prod_{j=0, j \ne h}^{d} (A - \lambda_j I)$. If Γ is k-walk-regular, we have that there are constant uv-entries for $I, A, A^2, \ldots$, provided that $\partial(u,v) \le k$. Observe that $P_h(A)$ is a polynomial of A, so we have that $m_{uv}(\lambda_h)$ is a constant independent of u, v. $\square$

Proposition 2 *Let Γ be a strongly connected normal digraph with predistance polynomials $p_0, p_1, \ldots, p_d$. If Γ is k-walk-regular, then $p_i(A) = A_i$, for $0 \le i \le k$.*

Proof Suppose Γ is a k-walk-regular digraph. The number of walks with length i from vertex u to vertex v at distance i $(0 \le i \le k)$ is $a_{uv}^l = a_i^l$ (a constant). Hence

$$A^l = a_0^l I + a_1^l A + a_2^l A_2 + \cdots + a_i^l A_i, \quad 0 \le i \le k, \tag{8}$$

where, necessarily, $a_i^i \ne 0$ and as already mentioned, $a_j^i = 0$ for any $j > i$. In matrix form

$$\begin{pmatrix} I \\ A \\ A^2 \\ \vdots \\ A^k \end{pmatrix} = \begin{pmatrix} a_0^0 & 0 & 0 & 0 & \ldots & 0 \\ a_0^1 & a_1^1 & 0 & 0 & \ldots & 0 \\ a_0^2 & a_1^2 & a_2^2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_0^k & a_1^k & a_2^k & a_3^k & \ldots & a_k^k \end{pmatrix} \begin{pmatrix} I \\ A \\ A_2 \\ \vdots \\ A_k \end{pmatrix}, \tag{9}$$

where $C := (a_i^l)$ is a lower triangular matrix. Since $a_i^i > 0$ for any $0 \le i \le k$, the matrix C is non-singular and its inverse C^{-1} is also a lower triangular matrix. Hence A_i is a polynomial, say, q_i of degree of i in A for any $0 \le i \le k$:

$$A_i = q_i(A) = \alpha_0^i I + \alpha_1^i A + \alpha_2^i A^2 + \cdots + \alpha_k^i A^k, \quad 0 \le i \le k, \tag{10}$$

where $\alpha_i^i \neq 0$. These polynomials are orthogonal with respect to the scalar product since

$$\langle q_i, q_j \rangle = \frac{1}{n}\mathrm{tr}(q_i(A)q_j(A)^*) = \frac{1}{n}\mathrm{tr}(A_i A_j^T) = 0, \quad i \neq j.$$

Moreover, notice that $A_i\mathbf{j} = q_i(A)\mathbf{j} = q_i(\lambda_0)\mathbf{j}$ since $\mathbf{j}$ is an eigenvector of λ_0, it is easy to see the number of vertices at distance i, $0 \le i \le k$, from a given vertex u is a constant through all the digraph: $n_i = |\Gamma_i^+(u)| = q_i(\lambda_0)$ for every $u \in V$. Thus

$$\|q_i\|^2 = \langle q_i, q_i \rangle = \frac{1}{n}\mathrm{tr}(q_i(A)q_i(A)^*) = \frac{1}{n}\mathrm{tr}(A_i A_i^{\mathrm{T}}) = q_i(\lambda_0).$$

Therefore, the obtained polynomials are, in fact, the (pre)distance polynomials $q_i = p_i$, $0 \le i \le k$, for the uniqueness of the predistance polynomials. □

From the result above it is immediate to have

Proposition 3 *Let Γ be a k-walk-regular digraph and a strongly connected normal digraph. Then the number of vertices at distance k from any given vertex is equal to $p_k(\lambda_0)$, for each $k = 0, 1, \ldots, t$.*

Proof By the Proposition 2 we have that $A_i = p_i(A)$, $i = 0, 1, \ldots, k$ and Γ is a λ_0-regular digraph. Thus we have that $\mathbf{j} = (1, 1, \ldots, 1)^{\mathrm{T}}$ is an eigenvector of A corresponding to the eigenvalue λ_0. Consequently, $A_k\mathbf{j} = p_k(A)\mathbf{j} = p_k(\lambda_0)\mathbf{j}$, which implies that $n_k = p_k(\lambda_0)$, $k = 0, 1, \ldots, t$. □

Theorem 3 *Let Γ be a strongly connected normal digraph with predistance polynomials $p_0, p_1, \ldots, p_d$. Then the following two statements are equivalent:*

(i) Γ *is k-walk-regular;*

(ii) $(p_j(A))_{uv} = 0$, *for $k+1 \le j \le d$, $\partial(u, v) = i \le k$.*

Proof If Γ is k-walk-regular, then

$$(p_i(A)E_h)_{uu} = p_i(\lambda_h)(E_h)_{uu} = p_i(\lambda_h)m_{uu}(\lambda_h) = p_i(\lambda_h) \cdot m_{0h},$$

for any h with $0 \le h \le d$. But if $\partial(u, v) = i \le k$, we have already known that $p_i(A) = A_i$ and then

$$\begin{aligned}(p_i(A)E_h)_{uu} = (A_iE_h)_{uu} &= \sum_{v \in V}(A_i)_{uv}(E_h)_{vu} = \sum_{v \in V}(A_i)_{uv}(\overline{E_h})_{uv} \\ &= \sum_{v \in V}(A_i)_{uv}\overline{(E_h)_{uv}} = \sum_{v \in \Gamma_i^+(u)} \overline{m_{uv}(\lambda_h)} = n_i\overline{m_{ih}},\end{aligned}$$

where we have used the invariance of the crossed local multiplicities $m_{uv}(\lambda_h) = m_{ih}$, and the number of vertices at distance i from any given vertex $n_i = p_i(\lambda_0)$. So

$$m_{ih} = \frac{m_h\overline{p_i(\lambda_h)}}{np_i(\lambda_0)}, \quad 0 \le i \le k,\ 0 \le h \le d.$$

Therefore,

$$\begin{aligned}(p_j(A))_{uv} &= \sum_{h=o}^{d} p_j(\lambda_h)(E_h)_{uv} = \sum_{h=o}^{d} p_j(\lambda_h)m_{ih} = \frac{1}{np_i(\lambda_0)}\sum_{h=0}^{d} m_h p_j(\lambda_h)\overline{p_i(\lambda_h)} \\ &= \frac{1}{np_i(\lambda_0)}\langle p_j, p_i\rangle = 0, \quad i \le k < j.\end{aligned}$$

Conversly, assume that (ii) holds and for every h, $0 \le h \le d$. Now we consider the expression of $P_h = \sum_{j=0}^{d} \beta_{hj} p_j$, where β_{hj} is the coefficient of P_h in terms of p_j. If $\partial(u,v) = i \le k$,

$$\begin{aligned}m_{uv}(\lambda_h) = (E_h)_{uv} = (P_h(A))_{uv} &= \sum_{j=0}^{d} \beta_{hj}(p_j(A))_{uv} \\ &= \sum_{j=0}^{k} \beta_{hj}(A_j)_{uv} + \sum_{j=k+1}^{d} \beta_{hj}(p_j(A))_{uv} = \beta_{hi}.\end{aligned}$$

Thus, the crossed multiplicities, $m_{uv}(\lambda_h) = \beta_{hi}$, only depend on the distance from u to v. $\square$

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