**$k$-Walk-Regular Digraphs**

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**Abstract** In this paper, we define a class of strongly connected digraph, called the $k$-walk-regular digraph, study some properties of it, provide its some algebraic characterization and point out that the 0-walk-regular digraph is the same as the walk-regular digraph discussed by Liu and Lin in 2010 and the $D$-walk-regular digraph is identical with the weakly distance-regular digraph defined by Comellas et al in 2004.

**Keywords** $k$-walk-regular digraph; predistance polynomial; the crossed uc-local multiplicity.

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1. Introduction

Let $\Gamma = (V, E)$ be a strongly connected digraph, with $V$ denoting the set of vertices and $E$ the set of arcs. If $(u, v) \in E$, we say that $u$ is adjacent to $v$ (or $v$ is adjacent from $u$). A walk of length $l$ in $\Gamma$ is a sequence $(u_0, u_1, \ldots, u_l)$ of vertices such that $(u_{i-1}, u_i) \in E$, $i = 1, 2, \ldots, l$, and a walk is closed if its first and last vertices are the same. If the vertices in a walk are distinct, we call it a path. We say a digraph is strongly connected if any two vertices can be joined by a path. The number of arcs traversed in the shortest walk from $u$ to $v$ is called the distance from $u$ to $v$, denoted by $\partial(u, v)$, and we call the value $D := \max\{\partial(u, v) | u, v \in V\}$ the diameter of $\Gamma$. Notice that what we consider is the oriented graphs, so $\partial(u, v)$ and $\partial(v, u)$ may not be equal. For any fixed integer $0 \leq k \leq D$, we will denote by $\Gamma^+(u)$ (respectively, $\Gamma^-(u)$) the set of vertices at distance $k$ from $u$ (respectively, the set of vertices from which $u$ is at distance $k$). Sometimes it is written, for short, $\Gamma^+(u)$ or $\Gamma^-(u)$ instead of $\Gamma^+(v)$ or $\Gamma^-(v)$, respectively. Thus the out-valency and in-valency of $u$ are $k^+(u) := |\Gamma^+(u)|$ and $k^-(u) := |\Gamma^-(u)|$. The digraph $\Gamma$ is $k$-regular if $k^+(u) = k^-(u) = k$ for every $u \in V$.

The adjacency matrix $A$ and the distance-$k$ matrix $A_k$, where $0 \leq k \leq D$, of $\Gamma$ are defined...
by

\[ A_{uv} := \begin{cases} 1, & \text{if } (u, v) \in E, \\ 0, & \text{otherwise}, \end{cases} \]

and

\[ (A_k)_{uv} := \begin{cases} 1, & \text{if } \partial(u, v) = k, \\ 0, & \text{otherwise}, \end{cases} \]

respectively. If \( AA^T = A^T A, \) then \( \Gamma \) is said to be normal. As for normal matrices, there are the following properties:

**Theorem 1** Let \( A \) be an \( n \times n \) complex matrix with eigenvalues \( \lambda_0, \lambda_1, \ldots, \lambda_{n-1} \). Then \( A \) is normal if and only if any of the following assertions holds:

(a) \( U^* AU = \Lambda \) for some matrix \( U \) such that \( UU^* = I \), and \( \Lambda = \text{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) \), where \( U^* \) is the transpose of \( U \)'s conjugate;

(b) \( A^* = p(A) \) for some polynomial \( p \in \mathbb{C}[x] \);

(c) \( \text{tr}(AA^*) = \sum_{i=0}^{n-1} |\lambda_i|^2 \).

Now assume that \( A \) has \( d + 1 \) distinct eigenvalues \( \lambda_0, \lambda_1, \ldots, \lambda_d \) and \( m(x) = (x - \lambda_0)(x - \lambda_1) \cdots (x - \lambda_d) \) is the minimal polynomial of \( A \). The spectrum of the digraph \( \Gamma \), denoted by \( \text{sp} \Gamma \), consists of the eigenvalues of \( A \), which might be not real since \( A \) is not symmetric, together with their (algebraic) multiplicities:

\[ \text{sp} \Gamma = \{ \lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, \ldots, \lambda_d^{m(\lambda_d)} \}, \]

where the superscripts stand for the multiplicities \( m_i = m(\lambda_i) \) for \( i = 0, 1, \ldots, n - 1 \) and \( \lambda_0 \) is the maximal eigenvalue and \( \lambda_i \neq \lambda_j \) if \( i \neq j \). In particular, \( m(\lambda_0) = 1 \) and \( m(\lambda_0) + m(\lambda_1) + \cdots + m(\lambda_d) = n \) (for details the readers can see [1]).

By the Perron-Frobenius theorem, \( \lambda_0 \) is simple and has a positive eigenvector \( \nu \). If \( \Gamma \) is \( k \)-regular, then we may pick \( \nu = j \), where \( j \) denotes the all 1-vector, and \( \lambda_0 = k \).

The adjacency algebra of \( \Gamma \) (also called Bose-Mesner algebra when it is closed under the Hadamard-or componentwise-product) is defined by \( \mathcal{A}(\Gamma) := \{ p(A) | p \in \mathbb{C}[x] \} \), where \( A \) is the adjacency matrix of \( \Gamma \). The dimension of \( \mathcal{A}(\Gamma) \), as a \( \mathbb{C} \)-vector space, equals the degree of the minimum polynomial \( m(x) \). It is obvious that \( \{ I, A, A^2, \ldots, A^d \} \) is a basis of the adjacency or Bose-Mesner algebra \( \mathcal{A}(\Gamma) \) and the dimension of the Bose-Mesner algebra is at least \( D + 1 \) since the powers \( I, A, A^2, \ldots, A^D \) are linearly independent. By Theorem 1(a), we know that the eigenvectors of a normal \( n \times n \) square matrix constitute an orthogonal basis of the vector space \( \mathbb{C}^n \), with inner product \( \langle u, v \rangle = u^* v \). For each polynomial \( p \in \mathbb{C}[x] \) we define \( p \) operates on the vector \( v \in \mathbb{C}^n \) by \( pv = p(A)v \). For each \( \lambda_i \), let \( U_i \) be the matrix whose columns form an orthonormal basis of the eigenspace \( V_i := \text{Ker}(A - \lambda_i I) \). Then the orthogonal projection onto \( V_i \) is represented by the matrix \( E_i = U_i U_i^* \), or alternatively, \( E_i = \frac{1}{\phi_i} \prod_{j=0,j\neq i}^d (A - \lambda_j I) \), where \( \phi_i = \prod_{j=0,j\neq i}^d (\lambda_i - \lambda_j) \). These matrices are called the principal idempotents of \( A \) and satisfy the following properties: \( E_i E_j = \delta_{ij} E_i, AE_i = \lambda_i E_i \). Also \( \{ E_0, E_1, \ldots, E_d \} \) is a basis of \( \mathcal{A}(\Gamma) \). Then we can give the orthogonal decomposition of the unitary vector \( e_u \) representing vertex \( u \)
as follows:

\[ e_u = z_u^0 + z_u^1 + \ldots + z_u^d, \]

where \( z_u^i = E_i e_u, \ i = 0, 1, \ldots, d. \)

Throughout this paper, we assume that \( \Gamma \) is a strongly connected digraph with order \( n = |V| \), size \( m = |E| \), diameter \( D \), normal adjacency matrix \( A \) and \( d + 1 \) distinct eigenvalues.

2. Main results

**Definition 1** \( \Gamma = (V, E) \) is said to be a \( k \)-walk-regular digraph, for a given integer \( k \) (\( 0 \leq k \leq D \)), if the number of walks of length \( l \), \( a_{l uv}^k = (A^l)_{uv} \), from vertex \( u \) to vertex \( v \) only depends on the distance from \( u \) to \( v \), provided that this distance does not exceed \( k \). In this case we just denote the number by \( a_{l k} \).

Thus, in particular, the 0-walk-regular digraph coincides with the walk-regular digraph, where the number of cycles of length \( l \) rooted at a given vertex is a constant through all the digraph defined by Liu and Lin in [5] and the \( D \)-walk-regular digraph is the same as weakly distance-regular digraph defined by Comellas in [2].

For a given digraph \( \Gamma \) with adjacency matrix \( A \), we consider the following scalar product in \( \mathbb{C}[x] \):

\[ \langle p, q \rangle = \frac{1}{n} \text{tr}(p(A)q(A^*)). \]

This product is well defined in the quotient ring \( \mathbb{C}[x]/(m(x)) \), where \( (m(x)) \) is the ideal generated by the minimum polynomial of \( \Gamma \), \( m(x) \).

**Proposition 1** If \( \Gamma \) is a normal digraph with spectrum \( \text{sp} \Gamma \) defined above, then we have

\[ \langle p, q \rangle = \frac{1}{n} \text{tr}(p(A)q(A^*)) = \frac{1}{n} \sum_{k=0}^{d} m_k(p(\lambda_k)q(\lambda_k)). \]

**Proof** A can be diagonalized by means of a unitary matrix, that is, \( U^*AU = D \) for some matrix \( U \) such that \( U^*U = I \) and \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \).

\[ \langle p, q \rangle = \frac{1}{n} \text{tr}(p(A)q(A^*)) = \frac{1}{n} \text{tr}(Up(\Lambda)U^*(Uq(\Lambda)U^*)) = \frac{1}{n} \text{tr}(Up(\Lambda)U^*Uq(\Lambda)U^*) = \frac{1}{n} \text{tr}(Up(\Lambda)q(\Lambda)U^*) = \frac{1}{n} \text{tr}(p(\Lambda)q(\Lambda)) \]

\[ = \frac{1}{n} \sum_{k=0}^{d} m_k(p(\lambda_k)q(\lambda_k)). \square \]

Notice that \( 1, x, x^2, \ldots, x^d \) are linearly independent in \( \mathbb{C}_d[x] \), then by using the Gram-Schmidt method and normalizing appropriately, one can immediately prove the existence and the uniqueness of an orthogonal system of polynomials \( \{p_k\}_{0 \leq k \leq d} \) called predistance polynomials which, for any \( 0 \leq h, k \leq d \), satisfy:

1. \( dgr(p_k) = k; \)
2. \( \langle p_h, p_k \rangle = 0, \) if \( h \neq k; \)
3. \( \| p_k \|^2 = p_k(\lambda_0). \)
Recall that, in a weakly distance-regular digraph, we have $D = d$ and such polynomials satisfy $p_k(A) = A_k$ ($0 \leq k \leq d$), where $A_k$ stands for the distance-$k$ matrix.

From the decomposition (1) we define the crossed $uv$-local multiplicity of eigenvalue $\lambda_k$ as $m_{uv}(\lambda_k) = (E_k)_{uv}$ (which is similar to the definition in [3]). Furthermore,

$$m_{uv}(\lambda_k) = (E_k)_{uv} = (E_k e_u, e_v) = (E_k e_u, E_k e_v) = (z_u^k, z_v^k), \quad u, v \in V.$$

Now, for a given $k$, $0 \leq k \leq d$, if the crossed $uv$-local multiplicities of $\lambda_h$, $m_{uv}(\lambda_h)$, only depend on the distance from $u$ to $v$, provided that $\partial(u, v) = i \leq k$, we denote $m_{uv}(\lambda_h) \triangleq m_{ih}$.

**Theorem 2** Let $\Gamma$ be a strongly connected normal digraph with predistance polynomials $p_0, p_1, \ldots, p_d$. Then the following statements are equivalent.

(i) $\Gamma$ is $k$-walk-regular;

(ii) The $uv$-local multiplicities of $\lambda_h$, $m_{uv}(\lambda_h)$, only depend on the distance from $u$ to $v$, provided that $\partial(u, v) = i \leq k$, that is, $m_{uv}(\lambda_h) = m_{ih}$.

**Proof** Since $A^l$ can be expressed as a linear combination of the idempotents $E_h : A^l = \sum_{h=0}^d \lambda_h^l E_h$, we have that the number of walks $a_{uv}^l$ can be computed in terms of the crossed $uv$-local multiplicities as

$$a_{uv}^l = (A^l)_{uv} = \sum_{h=0}^d \lambda_h^l (E_h)_{uv} = \sum_{h=0}^d m_{uv}(\lambda_h) \lambda_h^l.$$ 

Then if $m_{uv}(\lambda_h) = m_{ih}$ for any $u, v \in V$ such that $\partial(u, v) = i \leq k$, and $l \geq 0$, $a_{uv}^l = \sum_{h=0}^d m_{ih} \lambda_h^l$ is independent of $u, v$, provided that $\partial(u, v) = i \leq k$ and $\Gamma$ is $k$-walk-regular.

Conversly, the crossed $uv$-local multiplicities are $m_{uv}(\lambda_h) = (E_h)_{uv} = (P_h(A))_{uv}$, where $P_h(A) = \frac{1}{\partial_h} \prod_{j=0,j \neq h}^d (A - \lambda_j I)$. If $\Gamma$ is $k$-walk-regular, we have that there are constant $uv$-entries for $I, A, A^2, \ldots$, provided that $\partial(u, v) \leq k$. Observe that $P_h(A)$ is a polynomial of $A$, so we have that $m_{uv}(\lambda_h)$ is a constant independent of $u, v$. $\Box$

**Proposition 2** Let $\Gamma$ be a strongly connected normal digraph with predistance polynomials $p_0, p_1, \ldots, p_d$. If $\Gamma$ is $k$-walk-regular, then $p_i(A) = A_i$, for $0 \leq i \leq k$.

**Proof** Suppose $\Gamma$ is a $k$-walk-regular digraph. The number of walks of length $i$ from vertex $u$ to vertex $v$ at distance $i$ ($0 \leq i \leq k$) is $a_{uv}^i = a_i^i$ (a constant). Hence

$$A^i = a_0^i I + a_1^i A + a_2^i A^2 + \cdots + a_k^i A_k, \quad 0 \leq i \leq k, \quad (8)$$

where, necessarily, $a_i^i \neq 0$ and as already mentioned, $a_j^i = 0$ for any $j > i$. In matrix form

$$
\begin{pmatrix}
I \\
A \\
A^2 \\
\vdots \\
A^k
\end{pmatrix} =
\begin{pmatrix}
a_0^0 & 0 & 0 & 0 & \cdots & 0 \\
a_0^1 & a_1^1 & 0 & 0 & \cdots & 0 \\
a_0^2 & a_1^2 & a_2^2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_0^k & a_1^k & a_2^k & a_3^k & \cdots & a_k^k
\end{pmatrix}
\begin{pmatrix}
I \\
A \\
A^2 \\
\vdots \\
A^k
\end{pmatrix}, \quad (9)
$$
where \( C := (a_{ij}^r) \) is a lower triangular matrix. Since \( a_{ij}^r > 0 \) for any \( 0 \leq i \leq k \), the matrix \( C \) is non-singular and its inverse \( C^{-1} \) is also a lower triangular matrix. Hence \( A_i \) is a polynomial, say, \( q_i \) of degree of \( i \) in \( A \) for any \( 0 \leq i \leq k \):

\[
A_i = q_i(A) = a_0^r I + a_1^r A + a_2^r A^2 + \cdots + a_k^r A^k, \quad 0 \leq i \leq k,
\]

where \( a_i^r \neq 0 \). These polynomials are orthogonal with respect to the scalar product since

\[
\langle q_i, q_j \rangle = \frac{1}{n} \text{tr}(q_i(A)q_j(A)^*) = \frac{1}{n} \text{tr}(A_iA_j^T) = 0, \quad i \neq j.
\]

Moreover, notice that \( A_i j = q_i(A) j = q_i(\lambda_0) j \) since \( j \) is an eigenvector of \( \lambda_0 \), it is easy to see the number of vertices at distance \( i \), \( 0 \leq i \leq k \), from a given vertex \( u \) is a constant through all the digraph: \( n_i = |\Gamma^+_i(u)| = q_i(\lambda_0) \) for every \( u \in V \). Thus

\[
\|q_i\|^2 = \langle q_i, q_i \rangle = \frac{1}{n} \text{tr}(q_i(A)q_i(A)^*) = \frac{1}{n} \text{tr}(A_iA_i^T) = q_i(\lambda_0).
\]

Therefore, the obtained polynomials are, in fact, the (pre)distance polynomials \( q_i = p_i, 0 \leq i \leq k \), for the uniqueness of the predistance polynomials. \( \square \)

From the result above it is immediate to have

**Proposition 3** Let \( \Gamma \) be a \( k \)-walk-regular digraph and a strongly connected normal digraph. Then the number of vertices at distance \( k \) from any given vertex is equal to \( p_k(\lambda_0) \), for each \( k = 0, 1, \ldots, t \).

**Proof** By the Proposition 2 we have that \( A_i = p_i(A), i = 0, 1, \ldots, k \) and \( \Gamma \) is a \( \lambda_0 \)-regular digraph. Thus we have that \( j = (1,1, \ldots, 1)^T \) is an eigenvector of \( A \) corresponding to the eigenvalue \( \lambda_0 \). Consequently, \( A_k j = p_k(A) j = p_k(\lambda_0) j \), which implies that \( n_k = p_k(\lambda_0), k = 0, 1, \ldots, t \). \( \square \)

**Theorem 3** Let \( \Gamma \) be a strongly connected normal digraph with predistance polynomials \( p_0, p_1, \ldots, p_t \). Then the following two statements are equivalent:

(i) \( \Gamma \) is \( k \)-walk-regular;

(ii) \( (p_j(A))_{uv} = 0 \), for \( k + 1 \leq j \leq d \), \( \partial(u, v) = i \leq k \).

**Proof** If \( \Gamma \) is \( k \)-walk-regular, then

\[
(p_i(A)E_h)_{uv} = p_i(\lambda_h)(E_h)_{uv} = p_i(\lambda_h)m_{uu}(\lambda_h) = p_i(\lambda_h) \cdot m_{0h},
\]

for any \( h \) with \( 0 \leq h \leq d \). But if \( \partial(u, v) = i \leq k \), we have already known that \( p_i(A) = A_i \) and then

\[
(p_i(A)E_h)_{uv} = (A_iE_h)_{uv} = \sum_{v \in V}(A_i)_{uv}(E_h)_{vu} = \sum_{v \in V}(A_i)_{uv}(E_h)_{uv} = \sum_{v \in V}(A_i)_{uv}(\lambda_h)_{uv} = n_i m_{ih},
\]

where we have used the invariance of the crossed local multiplicities \( m_{uv}(\lambda_h) = m_{ih} \), and the
number of vertices at distance $i$ from any given vertex $n_i = p_i(\lambda_0)$. So

$$m_{ih} = \frac{m_h p_i(\lambda_h)}{np_i(\lambda_0)}, \quad 0 \leq i \leq k, \ 0 \leq h \leq d.$$ 

Therefore,

$$ (p_j(A))_{uv} = \sum_{h=0}^{d} p_j(\lambda_h)(E_h)_{uv} = \sum_{h=0}^{d} p_j(\lambda_h)m_{ih} = \frac{1}{np_i(\lambda_0)} \sum_{h=0}^{d} m_h p_j(\lambda_h)p_i(\lambda_h) 

= \frac{1}{np_i(\lambda_0)} (p_j, p_i) = 0, \quad i \leq k < j. $$

Conversely, assume that (ii) holds and for every $h$, $0 \leq h \leq d$. Now we consider the expression of $P_h = \sum_{j=0}^{d} \beta_{hj}p_j$, where $\beta_{hj}$ is the coefficient of $P_h$ in terms of $p_j$. If $\partial(u, v) = i \leq k$,

$$ m_{uv}(\lambda_h) = (E_h)_{uv} = (P_h(A))_{uv} = \sum_{j=0}^{d} \beta_{hj}(p_j(A))_{uv} 

= \sum_{j=0}^{k} \beta_{hj}(A_j)_{uv} + \sum_{j=k+1}^{d} \beta_{hj}(p_j(A))_{uv} = \beta_{hi}. $$

Thus, the crossed multiplicities, $m_{uv}(\lambda_h) = \beta_{hi}$, only depend on the distance from $u$ to $v$. □

References