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Minimality of Complex Exponential System

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Abstract A sufficient condition is obtained for the minimality of the complex exponential system $E(\Lambda, M) = \{z^l e^{\lambda_n z} : l = 0, 1, \ldots, m_n - 1;\ n = 1, 2, \ldots\}$ in the Banach space L^p_α consisting of all functions f such that $f^{-\alpha} \in L^p(\mathbb{R})$. Moreover, if the incompleteness holds, each function in the closure of the linear span of exponential system $E(\Lambda, M)$ can be extended to an analytic function represented by a Taylor-Dirichlet series.

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1. Introduction

Following Deng [1], a system $E = \{e_n : n = 1, 2, \ldots\}$ of elements of a Banach space X is called (i) incomplete in X if $\overline{\text{span}} E \neq X$; (ii) minimal in X if for all $n = 1, 2, \ldots$, $e_n \notin \overline{\text{span}}(E - \{e_n\})$, where span$E$ is the linear span of the system E and $\overline{\text{span}} E$ is the closure of span$E$ in X. The incompleteness of the system E in X is equivalent to the existence of a non-trivial functional f in the dual Banach space X^* of X which annihilates the system E, i.e., $f(e_n) = 0, n = 1, 2, \ldots$. The minimality of the system E in X is equivalent to the existence of a system of conjugate functionals $\{f_n : n = 1, 2, \ldots\}$ in X^*, i.e., $f_n(e_m) = \delta_{nm}$ (Kronneker delta, i.e., $\delta_{nn} = 1$, while $\delta_{nm} = 0$ for $n \neq m$). The system $\{f_n\}$ is also called a biorthogonal system of the system E.

Let $\alpha(t)$ be a continuous function $\mathbb{R}$ and satisfy

$$\lim_{t\to\infty} \frac{\alpha(t)}{t} = \infty, \quad a_0 = \limsup_{t\to-\infty} \frac{|\alpha(t)|}{|t|} < \infty. \tag{1}$$

Let

$$L^p_\alpha = \Big\{ f : \|f\|_\alpha = \Big(\int_{-\infty}^{\infty} |f(t) e^{-\alpha(t)}|^p \mathrm{d}t \Big)^{\frac{1}{p}} < \infty \Big\}, \quad 1 \le p < +\infty.$$

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Then L^p_α is a Banach space. Let $\Lambda = \{\lambda_n : n = 1, 2, \ldots\}$ be a sequence of distinct complex numbers in the open right half plane $\mathbb{C}_{a_0} = \{z = x + iy : x > a_0\}$ and satisfy

$$a_1 = \limsup_{n\to\infty} |\theta_n| < \frac{\pi}{2}. \tag{2}$$

Let $M = (m_n : n = 1, 2, \ldots)$ be a sequence of positive integers and suppose that there exists an increasing positive function $q(r)$ on $[0, \infty)$ satisfying

$$a_2 = \limsup_{r\to\infty} q(r)r^{-1} \log r < \infty \tag{3}$$

and

$$D(q) = \limsup_{r\to\infty} \frac{n(r + q(r)) - n(r)}{q(r)} < \infty, \tag{4}$$

where $n(t) = \sum_{|\lambda_n|\le t} m_n$ is the counting function of Λ and M. With these sequences Λ and M, we associate the system of complex exponentials

$$E(\Lambda, M) = \{t^{k-1}e^{\lambda_n t} : k = 1, 2, \ldots, m_n; n = 1, 2, \ldots\}.$$

The condition (1) guarantees that $E(\Lambda, M)$ is a subset of L^p_α. The author in [2] has obtained some results on completeness of $E(\Lambda, M)$ in L^p_α.

Theorem A ([2]) *Let $\alpha(t)$ be continuous on $\mathbb{R}$, convex on $[t_0, \infty)$ for some constant t_0 and satisfy (1). Suppose that $\Lambda = \{\lambda_n : n = 1, 2, \ldots\}$ is a sequence of distinct complex numbers in $\mathbb{C}_{a_0}$ satisfying (2) and $M = (m_n : n = 1, 2, \ldots)$ is a sequence of positive integers. If there exists a positive and increasing function $q(r)$ on $[0, \infty)$ such that (3) and (4) hold, then $E(\Lambda, M)$ is incomplete in L^p_α if and only if there exists $a \in \mathbb{R}$ such that*

$$J(a) = \int_0^{+\infty} \frac{\alpha(\lambda(t) + a)}{1 + t^2} \mathrm{d}t < \infty, \tag{5}$$

where

$$\lambda(r) = \begin{cases} 2 \displaystyle\sum_{|\lambda_n|\le r} \frac{m_n \cos\theta_n}{|\lambda_n|} , & \text{if } r \ge |\lambda_1|, \\ 0 , & \text{otherwise} . \end{cases} \tag{6}$$

In this paper, we study the minimality of $E(\Lambda, M)$ in L^p_α, the main conclusions are as follows:

Theorem *Let $\alpha(t)$ be continuous on $\mathbb{R}$, convex on $[t_0, \infty)$ for some constant t_0 and satisfy (1). Suppose that $\Lambda = \{\lambda_n : n = 1, 2, \ldots\}$ is a sequence of distinct complex numbers in $\mathbb{C}_{a_0}$ such that (2) and*

$$a_3 = \liminf_{n\to\infty} \frac{\inf\{\log |\lambda_k - \lambda_n| : k \ne n\}}{\log |\lambda_n|} > -\infty \tag{7}$$

hold. Suppose that $M = (m_n : n = 1, 2, \ldots)$ is a sequence of positive integers and there exists a positive and increasing function $q(r)$ on $[0, \infty)$ such that (3) and (4) hold. If $E(\Lambda, M)$ is incomplete in L^p_α, then $E(\Lambda, M)$ is minimal and each function $f \in \overline{\operatorname{span}}\, E(\Lambda, M)$ can be extended to an entire function $g(z)$ represented by a Taylor-Dirichlet series

$$g(z) = \sum_{n=1}^{\infty} \sum_{k=0}^{m_n-1} a_{n,k} z^k e^{\lambda_n z}. \tag{8}$$

Remark Some similar results were obtained in [3–5].

2. Proof of Theorem

In order to prove Theorem, we need the following technical lemma:

Lemma 1 ([1]) *Suppose that $\Lambda = \{\lambda_n = |\lambda_n|e^{i\theta_n} : n = 1, 2, \ldots\}$ is a sequence of distinct complex numbers in $\mathbb{C}_{a_0}$ satisfying (2), and $M = (m_n : n = 1, 2, \ldots)$ is a sequence of positive integers. If there exists a positive and increasing function $q(r)$ on $[0, \infty)$ such that (3) and (4) hold, then for each $b > 0$, the function*

$$G_b(z) = Q_b(z) \prod_{\mathrm{Re}\,\lambda_n > b} \Big(\frac{1 - \frac{z}{\lambda_n}}{1 + \frac{z}{\overline{\lambda}_n}}\Big)^{m_n} \exp\Big(\frac{2zm_n \cos\theta_n}{|\lambda_n|}\Big) \tag{9}$$

is analytic in the half-plane $\mathbb{C}_{-b} = \{z = x + iy : x > -b\}$ with zeros of orders m_n at each point $\lambda_n (n = 1, 2, \ldots)$ and satisfies the following inequality:

$$|G_b(z)| \le \exp\{|x|\lambda(2r) + A|x| + A\}, \quad z \in \mathbb{C}_{-b}, \tag{10}$$

where

$$Q_b(z) = \prod_{|\mathrm{Re}\,\lambda_n| \le b} \Big(\frac{z - \lambda_n}{z + b + 1}\Big)^{m_n}.$$

Moreover, for each positive constant A_0 and $\varepsilon_0 > 0$,

$$|G_b(z)| \ge \exp\{x\lambda(r) - A|x| - A\}, \quad z \in C(A_0, \varepsilon_0), \tag{11}$$

where $C(A_0, \varepsilon_0) = \{z \in \mathbb{C}_{-b} : |z - \lambda_n| \ge \delta_n, n = 1, 2, \ldots\}$, $\delta_n = \varepsilon_0 |\lambda_n|^{-A_0}$, $n = 1, 2, \ldots$.

Proof of Theorem By (7), there exist positive constants ε_0 and A_0 and disjoint open discs $D_n = \{z : |z - \lambda_n| < \delta_n\}$ $(n = 1, 2, \ldots)$, where $\delta_n = \frac{\varepsilon_0}{|\lambda_n|^{A_0}} : n = 1, 2, \ldots$. If $M(\Lambda, E)$ is incomplete in L^p_α, by Theorem A there exists a real number a such that (5) holds. If $J(a) < \infty$ for all a, where $J(a)$ is defined by (5) (for example, if $\lambda(r)$ is bounded, then $J(a) < \infty$ for all a), then there exists a sequence $\{r_n : n = 1, 2, \ldots\}$ with $r_{n+1} > 1 + 2r_n$ $(n = 1, 2, \ldots)$ such that

$$\int_{r_n}^{+\infty} \frac{\alpha(\lambda(t) + n)}{1 + t^2} \mathrm{d}t < \frac{1}{2^n}, \quad n = 1, 2, \ldots.$$

It follows that there exists a set $\widetilde{\Lambda} = \{\widetilde{\lambda}_n : n = 1, 2, \ldots\}$ such that the following conditions hold:

i) $\widetilde{\lambda}(r)$ is unbounded on $(0, \infty)$, where

$$\widetilde{\lambda}(r) = 2 \sum_{\widetilde{\lambda}_n \le r} \frac{1}{\widetilde{\lambda}_n} \ (r \ge \widetilde{\lambda}_1), \text{ and } \widetilde{\lambda}(r) = 0 \ (r < \widetilde{\lambda}_1);$$

ii) $\liminf_{n\to\infty}(\widetilde{\lambda}_{n+1} - \widetilde{\lambda}_n) > 0$;

iii) $\inf\{|\lambda_m - \widetilde{\lambda}_n| : m, n = 1, 2, \ldots\} > 0$ and

iv) $\int_0^{+\infty} \frac{\alpha(\lambda(t)+\widetilde{\lambda}(t))}{1+t^2} \mathrm{d}t < \infty$.

So we can assume that $\lambda(r)$ is unbounded (if necessary, replace $\lambda(r)$ by $\lambda(r) + \widetilde{\lambda}(r)$), and (5) holds for $a = 0$ in the following. Let $\varphi(t)$ be an even function such that $\varphi(t) = \alpha(\lambda(t))$ for $t \ge 0$.

Let $u(z)$ be the Poisson integral of $2\varphi(t)$, i.e.,

$$u(z)=\frac{x}{\pi}\int_{-\infty}^{+\infty}\frac{2\varphi(t)}{x^2+(y-t)^2}\mathrm{d}t. \tag{12}$$

Then $u(x+iy)$ is harmonic in the half-plane $\mathbb{C}_0=\{z=x+iy:x>0\}$ satisfying

$$u(z)\geq\frac{x}{\pi}\int_{|t|\geq|z|}\frac{2\varphi(|z|)}{x^2+(y-t)^2}\mathrm{d}t=\varphi(|z|),\quad x>0.$$

Therefore, there exists an analytic function $g(z)$ on $\mathbb{C}_0$ such that $\operatorname{Re} g(z)=u(z)\geq\varphi(|z|)$ $(x>0)$ and

$$\lim_{r\to\infty}r^{-1}\max\{\operatorname{Re} g(b+re^{i\theta}):|\theta|\leq\frac{\pi}{2}-\varepsilon\}=0, \tag{13}$$

for each $b>0$ and $\varepsilon\in(0,\frac{\pi}{2})$. Let $b>a_0+2$ and let

$$\varphi_b(z)=\frac{G_b(z)}{(1+z+b)^b}\exp\{-g(z+b)\}, \tag{14}$$

where $G_b(z)$ is defined by (9). Then $\varphi_b(z)$ is analytic in $\mathbb{C}_{-b}=\{z=x+iy:x>-b\}$. Since $\varphi(r)\geq(x-1)(\lambda(r)-a)-\alpha^*(x-1)$ for $r=|z|$ with $\lambda(r)+a>t_0$ (if $\lambda(r)$ is bounded, we can take a real number a such that $\lambda(r)+a>t_0$), by (14), there exists a positive constant A_2 such that

$$|\varphi_b(z)|\leq\frac{1}{1+|z|^2}\exp\{\alpha^*(x-1)+A_2x\},\quad x>-b \tag{15}$$

holds, where α^* is the Legendre transform of α (see [6]). Let $A_{n,j}$ be the coefficient of the principal part of the Laurent series for the function $\frac{e^{A_2z}}{\varphi_b(z)}$ in $D_n-\{\lambda_n\}$, i.e.,

$$\frac{e^{A_2z}}{\varphi_b(z)}=\sum_{j=1}^{m_n}\frac{A_{n,j}}{(z-\lambda_n)^j}+\widetilde{\varphi}_n(z), \tag{16}$$

where $\widetilde{\varphi}_n(z)$ is analytic in D_n. Then by Cauchy's formula,

$$A_{n,j}=\frac{1}{2\pi i}\int_{|z-\lambda_n|=\delta_n}\frac{e^{A_2z}}{\varphi_b(z)}(z-\lambda_n)^{j-1}\mathrm{d}z. \tag{17}$$

According to (11) and (13),

$$\max\{|A_{n,j}|:1\leq j\leq m_n\}\leq\exp\{-\operatorname{Re}\lambda_n\lambda(|\lambda_n|)+A\operatorname{Re}\lambda_n+A\}. \tag{18}$$

Note that A is independent of x and λ_n. Now consider the analytic functions in $\mathbb{C}_{-b}$

$$H_{n,k}(z)=\varphi_b(z)\exp\{-A_2z\}\sum_{l=1}^{m_n-k}\frac{A_{n,k+l}}{k!(z-\lambda_n)^l},\quad k=0,1,\ldots,m_n-1;\ n=1,2,\ldots. \tag{19}$$

By (15) and (18), we have, for $x>-b$,

$$|H_{n,k}(z)|\leq\frac{A}{1+|z|^2}\exp\{\alpha^*(x-1)-\operatorname{Re}\lambda_n\lambda(|\lambda_n|)+A\operatorname{Re}\lambda_n\}. \tag{20}$$

Let

$$h_{n,k}(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}H_{n,k}(iy)e^{-iyt}\mathrm{d}y.$$

Then $h_{n,k}(t)$ is bounded and continuous on $\mathbb{R}$. By Cauchy's formula,

$$h_{n,k}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} H_{n,k}(x+iy) e^{-(x+iy)t} \mathrm{d}y, \quad x > -b.$$

By (20) and the formula of the Legendre transform $(\alpha^*)^* = \alpha$, for $x > -b$, we obtain

$$|h_{n,k}(t) e^{\alpha(t)}| \le \exp\{A + A\mathrm{Re}\,\lambda_n - \mathrm{Re}\,\lambda_n \lambda(|\lambda_n|) - |t|\}. \tag{21}$$

The function $h_{n,k}(t)e^{xt}$ can be regarded as the Fourier transform of $H_{n,k}(x+iy)$. Consequently,

$$H_{n,k}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h_{n,k}(t) e^{tz} \mathrm{d}t, \quad \mathrm{Re}\, z > -b.$$

Next we will prove that

$$H_{n,k}^{(l)}(\lambda_j) = \delta_{nj}\delta_{kl}, \quad \text{i.e.,}\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\lambda_j t} t^l h_{n,k}(t) \mathrm{d}t = \delta_{nj}\delta_{kl}, \tag{22}$$

where $l \in \{0, 1, \ldots, m_j - 1\}$, $k \in \{0, 1, \ldots, m_n - 1\}$, $n, j \in \mathbb{N}$. It is obvious that if $j \ne n$, then $H_{n,k}^{(l)}(\lambda_j) = 0$ $(l = 0, 1, \ldots, m_j - 1)$. If $j = n$, then by (18), for $z \in D_n$,

$$\begin{aligned}
H_{n,k}(z) &= \varphi_b(z) e^{-A_2 z} \sum_{l=k+1}^{m_n} \frac{A_{n,l}}{k!(z-\lambda_n)^l} \\
&= \varphi_b(z) e^{-A_2 z} \frac{(z-\lambda_n)^k}{k!} \Big(\frac{e^{A_2 z}}{\varphi_b(z)} - \sum_{l=1}^{k} \frac{A_{n,l}}{(z-\lambda_n)^l} - \varphi_n(z) \Big) \\
&= \frac{(z-\lambda_n)^k}{k!} + \sum_{l=m_n}^{+\infty} B_{n,l} (z-\lambda_n)^l,
\end{aligned}$$

where $k = 0, 1, \ldots, m_n - 1$; $n = 1, 2, \ldots$. This proves (22). Define a linear functional $T_{n,k}$ on $\mathrm{span}\, E(\Lambda, M)$ by

$$T_{n,k}(P) = a_{n,k} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sum a_{j,l} h_{n,k}(t) t^l e^{\lambda_j t} \mathrm{d}t$$

for each exponential polynomial $P(t) = \sum_{j,l} a_{j,l} t^l e^{\lambda_j t} \in \mathrm{span}\, E(\Lambda, M)$. By (21),

$$|T_{n,k}(P)| \le 2||P||_\alpha \exp\{A + A\mathrm{Re}\,\lambda_n - \mathrm{Re}\,\lambda_n \lambda(|\lambda_n|)\}.$$

Hence $T_{n,k}$ is a bounded linear functional on $E(\Lambda, M)$, so by Hahn-Banach theorem $T_{n,k}$ can be extended to a bounded linear functional (denoted by $\overline{T}_{n,k}$) on L_α^p and

$$\|\overline{T}_{n,k}\| = \|T_{n,k}\| \le C_n = 2\exp\{A + A\mathrm{Re}\,\lambda_n - \mathrm{Re}\,\lambda_n \lambda(|\lambda_n|)\}. \tag{23}$$

So $\{\overline{T}_{n,k}; 1 \le k \le m_n, n = 1, 2, \ldots\}$ is a biorthogonal system of $E(\Lambda, M)$ in $(L_\alpha^p)^*$ and $E(\Lambda, M)$ is minimal in L_α^p. If $f \in \overline{\mathrm{span}}\, E(\Lambda, M)$, there exists a sequence of exponential polynomials

$$P_j(t) = \sum_{n=1}^{j} \sum_{k=0}^{m_n - 1} a_{n,k}^j t^k e^{\lambda_n t} \in \mathrm{span}\, E(\Lambda, M)$$

such that

$$\lim_{j \longrightarrow \infty} \|f - P_j\|_\alpha = 0, \tag{24}$$

where $a_{n,k}=\overline{T}_{n,k}(f)$. Since $|\overline{T}_{n,k}(f)|\le \|\overline{T}_{n,k}\|\|f\|_\alpha$ and $\lambda(r)$ is unbounded on $(0,\infty)$, by(21), the function $g(z)$ defined by (8) is an entire function. Note that

$$|a_{n,k}-a_{n,k}^j|=|\overline{T}_{n,k}(f)-\overline{T}_{n,k}(P_j)|\le C_n\|f-P_j\|_\alpha,\quad n=1,2,\ldots.$$

For any real numbers a,b $(a<b)$, we have

$$\begin{aligned}
&\Big(\int_a^b |(f(t)-g(t))e^{-\alpha(t)}|^p\mathrm{d}t\Big)^{\frac{1}{p}}\le \|f-P_j\|_\alpha+\|P_j-g\|_\alpha\\
&\le \|f-P_j\|_\alpha+\sum_{n=1}^{j}\sum_{k=0}^{m_n-1}|a_{n,k}-a_{n,k}^j|\Big(\int_a^b e^{Re\lambda_n t-p\alpha(t)}t^{kp}\mathrm{d}t\Big)^{\frac{1}{p}}+\\
&\sum_{n=j+1}^{+\infty}\sum_{k=0}^{m_n-1}|a_{n,k}|\Big(\int_a^b e^{\mathrm{Re}\,\lambda_n t-p\alpha(t)}t^{kp}\mathrm{d}t\Big)^{\frac{1}{p}}\\
&\le \|f-P_j\|_\alpha\Big[1+\sum_{n=1}^{j}\sum_{k=0}^{m_n-1}\|\overline{T}_{n,k}\|e^{\mathrm{Re}\,\lambda_n b}(1+|a|+|b|)^k\Big(\int_{-\infty}^{+\infty}e^{-p\alpha(t)}\mathrm{d}t\Big)^{\frac{1}{p}}\Big]+\\
&\|f\|_\alpha\sum_{n=j+1}^{+\infty}\sum_{k=0}^{m_n-1}\|\overline{T}_{n,k}\|e^{\mathrm{Re}\,\lambda_n b}(1+|a|+|b|)^k\Big(\int_{-\infty}^{+\infty}e^{-p\alpha(t)}\mathrm{d}t\Big)^{\frac{1}{p}}.
\end{aligned}$$

Letting $j\to\infty$, by (21) and (24), we obtain that $f(x)=g(x)$ for each $x\in\mathbb{R}$. This completes the proof of Theorem. □

References

[1] DENG Guantie. *Incompleteness and minimality of complex exponential system* [J]. Sci. China Ser. A, 2007, **50**(10): 1467–1476.

[2] FAN Xiequan. *Three topics on the completeness of three function systems* [D]. Master thesis of Beijing Normal University, 2009.

[3] DENG Guantie. *Weighted exponential polynomial approximation* [J]. Sci. China Ser. A, 2003, **46**(2): 280–287.

[4] DENG Guantie. *On weighted polynomial approximation with gaps* [J]. Nagoya Math. J., 2005, **178**: 55–61.

[5] GAO Zhiqiang, DENG Guantie. *On weighted approximation by lacunary polynomials on the rays emerging from the origin* [J]. Studia Sci. Math. Hungar., 2008, **45**(2): 197–205.

[6] ROCKAFELLAR R. *Convex Analysis* [M]. Princeton Univ. Press, Princeton, 1970.