Journal of Mathematical Research & Exposition
Jul., 2011, Vol. 31, No. 4, pp. 698–704
DOI:10.3770/j.issn:1000-341X.2011.04.015
Http://jmre.dlut.edu.cn

Value Distribution of Meromorphic Solutions of Some q-Difference Equations

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Abstract In this paper, we consider the value distribution of meromorphic solutions of order zero of some kind of q-difference equations and examples are also given to elaborate our results.

Keywords q-difference equation; meromorphic solution; zero-order.

Document code A
MR(2010) Subject Classification 30D35; 39B32
Chinese Library Classification O174.5

1. Introduction and results

In this paper, a meromorphic function means that the function is meromorphic in the whole complex plane. We also assume that the reader is familiar with the usual notations of Nevanlinna theory [1–3]. Moreover, the standard definition of logarithmic density can be found in [4].

Recently, a number of papers [5–17] focused on the complex difference and complex q-difference. These papers also investigated the existence and the growth of meromorphic solutions of difference equations and q-difference equations.

The classical paper on the Schröder equation

$$f(qz) = R(f(z)),$$

where $q \in \mathbb{C}$, $|q| \neq 0, 1$, and $R(f)$ is a rational function in f, is due to Ritt [18].

In the important collection [19] of research problems, Rubel raised the question: What can be said about the more general equation

$$f(qz) = R(z, f(z)), \tag{1.1}$$

where $R(z, f)$ is rational in both variables?

Received November 9, 2009; Accepted April 18, 2011
Supported by the National Natural Science Foundation of China (Grant No. 10871076), the Youth Science Foundation of Education Bureau of Jiangxi Province (Grant No. GJJ11072) and the Natural Science Foundation of Jiangxi Province (Grant No. 2009GQS0013).
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Gundersen et al. obtained the following result in [11].

Theorem A *Suppose that f is a transcendental meromorphic solution of an equation of the form (1.1), with $|q| > 1$ and meromorphic coefficients of growth $S(r, f)$. Then we have that $\rho(f) = \frac{\log d}{\log |q|}$, where $d = \deg_f R$.*

An important paper relative to the above problem is due to Bergweiler [7]. The authors considered linear q-difference equation

$$\sum_{j=0}^{n} a_j(z) f(q^j z) = Q(z), \tag{1.2}$$

where $q \in \mathbb{C}$, $|q| \neq 0, 1$, $n \in \mathbb{N}$, the coefficients $a_j(z)$, $j = 0, 1, \dots, n$ and $Q(z)$ are rational functions, and $a_0(z) \not\equiv 0$, $a_n(z) \equiv 1$. It was shown that the equation (1.2) has some similarity to the genearalized Schröder equation (1.1), but it has also somewhat different aspects. Two important results on the growth of meromorphic solutions were proved in [7] as follows.

Theorem B *All meromorphic solutions of (1.2) satisfy $T(r, f) = O((\log r)^2)$.*

Theorem C *All transcendental meromorphic solutions of (1.2) satisfy $(\log r)^2 = O(T(r, f))$.*

Recently, we considered more general q-difference equations than the above equations in [17], which are of the forms

$$\sum_{j=1}^{n} c_j(z) f(q^j z) = R(z, f(z)) = \frac{P(z, f(z))}{Q(z, f(z))}$$

and

$$\frac{\sum_{\lambda \in I} d_\lambda(z) f(qz)^{i_{\lambda,1}} f(q^2 z)^{i_{\lambda,2}} \cdots f(q^n z)^{i_{\lambda,n}}}{\sum_{\mu \in J} e_\mu(z) f(qz)^{j_{\mu,1}} f(q^2 z)^{j_{\mu,2}} \cdots f(q^n z)^{j_{\mu,n}}} = R(z, f(z)) = \frac{P(z, f(z))}{Q(z, f(z))}.$$

Note that all meromorphic solutions of Riccati q-difference equation and linear q-difference equation, both with rational coefficients, are of order zero, respectively by Theorems A and B. This fact shows that it is of great importance to investigate meromorphic solutions of order zero of q-difference equations.

In this paper, we consider the value distribution of meromorphic solutions of order zero of some kind of q-difference equations.

Theorem 1 *Suppose that f is a non-constant meromorphic solution of order zero of a q-difference equation of the form*

$$\begin{aligned} \sum_{\lambda \in I} c_\lambda(z) f(qz)^{i_{\lambda,1}} f(q^2 z)^{i_{\lambda,2}} \cdots f(q^n z)^{i_{\lambda,n}} &= \frac{P(z, f(z))}{Q(z, f(z))} \\ &= \frac{a_k(z)(f(z))^k + a_{k+1}(z)(f(z))^{k+1} + \cdots + a_{s-1}(z)(f(z))^{s-1} + a_s(z)(f(z))^s}{b_0(z) + b_1(z) f(z) + \cdots + b_t(z)(f(z))^t}, \end{aligned} \tag{1.3}$$

where $I = \{(i_{\lambda,1}, i_{\lambda,2}, \dots, i_{\lambda,n})\}$ is a finite index set, and $i_{\lambda,1} + i_{\lambda,2} + \cdots + i_{\lambda,n} = \sigma > 0$ for all $\lambda \in I$, and $q \in \mathbb{C}\setminus \{0, 1\}$. Moreover, suppose that $0 \le k \le s$, $a_k(z) a_s(z) b_t(z) \not\equiv 0$, that $P(z, f)$ and $Q(z, f)$ have no common factors, and that all meromorphic coefficients in (1.3) are of growth *of $o(T(r,f))$ on a set of logarithmic density 1. If*

$$\max\{t, s-\sigma\} > \min\{\sigma, k\}, \tag{1.4}$$

then

$$N(r,f) \neq o(T(r,f)) \tag{1.5}$$

on any set of logarithmic density 1.

Theorem 1 shows that f, the meromorphic solution of (1.3) satisfying all conditions in Theorem 1, has a relatively large number of poles to some extent. By strengthening the condition (1.4) in Theorem 1, we obtain the following stronger assertion, which means that the pole density of f can be comparable to the growth of f to some extent.

Theorem 2 *Suppose that*

$$\max\{t, s-\sigma\} \geq \min\{\sigma, k\} + \sigma \tag{1.6}$$

and other assumptions in Theorem 1 hold. Then

$$N(r,f) = T(r,f) + o(T(r,f)) \tag{1.7}$$

holds on a set of logarithmic density 1.

The following Example is an example on Theorems 1 and 2.

Example The function

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{2^{\frac{1}{2}n(n+1)}}$$

is a transcendental entire function of order zero [20]. Since it satisfies the equation

$$f(2z) = z + zf(z),$$

we have by Theorem B that

$$T(r, f+1) = T(r,f) = O((\log r)^2). \tag{1.8}$$

Set $g(z) = \frac{1}{f(z)+1}$, then it satisfies the equation

$$g(2z) + g(4z) = \frac{z(2z+1)g(z) + 2(z+1)g^2(z)}{2z^3 + z(4z+1)g(z) + (2z+1)g^2(z)},$$

where $s = t = 2$, $k = \sigma = 1$, showing that g satisfies the conditions of Theorems 1 and 2. By (1.8) and the details of [21, p.47-49], we have that

$$N(r,g) = N(r, \frac{1}{f(z)+1}) = (1+o(1))T(r, f+1) = (1+o(1))T(r,g),$$

showing that g satisfies the results of Theorems 1 and 2. Therefore, Theorems 1 and 2 may hold.

Remark 1 (1) Clearly, the equation (1.1) is a special case of the equation (1.3), where $\sigma = n = 1$.

(2) The condition (1.4) cannot be omitted in Theorem 1. For example, we consider the function

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{2^{\frac{1}{2}n(n+1)}}$$

in the above Example again. It satisfies the equation

$$f(2z)f(4z) = 2z^2\left(1 + z + (2z+1)f(z) + zf^2(z)\right),$$

where $s = 2$, $k = t = 0$, $\sigma = 2$ contradict (1.4). Clearly, (1.5) fails for this case. But the condition (1.4) is not necessary for Theorem 1. For example, the function

$$f(z) = \frac{1}{\prod_{i=0}^{\infty}(1 - q^i z)}, \quad 0 < |q| < 1$$

is a transcendental meromorphic function [15]. On the one hand, it satisfies the q-difference equation

$$f(qz) = (1 - z)f(z),$$

where $s = k = \sigma = 1$, $t = 0$ contradict (1.4). But by the details of the proofs of Theorems B and C, we have

$$M_1(\log^2 r) \le N(r, f) \le M_2(\log^2 r),$$

and

$$K_1(\log^2 r) \le T(r, f) \le K_2(\log^2 r),$$

for some constants $M_2 \ge M_1 > 0$, $K_2 \ge K_1 > 0$. Thus, (1.5) holds without any exceptional set.

(3) The assumption concerning meromorphic coefficients in Theorems 1 and 2 cannot be omitted. For example, the function $f(z) = 1 - z$ satisfies the equation

$$f(qz) = \frac{f(z) - qz(1-z)}{f(z)}.$$

Clearly, the coefficient $-qz(1 - z)$ is not of growth $o(T(r, f))$. Thus, though the assumptions (1.4) and (1.6) are satisfied, (1.5) and (1.7) fail.

2. Lemmas for proofs of theorems

Lemma 1 ([6]) *Let f be a non-constant zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$. Then there holds*

$$m(r, \frac{f(qz)}{f(z)}) = o(T(r, f))$$

on a set of logarithmic density 1.

Lemma 2 ([2]) (Valiron-Mohon'ko) *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*

$$R(z, f(z)) = \frac{\sum_{i=0}^{m} a_i(z) f(z)^i}{\sum_{j=0}^{n} b_j(z) f(z)^j}$$

with meromorphic coefficients $a_i(z), b_j(z)$, the characteristic function of $R(z, f(z))$ satisfies

$$T(r, R(z, f(z))) = dT(r, f) + O(\Psi(r)),$$

where $d = \max\{m, n\}$ *and* $\Psi(r) = \max_{i,j}\{T(r, a_i), T(r, b_j)\}$.

Lemma 3 ([4]) *If* $T : \mathbb{R}^+ \to \mathbb{R}^+$ *is an increasing function such that*

$$\overline{\lim_{r\to\infty}} \frac{\log T(r)}{\log r} = 0,$$

then the set

$$E = \{r : T(C_1 r) \geq C_2 T(r)\}$$

has logarithmic density 0 for all $C_1 > 1$ *and* $C_2 > 1$.

Remark 2 ([7]) We shall also use the observation that

$$N(r, f(qz)) = N(|q|r, f) + O(1)$$

holds for any meromorphic function f and any non-zero complex constant q.

3. Proof of Theorem 1

To prove our theorems, we propose to follow the main idea in the proof of the case of difference equations [22]. Denote

$$U_q(z, f(z)) = \sum_{\lambda \in I} c_\lambda(z) f(qz)^{i_{\lambda,1}} f(q^2 z)^{i_{\lambda,2}} \cdots f(q^n z)^{i_{\lambda,n}}.$$

Noting the assumption concerning the coefficients and the assumption that $i_{\lambda,1}+i_{\lambda,2}+\cdots+i_{\lambda,n} = \sigma$ for all $\lambda \in I$, we have by Lemma 1 that

$$m(r, \frac{U_q(z, f)}{f^\sigma}) = o(T(r, f)) \tag{3.1}$$

on a set of logarithmic density 1. Moreover, by applying Lemma 2 to both sides of (1.3), we have that

$$T(r, \frac{U_q(z, f)}{f^\sigma}) = T(r, \frac{P(z, f)}{f^\sigma Q(z, f)}) = T(r, \frac{f^k(a_k + \cdots + a_s f^{s-k})}{f^\sigma Q(z, f)}) = dT(r, f) + o(T(r, f)) \tag{3.2}$$

on a set of logarithmic density 1, where

$$d = \max\{t + \sigma, s\} - \min\{\sigma, k\} = \max\{t, s - \sigma\} - \min\{\sigma, k\} + \sigma \tag{3.3}$$

by the assumption that $P(z, f)$, $Q(z, f)$ have no common factors. By combining (3.1)–(3.3) and (1.4), we have that

$$N(r, \frac{U_q(z, f)}{f^\sigma}) = dT(r, f) + o(T(r, f)) \geq (\sigma + 1)T(r, f) + o(T(r, f)) \tag{3.4}$$

on a set of logarithmic density 1.

Suppose now on the contrary to the assertion of Theorem 1 that

$$N(r, f) = o(T(r, f)) \tag{3.5}$$

on a set of logarithmic density 1. Since f is of order zero, we have by Lemma 3 that

$$N(Cr, f) = (1 + o(1))N(r, f), \quad C > 1 \tag{3.6}$$

on a set of logarithmic density 1. Thus, by (3.6), the assumption (3.5) and Remark 2, we have that

$$N(r,U_q(z,f)) \le \sigma N\left(\max\{|q|,|q|^n\}r,f\right)+o(T(r,f)) \le \sigma(1+o(1))N(r,f)+o(T(r,f)) = o(T(r,f)) \tag{3.7}$$

on a set of logarithmic density 1, whenever $|q| \le 1$ or $|q| > 1$. Furthermore, we have by (3.7) that

$$\begin{aligned} N(r,\frac{U_q(z,f)}{f^\sigma}) &\le N(r,U_q(z,f)) + \sigma N(r,\frac{1}{f}) = \sigma N(r,\frac{1}{f}) + o(T(r,f)) \\ &\le \sigma T(r,f) + o(T(r,f)) \end{aligned} \tag{3.8}$$

on a set of logarithmic density 1. Thus, we have a contradiction that (3.4) and (3.8) hold simultaneously on a set of logarithmic density 1. Thus, (1.5) holds on any set of logarithmic density 1. □

4. Proof of Theorem 2

By (3.6), (3.7) and the assumption concerning the coefficients, we have that

$$\begin{aligned} N(r,\frac{U_q(z,f)}{f^\sigma}) &\le \sigma((1+o(1))N(r,f)+N(r,\frac{1}{f}))+o(T(r,f)) \\ &\le \sigma(2T(r,f)-m(r,f))+o(T(r,f)) \end{aligned} \tag{4.1}$$

on a set of logarithmic density 1.

On the other hand, we have by (3.1) and (3.2) that

$$N(r,\frac{U_q(z,f)}{f^\sigma}) = dT(r,f)+o(T(r,f)) \tag{4.2}$$

on a set of logarithmic density 1, where d is defined by (3.3).

Thus, we have by (4.1), (4.2) and (1.6) that

$$m(r,f) = o(T(r,f)),$$

that is

$$N(r,f) = T(r,f)+o(T(r,f))$$

on a set of logarithmic density 1. □

Acknowledgement The authors are grateful to the referees and editors for helpful suggestions to improve the readability of the paper.

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