Journal of Mathematical Research & Exposition
Jul., 2011, Vol. 31, No. 4, pp. 749–756
DOI:10.3770/j.issn:1000-341X.2011.04.022
Http://jmre.dlut.edu.cn

Viscosity Approximation Method for Infinitely Many Asymptotically Nonexpansive Maps in Banach Spaces

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Abstract In the framework of reflexive Banach spaces satisfying a weakly continuous duality map, the author uses the viscosity approximation method to obtain the strong convergence theorem for iterations with infinitely many asymptotically nonexpansive mappings. The main results obtained in this paper improve and extend some recent results.

Keywords asymptotically nonexpansive mapping; Gauge function; weakly continuous duality map.

Document code A
MR(2010) Subject Classification 47H09
Chinese Library Classification O177.91

1. Introduction

Let E be a real Banach space with the dual space E^*, and C be a nonempty closed convex subset of E. Denote by $F(T)$ the fixed point set of a mapping T. If $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp., $x_n \rightharpoonup x$) will denote strong (resp., weak) convergence of the sequence $\{x_n\}$ to a point $x \in E$.

Recall that a mapping $T : C \to C$ is said to be asymptotically nonexpansive mapping with the sequence $\{k_n\}$ if there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $k_n \to 1$ such that $\|T^n x - T^n y\| \leqslant k_n \|x - y\|$ for all $x, y \in C$ and $n \in \mathbb{N}$, where $T^0 = I$, the identity mapping, and $\mathbb{N} = \{0, 1, 2, \ldots\}$ denotes the set of nonnegative integers.

In the framework of real reflexive Banach spaces with a weakly continuous duality mapping J_φ associated with a gauge φ, Ceng, Xu and Yao [1] in 2008 studied the following implicit and explicit iterations with a finite family of asymptotically nonexpansive self-mappings $\{T_i\}_{i=1}^N$ associated with the real sequence $\{k_n\} \subset [1, +\infty)$:

$$x_n = (1 - \frac{1}{k_n})x_n + \frac{1 - t_n}{k_n} f(x_n) + \frac{t_n}{k_n} T_{r_n}^n x_n \tag{1.1}$$

and

$$x_{n+1} = (1 - \frac{1}{k_n})x_n + \frac{1 - t_n}{k_n} f(x_n) + \frac{t_n}{k_n} T_{r_n}^n x_n, \tag{1.2}$$

Received December 21, 2009; Accepted January 12, 2011
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where f is a contractive self-mapping, $r_n = n \pmod N$, with the mod function taking values in $\{1, 2, \dots, N\}$. Under the following assumptions:

$$0 < t_n < \frac{1-\alpha}{k_n - \alpha}, \quad \lim_{n\to\infty} t_n = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{k_n - 1}{1 - t_n} = 0, \tag{1.3}$$

Ceng, Xu and Yao [1] used the viscosity approximation technique to obtain a strong convergence theorem for the iterative schemes (1.1) and (1.2).

Up to now, many authors have been studying the similar iterative algorithms, such as Chang, Tan and Lee, et al. [8], Yao and Liou [9] and so on.

Motivated by [1] and Chang, Lee and Chan [2], we obtain in this paper the strong convergence theorem for the following iteration with infinitely many asymptotically nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_n^n x_n. \tag{1.4}$$

Particularly, if $\{T_n\}_{n=1}^{\infty} = \{T_1, T_2, \dots, T_N, T_1, T_2, \dots, T_N, T_1, \dots\}$, a finite family of asymptotically nonexpansive mappings $\{T_n\}_{n=1}^{N}$, then the iterations (1.4) becomes the explicit iteration

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_{r_n}^n x_n.$$

In addition, another purpose of this paper is to replace the implicit (1.3) with some simple conditions. We may assume $\lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0$ to replace (1.3), for example, $\alpha_n = \frac{1}{n+2}$ and $k_n = \frac{(n+2)^2+1}{(n+2)^2}$.

In this paper, we assume that E is a real reflexive Banach space with (topological) dual E^*. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing continuous function such that $\varphi(t) = 0$ if and only if $t = 0$. Such a function φ is known as a gauge with which we can define a duality map $J_\varphi : E \to 2^{E^*}$ by

$$J_\varphi(x) = \{j \in E^* : \langle x, j\rangle = \|x\|\varphi(\|x\|) \ \text{ and } \ \|j\| = \varphi(\|x\|)\}.$$

From Browder [3], we know that E has a weakly continuous duality map if there is a gauge φ such that the duality map J_φ is weakly sequently continuous from E, equipped with the weak topology, to E^*, equipped with the weak* topology. For instance, for each $1 < p < \infty$, the space l^p has a weakly continuous duality map with gauge $\varphi(t) = t^{p-1}$. In the case that $\varphi(t) = t$ for all $t \in \mathbb{R}^+$, we write the associated duality map as J, called the normalized duality map. In the framework of reflexive Banach spaces, J can be considered as single-valued. Indeed, we know from Asplund Theorem, each reflexive Banach space E can be re-normed with an equivalent norm such that both E and E^* are strictly convex. Thereby, each reflexive Banach space can be seen as a smooth Banach space, which implies J is single-valued.

Lemma 1.1 ([4]) *If a real sequence $\{a_n\} \subset [0, +\infty)$ satisfies $a_{n+1} \leqslant (1 - \gamma_n)a_n + \delta_n$, $\forall n \geqslant n_0$, where real sequences $\{\gamma_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfy the following conditions:*

$$\lim_{n\to\infty} \gamma_n = 0; \ \ \sum_{n=1}^{\infty} \gamma_n = \infty; \ \ \limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leqslant 0 \ \ \text{or} \ \ \sum_{n=1}^{\infty} |\delta_n| < \infty.$$

Then we have $\lim_{n\to\infty} a_n = 0$.

Lemma 1.2 ([5]) *Let E be a Banach space satisfying a weakly continuous duality map, C be a nonempty closed convex subset of E, and $T : C \to C$ be an asymptotically nonexpansive mapping with a fixed point. Then $I - T$ is semi-closed at zero, i.e., for each sequence $\{x_n\}$ in C, if $\{x_n\}$ converges weakly to $q \in C$ and $\{(I-T)x_n\}$ converges strongly to 0, then $(I-T)q = 0$.*

For the normalized duality map J, the following inequalities hold [6]:

$$\|x\|^2 + 2\langle y, j(x)\rangle \leqslant \|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall\, j(x) \in J(x),\ j(x+y) \in J(x+y).$$

If J_φ is the duality map associated with the gauge φ, then we have

Lemma 1.3 ([7]) *Let E be a real Banach space, and J_φ be the duality map associated with the gauge φ. Set, for $t \geqslant 0$,*

$$\Phi(t) = \int_0^t \varphi(s)\mathrm{d}s.$$

We have the following conclusions:

(i) For all $x, y \in E$ and $j \in J_\varphi(x+y)$, $\Phi(\|x+y\|) \leqslant \Phi(\|x\|) + \langle y, j\rangle$. In particular, for $x, y \in E$ and $j \in J(x+y)$, $\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, j\rangle$.

(ii) For $\lambda \in \mathbb{R}$ and for nonzero $x \in E$, $J_\varphi(\lambda x) = \operatorname{sgn}(\lambda)(\varphi(|\lambda|\|x\|)/\|x\|)J(x)$.

2. Main results

Firstly, we may give the assumptions on the framework of spaces. Assume that E is a real reflexive Banach space with a weakly continuous duality mapping J_φ associated with a gauge φ. In addition, we also assume $\lim_{t\to 0+} \frac{\varphi(t)}{t} = d \in (0, +\infty]$.

There exist plenty of spaces satisfying all the assumptions above. For example, each l^p $(1 < p \leqslant 2)$ satisfies all the assumptions above. In addition, each real reflexive Banach space with the weakly continuous normalized duality mapping J also satisfies all the assumptions above, for in this case $\varphi(t) = t$ for all $t \in \mathbb{R}^+$. It is also known that all Hilbert spaces admit the weakly continuous normalized duality mapping J.

Assume $\{T_n\}$ is an infinite family of asymptotically nonexpansive mappings of C into E. Recall that $\{T_n\}$ is a family of asymptotically nonexpansive mappings with a common sequance $\{k_n\}$, if there exists a common sequance $\{k_n\} \subset [1, +\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for each i,

$$\|T_i^n x - T_i^n y\| \leqslant k_n\|x - y\|, \quad \forall\, x, y \in C.$$

Particularly, when $\{T_n\}$ is a finite family of asymptotically nonexpansive mappings, $\{T_n\}$ must have such a common sequance $\{k_n\}$. Throughout this section, we also assume the sequence $\{T_n\}_{n=1}^\infty$ has such a common sequence $\{k_n\}$. We shall consider both the implicit iteration:

$$z_n = \alpha_n f(z_n) + (1-\alpha_n)T_n^n z_n, \quad \forall\, n \geqslant N, \tag{2.1}$$

and the explicit iteration

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_n^n x_n, \end{cases} \quad \forall\, n \in \mathbb{N}. \tag{2.2}$$

Theorem 2.1 *Suppose that $\{T_n\}_{n=1}^{\infty}$ is an infinite family of asymptotically nonexpansive mappings of C into C, having a common sequence $\{k_n\}$. Assume, $\sum_{n=1}^{\infty}(k_n-1)<\infty$ and $F=:\bigcap_{i=1}^{\infty}F(T_i)\neq\emptyset$. Suppose that $\{z_n\}\subset C$ is defined by the implicit iteration (2.1), where $N\in\mathbb{N}$ is a constant, $f:C\to C$ is a given contractive mapping with the contractive constant $0<\alpha<1$, $\{\alpha_n\}\subset(0,1)$ is a real sequence with $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=0}^{\infty}\alpha_n=\infty$ and $\lim_{n\to\infty}\frac{k_n-1}{\alpha_n}=0$.*

Then the sequence $\{z_n\}$ defined by (2.1) converges strongly to the unique solution $z\in F$ for the following variational inequality in F:

$$\langle (f-I)z, J(x-z)\rangle\leqslant 0,\ \forall\, x\in F \tag{2.3}$$

if and only if

$$\|z_n-T_iz_n\|\to 0,\quad \textit{for all } i\in\mathbb{N}. \tag{2.4}$$

Proof Firstly, it is not difficult to prove that there exists a natural number N such that for any $n\geqslant N$, there exists a unique element $z_n\in C$ satisfying (2.1).

Secondly, we verify the necessity of Theorem 2.1.

Indeed, if $\{z_n\}$ converges strongly to $z\in F$, we have

$$\|z_n-T_iz_n\|\leqslant\|z_n-z\|+\|T_iz-T_iz_n\|\leqslant(1+k_1)\|z_n-z\|\to 0. \tag{2.5}$$

Finally, we verify the sufficiency of Theorem 2.1.

At first, we prove that $\{z_n\}$ defined by (2.1) is bounded.

Indeed, for any given $x\in F$, it follows from Lemma 1.3, (2.1), $\Phi(0)=0$ and the convexity of Φ (so that $\Phi(ts)\leqslant t\Phi(s)$ for all $t\in[0,1]$) that

$$\Phi(\|z_n-x\|)\leqslant(\alpha_n\alpha+(1-\alpha_n)k_n)\Phi(\|z_n-x\|)+\alpha_n\langle f(x)-x, J_\varphi(z_n-x)\rangle.$$

Then we have

$$\frac{1-\alpha_n\alpha-(1-\alpha_n)k_n}{\alpha_n}\Phi(\|z_n-x\|)\leqslant\langle f(x)-x, J_\varphi(z_n-x)\rangle,$$

$$(1-\alpha-\frac{k_n-1}{\alpha_n})\Phi(\|z_n-x\|)\leqslant(k_n-\alpha-\frac{k_n-1}{\alpha_n})\Phi(\|z_n-x\|)\leqslant\langle f(x)-x, J_\varphi(z_n-x)\rangle.$$

It follows from $\lim_{n\to\infty}\frac{k_n-1}{\alpha_n}=0$ that there exists a natural number n_1 such that

$$\frac{k_n-1}{\alpha_n}<\frac{1-\alpha}{2},\quad\forall\, n\geqslant n_1. \tag{2.6}$$

Thus,

$$\Phi(\|z_n-x\|)\leqslant\frac{2}{1-\alpha}\langle f(x)-x, J_\varphi(z_n-x)\rangle,\ \forall\, n\geqslant n_1. \tag{2.7}$$

Noticing that (2.7) holds actually for all duality map J_φ, we may take the normalized duality J in particular (in this case $\Phi(t)=\frac{t^2}{2}$). Then we get by (2.7)

$$\|z_n-x\|^2\leqslant\frac{4}{1-\alpha}\langle f(x)-x, J(z_n-x)\rangle\leqslant\frac{4}{1-\alpha}\|f(x)-x\|\cdot\|z_n-x\|,\ \forall\, n\geqslant n_1,$$

which implies $\{z_n\}$ is bounded. Hence, both $\{f(z_n)\}$ and $\{T_n^nz_n\}$ are bounded. It follows by the reflexivity of Banach space E that there exists a weakly convergent subsequence $\{z_{n_i}\}\subset\{z_n\}$

such that

$$z_{n_i} \rightharpoonup z \in C, \quad \text{as} \quad i \to \infty.$$

Then by (2.4) and Lemma 1.2, we get $z \in F$.

Next, we shall prove that z is just the unique solution of variational inequality (2.3) in F.

Indeed, for any given $x \in F$, it follows by (2.1) that

$$\Phi(\|z_n - x\|) \leqslant (\alpha_n + (1-\alpha_n)k_n)\Phi(\|z_n - x\|) + \alpha_n\langle f(z_n) - z_n, J_\varphi(z_n - x)\rangle$$

Thus,

$$\langle z_n - f(z_n), J_\varphi(z_n - x)\rangle \leqslant \frac{(1-\alpha_n)(k_n - 1)}{\alpha_n}\Phi(\|z_n - x\|). \tag{2.8}$$

Now in (2.8), by replacing z_n with z_{n_i}, passing through the limit as $i \to \infty$, and by the boundedness of $\{z_n\}$, $\lim_{n\to\infty}\alpha_n = 0$, and $\lim_{n\to\infty}\frac{k_n - 1}{\alpha_n} = 0$, we conclude

$$\limsup_{i\to\infty}\langle z_{n_i} - f(z_{n_i}), J_\varphi(z_{n_i} - x)\rangle \leqslant 0, \ \forall\, x \in F. \tag{2.9}$$

Taking $x = z$ in (2.7), by $z_{n_i} \rightharpoonup z$ and the weak continuity of J_φ, we can get that

$$\Phi(\|z_{n_i} - z\|) \leqslant \frac{2}{1-\alpha}\langle f(z) - z, J_\varphi(z_{n_i} - z)\rangle \to 0, \quad \text{as} \ \ i \to \infty.$$

This implies

$$z_{n_i} \to z, \quad \text{as} \ \ i \to \infty. \tag{2.10}$$

It follows by (2.9), (2.10) and the weak continuity of J_φ that

$$\langle z - f(z), J_\varphi(z - x)\rangle = \limsup_{i\to\infty}\langle z_{n_i} - f(z_{n_i}), J_\varphi(z_{n_i} - x)\rangle \leqslant 0, \ \forall\, x \in F, \tag{2.11}$$

which implies z is a solution of the variational inequality (2.3) in F since $J(z-x)$ is a positive-scalar multiple of $J_\varphi(z-x)$ by virtue of Lemma 1.3.

Next, we prove that z is the unique solution of the variational inequality (2.3) in F.

Indeed, if u is another solution of variational inequality (2.3) in F, we can get by (2.3) the following two inequalities at the same time:

$$\langle f(u) - u, J(z-u)\rangle \leqslant 0; \quad \langle f(z) - z, J(u-z)\rangle \leqslant 0.$$

By adding up the two inequalities above, we get

$$(1-\alpha)\|u - z\|^2 \leqslant \langle (I-f)u - (I-f)z, J(u-z)\rangle \leqslant 0. \tag{2.12}$$

Thus, we have $u = z$, and hence z is the unique solution of the variational inequality (2.3) in F.

Finally, we can prove $z_n \to z$ as $n \to \infty$ since $\{z_n\}$ is sequentially compact and each cluster point of $\{z_n\}$ equals z. This completes the proof of Theorem 2.1. □

Next we turn to study the explicit scheme (2.2). We need the strong convergence of implicit scheme (2.1) to prove the strong convergence of the explicit scheme (2.2). This is quite a routine argument (see e.g., Ceng Xu and Yao [1]). In the following Theorem, we may similarly assume the condition (2.4) holds, which implies $z_n \to z$ as $n \to \infty$. In this section, we denote by z the unique solution in $F(T)$ for the variational inequality (2.3).

Theorem 2.2 *Suppose that $\{T_n\}_{n=1}^\infty$ is an infinite family of asymptotically nonexpansive* *mappings of C into C, having a common sequence $\{k_n\}$. Assume, $\sum_{n=1}^{\infty}(k_n-1)<\infty$ and $F =: \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose that $\{x_n\} \subset C$ is generated by the explicit iteration (2.2), where $f : C \to C$ is a given contractive mapping with the contractive constant $0<\alpha<1$, $\{\alpha_n\} \subset (0,1)$ is a real sequence with $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$ and $\lim_{n\to\infty}\frac{k_n-1}{\alpha_n} = 0$. Assume, in addition, the condition (2.4) holds (or $z_n \to z$).*

Then the sequence $\{x_n\}$ generated by (2.2) converges strongly to the unique solution $z \in F$ for the variational inequality (2.3) in F if and only if

$$\|x_n - T_i x_n\| \to 0, \quad \text{for all } \ i \in \mathbb{N}. \tag{2.13}$$

Proof Similarly to (2.5), we can easily verify the necessity of Theorem 2.2. In what follows, we verify the sufficiency of Theorem 2.1 by eight steps.

Step 1. We claim that $\{x_n\}$ generated by iteration (2.2) is bounded.

Indeed, considering of $z \in F$, we get by (2.2)

$$\|x_{n+1} - z\| \leqslant (k_n(1-\alpha_n) + \alpha\alpha_n)\|x_n - z\| + \alpha_n\|f(z) - z\|. \tag{2.14}$$

Similarly, there exists some natural number n_0 such that

$$\frac{k_n - 1}{\alpha_n} < \frac{1-\alpha}{2}, \quad \forall n \geqslant n_0. \tag{2.15}$$

Denote

$$\mu_n = k_n - 1, \quad \forall n \in \mathbb{N}, \tag{2.16}$$

then we get by (2.15) and (2.16)

$$(1-\alpha_n)k_n + \alpha\alpha_n \leqslant 1 - \alpha_n \cdot \frac{1-\alpha}{2}, \quad \forall n \geqslant n_0. \tag{2.17}$$

Then it follows by (2.14) and (2.17) that

$$\|x_{n+1} - z\| \leqslant \max\{\|x_n - z\|, \frac{2\|f(z)-z\|}{1-\alpha}\}, \quad \forall n \geqslant n_0.$$

Mathematical induction yields

$$\|x_{n+1} - z\| \leqslant \max\{\|x_{n_0} - z\|, \frac{2\|f(z)-z\|}{1-\alpha}\}, \quad \forall\, n \geqslant n_0,$$

which implies $\{x_n\}_{n=0}^{\infty}$ is bounded. Hence, both $\{T_n^n x_n\}$ and $\{f(x_n)\}$ are bounded. Thereby, we can assume that there exists a constant $M > 0$ such that

$$\|f(x_n)\| + \|T_n^n x_n\| + \|x_n\| \leqslant M, \quad \forall n \geqslant 0. \tag{2.18}$$

Obviously, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that

$$\limsup_{n\to\infty}\langle (f-I)z, J_\varphi(x_n - z)\rangle = \lim_{i\to\infty}\langle (f-I)z, J_\varphi(x_{n_i} - z)\rangle. \tag{2.19}$$

Moreover, it follows by the boundedness of $\{x_{n_i}\}$ and the reflexivity of Banach space E that there exists a subsequence $\{x_{n_k}\} \subset \{x_{n_i}\}$ such that

$$x_{n_k} \rightharpoonup p \in C, \quad \text{as} \quad k \to \infty. \tag{2.20}$$

Then we know by (2.20), (2.13) and Lemma 1.2 that $p \in F$.

Step 2. We claim that

$$\limsup_{n\to\infty}\langle (f-I)z, J_\varphi(x_n-z)\rangle \leqslant 0. \tag{2.21}$$

Indeed, in view of $p\in F$ and by replacing x with p in (2.11), we get

$$\langle (f-I)z, J_\varphi(p-z)\rangle \leqslant 0.$$

Then one can get by (2.20) and the weak continuity of J_φ that

$$\limsup_{n\to\infty}\langle (f-I)z, J_\varphi(x_n-z)\rangle = \langle (f-I)z, J_\varphi(p-z)\rangle \leqslant 0. \tag{2.22}$$

Step 3. For any $t_0>0$, we claim $\lim_{t\to t_0}\frac{\varphi(t)}{t}$ is a positive real number.

Indeed, since $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is a strictly increasing continuous function such that $\varphi(t)=0$ if and only if $t=0$, we know that $\frac{\varphi(t)}{t}$ is continuous in any bounded closed subinterval of $(0,+\infty)$. Thus, we have $\lim_{t\to t_0}\frac{\varphi(t)}{t}=\frac{\varphi(t_0)}{t_0}>0$.

Step 4. We shall prove

$$\limsup_{n\to\infty}\langle (f-I)z, J(x_{n+1}-z)\rangle \leqslant 0.$$

Indeed, the boundedness of $\{x_n\}$ yields the boundedness of the real sequence $\{\langle (f-I)z, J(x_{n+1}-z)\rangle\}$, and hence $\limsup_{n\to\infty}\langle (f-I)z, J(x_{n+1}-z)\rangle$ is some constant.

It is obvious that there exists a subsequence, say, $\{x_{n_k}\}\subset\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle (f-I)z, J(x_n-z)\rangle = \lim_{k\to\infty}\langle (f-I)z, J(x_{n_k}-z)\rangle. \tag{2.23}$$

From (2.22), (2.23) and Lemma 1.3(ii), we have

$$\limsup_{k\to\infty}\frac{\varphi(\|x_{n_k}-z\|)}{\|x_{n_k}-z\|}\lim_{k\to\infty}\langle (f-I)z, J(x_{n_k}-z)\rangle \leqslant \limsup_{n\to\infty}\langle (f-I)z, J_\varphi(x_n-z)\rangle \leqslant 0. \tag{2.24}$$

It is obvious that there exists a subsequence $\{x_{n_i}\}\subset\{x_{n_k}\}$ such that

$$\limsup_{k\to\infty}\frac{\varphi(\|x_{n_k}-z\|)}{\|x_{n_k}-z\|} = \lim_{i\to\infty}\frac{\varphi(\|x_{n_i}-z\|)}{\|x_{n_i}-z\|}. \tag{2.25}$$

Obviously, we can see it by Step 3 and the assumption $\lim_{t\to 0+}\frac{\varphi(t)}{t}=d\in(0,+\infty]$ that

$$(\lim_{i\to\infty}\frac{\varphi(\|x_{n_i}-z\|)}{\|x_{n_i}-z\|})\in(0,+\infty].$$

Then, by (2.23)–(2.25), we have proved

$$\limsup_{n\to\infty}\langle (f-I)z, J(x_{n+1}-z)\rangle \leqslant 0.$$

Denote

$$\eta_n=\max\{\langle (f-I)z, J(x_{n+1}-z)\rangle, 0\},\quad \forall\, n\in\mathbb{N}.$$

Step 5. We claim that $\eta_n\geqslant 0$, $\forall n\in\mathbb{N}$, and that $\eta_n\to 0$, as $n\to\infty$. Indeed, for arbitrarily given $\varepsilon>0$, there correspondingly exists n_ε such that $\langle (f-I)z, J(x_{n+1}-z)\rangle<\varepsilon$ for all $n\geqslant n_\varepsilon$. Then we know that $0\leqslant\eta_n<\varepsilon$ for all $n\geqslant n_\varepsilon$. Hence, one can get $\eta_n\to 0$ by the arbitrariness of ε.

Step 6. We prove $x_n \to z \in F$, as $n \to \infty$. Indeed, it follows by (2.2) that

$$\|x_{n+1}-z\|^2 \leqslant (1-\alpha_n)^2 k_n^2 \|x_n-z\|^2 + \alpha_n\alpha(\|x_n-z\|^2+\|x_{n+1}-z\|^2) + 2\alpha_n\eta_n.$$

Thereby,

$$\|x_{n+1}-z\|^2 \leqslant \frac{(1-\alpha_n)^2 k_n^2 + \alpha\alpha_n}{1-\alpha\alpha_n}\|x_n-z\|^2 + \frac{2\alpha_n\eta_n}{1-\alpha}. \tag{2.26}$$

Denote $\nu_n = k_n - 1$, $\forall\, n \in \mathbb{N}$. It is obvious that $\lim_{n\to\infty}\frac{\nu_n}{\alpha_n} = 0$, $\lim_{n\to\infty}\nu_n = 0$ and $\nu_n \geqslant 0$, $\forall\, n \in \mathbb{N}$. Thus, there exists a natural number n_2 such that

$$0 \leqslant \frac{\nu_n}{\alpha_n} < \frac{1-\alpha}{2} \quad \text{and} \quad 1 \leqslant k_n \leqslant 2, \quad \text{as} \quad n \geqslant n_2.$$

Thus, we have

$$(1-\alpha_n)^2 k_n^2 + \alpha\alpha_n \leqslant (1-\alpha\alpha_n) - \alpha_n(1-\alpha) + (4\alpha_n^2+\nu_n^2), \quad \text{as} \quad n \geqslant n_2.$$

Hence, we can get by (2.26) and the boundedness of $\{x_n\}$

$$\|x_{n+1}-z\|^2 \leqslant (1-(1-\alpha)\alpha_n)\|x_n-z\|^2 + \frac{2\alpha_n\eta_n + (4\alpha_n^2+\nu_n^2)(M+\|z\|)^2}{1-\alpha} \quad \text{as} \quad n \geqslant n_2.$$

Now, taking $\gamma_n = (1-\alpha)\alpha_n$, $\delta_n = \frac{2\alpha_n\eta_n+(4\alpha_n^2+\nu_n^2)(M+\|z\|)^2}{1-\alpha}$, we have proved $\|x_n-z\|^2 \to 0$ by Lemma 1.1. Therefore, the proof of Theorem 2.2 is completed. □

Remark Particularly, when $\{T_n\}_{n=1}^{\infty} = \{T_1, T_2, \ldots, T_N, T_1, T_2, \ldots, T_N, T_1, \ldots\}$, a finite family of asymptotically nonexpansive mappings $\{T_n\}_{n=1}^{N}$, then the iterations (2.1) and (2.2) become the implicit iteration $x_n = \alpha_n f(x_n) + (1-\alpha_n)T_{r_n}^n x_n$ and the explicit iteration $x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_{r_n}^n x_n$. Thus, our main results improve and extend the corresponding main results of [1], [2] and some references therein.

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