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Upper Locating-Domination Numbers of Cycles

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Abstract A set D of vertices in a graph $G=(V,E)$ is a locating-dominating set (LDS) if for every two vertices u,v of $V\setminus D$ the sets $N(u)\cap D$ and $N(v)\cap D$ are non-empty and different. The locating-domination number $\gamma_{\rm L}(G)$ is the minimum cardinality of an LDS of G, and the upper-locating domination number $\Gamma_{\rm L}(G)$ is the maximum cardinality of a minimal LDS of G. In the present paper, methods for determining the exact values of the upper locating-domination numbers of cycles are provided.

Keywords locating-domination number; upper locating-domination number; cycle.

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1. Introduction

All graphs considered in this paper are finite simple graphs, that is, undirected graphs without loops or multiple edges. We in general follow [4] for notation and graph theory terminology. Let $G=(V,E)$ be a simple graph with vertex set V and edge set E. For a vertex $v\in V$, the open neighborhood $N(v)$ of v consists of the vertices adjacent to v; the closed neighborhood of v is $N[v]=N(v)\cup\{v\}$. Also, let $d_G(v)=|N(v)|$ be the degree of v and $\delta(G)$ denote the minimum degree of graph G.

A set $D\subseteq V$ is a *dominating set* if every vertex of $V\setminus D$ has at least one neighbor in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. A dominating set $D\subseteq V$ is a locating-dominating set (LDS) if every two vertices u,v of $V\setminus D$ satisfy $N(u)\cap D\neq N(v)\cap D$. The locating-domination number $\gamma_{\rm L}(G)$ is the minimum cardinality of an LDS of G, and the upper locating-domination number, denoted by $\Gamma_{\rm L}(G)$, is the maximum cardinality of a minimal LDS of G. A minimal LDS with maximum cardinality is called a $\Gamma_{\rm L}(G)$-set. Locating domination was introduced by Slater [5, 6]. For further studies on locating-domination we refer to [1], [2] and [3].

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So far as we know, no work has been done on the upper locating- domination number, except for the recent work by Mustapha Chellali, et al [3], in which, the authors presented the exact value of $\Gamma_{\mathrm{L}}(G)$ for G a path. In this paper, we determine the exact values of $\Gamma_{\mathrm{L}}(G)$ for G a cycle.

2. Upper locating-domination numbers of cycles

The upper locating-domination number of any a path has been given as follows.

Theorem 1 ([3]) *For every path P_n,*

$$\Gamma_{\mathrm{L}}(P_n) = \begin{cases} 4k, & \text{if } n = 7k; \\ 4k+1, & \text{if } n = 7k+1 \text{ or } n = 7k+2; \\ 4k+2, & \text{if } n = 7k+3 \text{ or } n = 7k+4; \\ 4k+3, & \text{if } n = 7k+5; \\ 4k+4, & \text{if } n = 7k+6. \end{cases}$$

On the basis of Theorem 1, we compute the values of upper locating-domination numbers of cycles, which is shown as follows.

Theorem 2 *For every cycle C_n with $n \geq 4$,*

$$\Gamma_{\mathrm{L}}(C_n) = \begin{cases} 4k, & \text{if } n = 7k \text{ or } n = 7k+1; \\ 4k+1, & \text{if } n = 7k+2 \text{ or } n = 7k+3; \\ 4k+2, & \text{if } n = 7k+4 \text{ or } n = 7k+5; \\ 4k+3, & \text{if } n = 7k+6. \end{cases}$$

To prove Theorem 2, we first give two lemmas as follows.

Lemma 3 *For every cycle C_n with $n \geq 4$,*

$$\Gamma_{\mathrm{L}}(C_n) \geq \begin{cases} 4k, & \text{if } n = 7k \text{ or } n = 7k+1; \\ 4k+1, & \text{if } n = 7k+2 \text{ or } n = 7k+3; \\ 4k+2, & \text{if } n = 7k+4 \text{ or } n = 7k+5; \\ 4k+3, & \text{if } n = 7k+6. \end{cases}$$

Proof Suppose $C_n = v_1 v_2 \cdots v_n v_1$. When $n = 7k$ or $n = 7k+1$, let

$$D = \bigcup_{i=0}^{k-1} \{v_{7i+1}, v_{7i+2}, v_{7i+5}, v_{7i+6}\}.$$

First, we can easily find that D is an LDS of C_n and $|D| = 4k$. Also, D is minimal. In fact, for any a vertex $v \in D$, $D \setminus \{v\}$ is no longer an LDS of C_n. This means that $\Gamma_{\mathrm{L}}(C_n) \geq 4k$.

When $n = 7k+2$ or $n = 7k+3$, let

$$D = \{v_{7k+1}\} \cup \bigcup_{i=0}^{k-1} \{v_{7i+1}, v_{7i+2}, v_{7i+5}, v_{7i+6}\}.$$

It is not hard for us to find that D is a minimal LDS, and $|D| = 4k+1$, which means that $\Gamma_{\mathrm{L}}(C_n) \geq 4k+1$.

When $n = 7k+4$, let

$$D = \{v_1, v_2, v_5, v_6, v_9, v_{10}\} \cup \bigcup_{i=1}^{k-1} \{v_{7i+5}, v_{7i+6}, v_{7i+9}, v_{7i+10}\}.$$

It is not hard for us to verify that D is a minimal LDS, and $|D| = 4k+2$, which means that $\Gamma_{\mathrm{L}}(C_n) \geq 4k+2$.

When $n = 7k+5$, we distinguish two cases. If $k = 0$, it can be easily verified that $\Gamma_{\mathrm{L}}(C_5) = 2 \geq 4 \cdot 0 + 2$. If $k \geq 1$, let

$$D = \{v_1, v_2, v_5, v_6, v_9, v_{10}\} \cup \bigcup_{i=1}^{k-1} \{v_{7i+6}, v_{7i+7}, v_{7i+10}, v_{7i+11}\}.$$

One can easily verify that D is a minimal LDS, and $|D| = 4k+2$, which means that $\Gamma_{\mathrm{L}}(C_n) \geq 4k+2$.

When $n = 7k+6$, we distinguish two cases. If $k = 0$, it can be easily verified that $\Gamma_{\mathrm{L}}(C_6) = 3 \geq 4 \cdot 0 + 3$. If $k \geq 1$, let

$$D = \{v_1, v_2, v_5, v_6, v_8, v_9, v_{12}\} \cup \bigcup_{i=2}^{k} \{v_{7i}, v_{7i+1}, v_{7i+4}, v_{7i+5}\}.$$

It is not difficult to find that D is a minimal LDS, and $|D| = 4k+3$. This implies that $\Gamma_{\mathrm{L}}(C_n) \geq 4k+3$. We have finished the proof of Lemma 3. □

Lemma 4 *For every cycle C_n with $n \geq 4$, $\Gamma_{\mathrm{L}}(C_n) \leq \Gamma_{\mathrm{L}}(P_{n-1})$.*

Proof First we can easily verify that the conclusion of Lemma 4 holds for $n \leq 12$. In fact, the values of $\Gamma_{\mathrm{L}}(C_n)$ for $4 \leq n \leq 12$ are as follows.

n	4	5	6	7	8	9	10	11	12
$\Gamma_{\mathrm{L}}(C_n)$	2	2	3	4	4	5	5	6	6

So we may assume that $n \geq 13$ in what follows. We claim that every $\Gamma_{\mathrm{L}}(C_n)$-set contains two consecutive vertices of C_n. For otherwise, suppose to the contrary that there exists a $\Gamma_{\mathrm{L}}(C_n)$-set D such that any two vertices of D are not consecutive. Then $|D| \leq \lfloor \frac{n}{2} \rfloor$. Noticing $n \geq 13$, we can prove, by some simple computations, that

$$|D| \leq \left\lfloor \frac{n}{2} \right\rfloor \leq \begin{cases} \lfloor \frac{7k+1}{2} \rfloor < 4k, & \text{if } n = 7k \text{ or } n = 7k+1; \\ \lfloor \frac{7k+3}{2} \rfloor < 4k+1, & \text{if } n = 7k+2 \text{ or } n = 7k+3; \\ \lfloor \frac{7k+5}{2} \rfloor < 4k+2, & \text{if } n = 7k+4 \text{ or } n = 7k+5; \\ \lfloor \frac{7k+6}{2} \rfloor < 4k+3, & \text{if } n = 7k+6. \end{cases}$$

But this contradicts Lemma 3. Suppose D is a $\Gamma_{\mathrm{L}}(C_n)$-set, and assume $v_1, v_2 \in D$ without loss of generality. Noticing that no three consecutive vertices are in D by the minimality of D, we have $v_n \notin D$. Let P_{n-1} be the path resulting from C_n by removing the vertex v_n. Then D is a minimal LDS of P_{n-1}, and therefore $\Gamma_{\mathrm{L}}(C_n) = |D| \leq \Gamma_{\mathrm{L}}(P_{n-1})$ as desired. □

Proof of Theorem 2 By Theorem 1,

$$\Gamma_{\rm L}(P_{n-1}) = \begin{cases} 4k, & \text{if } n = 7k \text{ or } n = 7k+1; \\ 4k+1, & \text{if } n = 7k+2 \text{ or } n = 7k+3; \\ 4k+2, & \text{if } n = 7k+4 \text{ or } n = 7k+5; \\ 4k+3, & \text{if } n = 7k+6. \end{cases}$$

Then, Theorem 2 follows from Lemmas 3 and 4. □

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